The Riemann Hypothesis and Its Proof

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Received: June 20, 2018 Accepted: August 21, 2018 Online Published: September 6, 2018

doi:10.5539/jmr.v10n5p104 URL: https://doi.org/10.5539/jmr.v10n5p104

Abstract

We have presented a proof for the Riemann hypothesis by using the integral representative of the Riemann zeta function, the Dirichlet eta function, the Dirichlet lambda function and some partial sums. Here, we have mainly focused on $\left|\lim_{s\to s_c} \dfrac{\zeta(s)}{\zeta(1-s)}\right|$. This limit has to be finite and nonzero with any complex value in the critical strip ($0 < \Re(s_c) < 1$).

Keywords: Riemann hypothesis, proof, Riemann zeta function

We have presented this study in three parts. The first part mainly belongs the well-known information. In second part, three preparatory works have been introduced to support the proof. In last part, the proof has been handled directly.

PART I: It has comprised of Sections 1, 2, 3 and 4.

1. Introduction

The Riemann zeta function (Bombieri, 2000) is defined as below in the half-plane with the complex variable $s = \sigma + it$. It is an **absolutely convergent** series by $1 < \Re(s)$. The $\Re(s)$ is the real part of the complex variable.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} \cdots \qquad \text{for } 1 < \Re(s) \tag{1.1}$$

As shown by **Riemann**, $\zeta(s)$ extends to the whole complex plane $\mathbb{C}$ as a meromorphic function by analytic continuation and satisfies the functional equation:

$$\zeta(s)\,\Gamma\left(\frac{s}{2}\right) = \zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{s-1/2} \qquad \text{for } \mathbf{s \neq 1} \tag{1.2}$$

Also, if we handle another functional equation (Saidak & Zvengrowski, 2003, p146),

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s) \qquad \text{for } \mathbf{s \neq 1} \tag{1.3}$$

we can see that the Riemann zeta function $\zeta(s)$ has zeros at the negative even integers ($-2, -4, -6, ...$). These are called as the trivial zeros. The Riemann hypothesis which proposed by Riemann in 1859 is concerned with the non-trivial zeros (Bombieri, 2000). Let us denote these non-trivial zeros as $s_\circ = \sigma_\circ + it_\circ$.

The Riemann hypothesis: The real part of any non-trivial zeros of the Riemann zeta function is $1/2$. It means, $\zeta(s_\circ) = 0 \Rightarrow \sigma_\circ = 1/2$. It is one of the most important unsolved problems in mathematics which will help to illuminate the distribution of the prime numbers .

2. The Gamma Function

The gamma function is defined for any complex numbers ´s´ except the non-positive integers. So, it is analytic everywhere except at $s = 0, -1, -2, -3 ...$

Some well-known features of the gamma function (Milicic's notes, p5):

- It is expressed with the positive real part via the improper integral:

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx \qquad \text{for } 0 < \Re(s)$$

- It satisfies the factorial by positive integers as $\Gamma(n) = (n-1)!$
- It satisfies the functional equation as:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \qquad \text{for all ´}\mathbf{s}\text{´ except the integers}$$

- It has no zeros.

Remark 1. *Depending on the above properties, in the region* $0 < \Re(s) < 1$, *we can say that* $\Gamma(s)$ *and* $\Gamma(1-s)$ *have no zeros, no poles and they are finite.*

3. The Riemann Zeta Function in the Region $0 < \Re(s) < 1$

In 1914 Hardy proved that $\zeta(1/2+it)$ has infinitely many zeros. Also, Xavier verified the Riemann Hypothesis numerically on the first 10^{13} zeros. See also our another work about the zeros of the Riemann zeta function (Gunes, 2018).

Remark 2. *If* $\zeta(s)$ *is non-trivially equal to zero, the real part must satisfy the following condition which had been already proven (Titchmarsh, 1986, p30):*

$$0 < \sigma_\circ < 1 \qquad \textit{for } s_\circ = \sigma_\circ + it_\circ$$

Theorem 1. *If one of the Riemann zeta functions which in* $\zeta(s)$, $\zeta(1-s)$ *and* $\zeta(\overline{s})$ *is non-trivially equal to zero; then all of them are equal to zero.*

$$\zeta(s) = 0 \Leftrightarrow \zeta(1-s) = 0 \Leftrightarrow \zeta(\overline{s}) = 0 \tag{3.1}$$

Proof. Let us show it easily in the $0 < \Re(s) < 1$ region:

If we set the functional equation in 1.3 equal to zero:

$$\zeta(s) = 0 \quad \Rightarrow \quad 2^s \pi^{s-1} \sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s) = 0$$

The following result is obtained for $0 < \Re(s) < 1$:

$$\Gamma(1-s)\zeta(1-s) = 0$$

And we know $\Gamma(1-s) \neq 0$ from Remark 1. Consequently, $\zeta(1-s) = 0$.

On the other hand, the Schwarz reflection principle in complex analysis (Cartan, 1995) says:

$$\zeta(\overline{s}) = \overline{\zeta(s)}$$

Thus, we can write accordingly: $\zeta(s) = 0 \Leftrightarrow \zeta(\overline{s}) = 0$

□

4. The Dirichlet Functions

Before the Dirichlet functions, let us define a new **absolutely convergent** function as $q(s)$:

$$q(s) = \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} \cdots \qquad \text{for } 1 < \Re(s)$$

The relation between it and the Riemann zeta function looks obviously:

$$q(s) = \frac{1}{2^s} \sum_{n=1}^{\infty} \frac{1}{n^s} = 2^{-s}\zeta(s) \qquad \text{for } 1 < \Re(s) \tag{4.1}$$

4.1 The Dirichlet Lambda Function

It is a well-known **absolutely convergent** series.

$$\lambda(s) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} = \frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} \cdots \qquad \text{for} \quad 1 < \Re(s)$$

Because the Riemann zeta function is absolutely convergent for $1 < \Re(s)$, we can rearrange it:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} + \sum_{n=1}^{\infty} \frac{1}{(2n)^s} \qquad \text{for} \ 1 < \Re(s)$$

$$\lambda(s) = \zeta(s) - q(s) \qquad \text{for } 1 < \Re(s)$$

Thus, we see that the Dirichlet lambda function satisfies:

$$\lambda(s) = (1 - 2^{-s})\zeta(s) \qquad \text{for } 1 < \Re(s) \tag{4.2}$$

4.2 The Dirichlet Eta Function

It is also a well-known **convergent** series:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} \cdots \qquad \text{for} \ 0 < \Re(s)$$

PART II: We will present three preparatory works (Sections 5, 6 and 7) in this part before the proof attempt.

5. The First Work

- By an analytic continuation, the Riemann zeta function satisfies the Dirichlet eta function through the following well-known equation (Roman, & Minac, 2009, p6):

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}} \qquad \text{for} \ 0 < \Re(s) \text{ and } s \neq 1 \tag{5.1}$$

  It is very clear that the term $1 - 2^{1-s}$ above is finite and nonzero when $0 < \Re(s) < 1$. So we can write by the non-trivial zeros (which $s_\circ = \sigma_\circ + it_\circ$ with the feature $0 < \sigma_\circ < 1$):

$$\zeta(s_\circ) = 0 \Leftrightarrow \eta(s_\circ) = 0 \quad \text{Then with Theorem 1:}$$

$$\eta(s_\circ) = 0 \Leftrightarrow \eta(1 - s_\circ) = 0 \Leftrightarrow \eta(\overline{s_\circ}) = 0 \tag{5.2}$$

- Now here, let us consider *a positive integer which is sufficiently large*. Let us denote it as *´k´* **.** Then, let us consider another value as *´m´* with $m = 2k$ feature and let us write the following partial sum:

$$\eta_m(s) = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} \cdots - \frac{1}{m^s} \qquad \text{for} \ 0 < \Re(s) \tag{5.3}$$

- Thus, we can rewrite the Riemann zeta function with eq. 5.3 as below:

$$\zeta(s) = \lim_{m \to \infty} \frac{\eta_m(s)}{1 - 2^{1-s}} \qquad \text{for } 0 < \Re(s) \text{ and } s \neq 1 \tag{5.4}$$

- With the $m = 2k$ feature; we can rearrange the following partial sum in the region $0 < \Re(s) < 1$. We know that for a finite sum, a rearrangement of its terms does not change the value of the sum.

$$\sum_{n=1}^{m}\frac{1}{n^s} = \sum_{n=1}^{k}\frac{1}{(2n-1)^s} + \sum_{n=1}^{k}\frac{1}{(2n)^s} \qquad \text{for } 0 < \Re(s) < 1 \tag{5.5}$$

- We can also rearrange the partial series $\eta_m(s)$ for $0 < \Re(s) < 1$ as:

$$\eta_m(s) \Rightarrow \eta_{m_r}(s), \ \text{ it means } \ \eta_{m_r}(s) = \sum_{n=1}^{k}\frac{1}{(2n-1)^s} - \sum_{n=1}^{k}\frac{1}{(2n)^s} \tag{5.6}$$

6. The Second Work

Let us write for $\mathbf{1} < \Re(\mathbf{s})$:

$$\zeta(s) + \frac{1}{1-s} = \sum_{n=1}^{\infty}\frac{1}{n^s} - \int_1^{\infty}\frac{1}{x^s}\,dx \qquad \text{for } 1 < \Re(s)$$

Then, we can write the following formula (see also Elkies notes, p1):

$$\zeta(s) = \sum_{n=1}^{\infty}\left(\frac{1}{n^s} - \int_n^{n+1}\frac{1}{x^s}\,dx\right) - \frac{1}{1-s} \qquad \text{for } 0 < \Re(s) \text{ and } s \neq 1 \tag{6.1}$$

Meanwhile, after this stage, let us focus to work only in the critical strip $\mathbf{0} < \Re(\mathbf{s}) < \mathbf{1}$. Now, let us consider the above-Riemann zeta function until the ´m´ integer which was sufficiently large:

$$\zeta_m(s) = \sum_{n=1}^{m}\left(\frac{1}{n^s} - \int_n^{n+1}\frac{1}{x^s}\,dx\right) - \frac{1}{1-s} \qquad \text{for } 0 < \Re(s) < 1 \tag{6.2}$$

Because it is a partial sum now, we can rearrange it as below:

$$\zeta_{m_r}(s) = \sum_{n=1}^{m}\frac{1}{n^s} - \left(\int_1^{2}\frac{1}{x^s}\,dx + \int_2^{3}\frac{1}{x^s}\,dx + \cdots \int_{m-1}^{m}\frac{1}{x^s}\,dx + \int_m^{m+1}\frac{1}{x^s}\,dx\right) - \frac{1}{1-s}$$

$$\zeta_{m_r}(s) = \sum_{n=1}^{m}\frac{1}{n^s} - \int_0^{m+1}\frac{1}{x^s}\,dx \qquad \text{for } 0 < \Re(s) < 1 \tag{6.3}$$

Right after, we can write the functions $\mathbf{q}(\mathbf{s})$ and $\lambda(\mathbf{s})$ as below:

$$q(s) + \frac{1}{2^s(1-s)} = \sum_{n=1}^{\infty}\frac{1}{(2n)^s} - \int_1^{\infty}\frac{1}{(2x)^s}\,dx \qquad \text{for } 1 < \Re(s)$$

Then for $0 < \Re(s) < 1$:

$$q(s) = \sum_{n=1}^{\infty}\left(\frac{1}{(2n)^s} - \int_n^{n+1}\frac{1}{(2x)^s}\,dx\right) - \frac{1}{2^s(1-s)} \tag{6.4}$$

$$q_k(s) = \sum_{n=1}^{k}\left(\frac{1}{(2n)^s} - \int_n^{n+1}\frac{1}{(2x)^s}\,dx\right) - \frac{1}{2^s(1-s)} \tag{6.5}$$

$$q_{k_r}(s) = \sum_{n=1}^{k}\frac{1}{(2n)^s} - \int_0^{k+1}\frac{1}{(2x)^s}dx \tag{6.6}$$

By the same base, the other function:

$$\lambda(s) + \frac{1}{2(1-s)} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} - \int_1^{\infty} \frac{1}{(2x-1)^s}\, dx \qquad \text{for } 1 < \Re(s)$$

Then for $0 < \Re(s) < 1$:

$$\lambda(s) = \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^s} - \int_n^{n+1} \frac{1}{(2x-1)^s}\, dx \right) - \frac{1}{2(1-s)} \tag{6.7}$$

$$\lambda_k(s) = \sum_{n=1}^{k} \left(\frac{1}{(2n-1)^s} - \int_n^{n+1} \frac{1}{(2x-1)^s}\, dx \right) - \frac{1}{2(1-s)} \tag{6.8}$$

$$\lambda_{k_r}(s) = \sum_{n=1}^{k} \frac{1}{(2n-1)^s} - \int_{1/2}^{k+1} \frac{1}{(2x-1)^s}\, dx \tag{6.9}$$

Remark 3. *Let us be aware that if we intend to handle the function $\zeta_m(s)$ with the infinity for $0 < \Re(s) < 1$; we can re-consider equation 6.2 as below:*

$$\lim_{m\to\infty} \zeta_m(s) = \lim_{m\to\infty} \left(\sum_{n=1}^{m} \left(\frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s}\, dx \right) - \frac{1}{1-s} \right) = \zeta(s)$$

On the other hand, we have to avoid dealing $\zeta_{m_r}(s)$ in 6.3 directly with infinity.

$$\zeta_{m_r}(s) = \sum_{n=1}^{m} \frac{1}{n^s} - \int_0^{m+1} \frac{1}{x^s}\, dx$$

However, we can handle $\zeta_{m_r}(s)$ with infinity after re-transforming it to $\zeta_m(s)$. Because they are partial sums, we can transform each other as $\zeta_{m_r}(s) \Leftrightarrow \zeta_m(s)$:

$$\sum_{n=1}^{m} \frac{1}{n^s} - \int_0^{m+1} \frac{1}{x^s}\, dx \quad \Leftrightarrow \quad \sum_{n=1}^{m} \left(\frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s}\, dx \right) - \frac{1}{1-s}$$

By the same way, in the region $0 < \Re(s) < 1$, we can re-consider the functions $\eta_m(s)$, $q_k(s)$ and $\lambda_k(s)$ in 5.3, 6.5 and 6.8 with the infinity as below:

$$\lim_{m\to\infty} \eta_m(s) = \lim_{m\to\infty} \left(\sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^s} \right) = \eta(s)$$

$$\lim_{k\to\infty} q_k(s) = \lim_{k\to\infty} \left(\sum_{n=1}^{k} \left(\frac{1}{(2n)^s} - \int_n^{n+1} \frac{1}{(2x)^s}\, dx \right) - \frac{1}{2^s(1-s)} \right) = q(s)$$

$$\lim_{k\to\infty} \lambda_k(s) = \lim_{k\to\infty} \left(\sum_{n=1}^{k} \left(\frac{1}{(2n-1)^s} - \int_n^{n+1} \frac{1}{(2x-1)^s}\, dx \right) - \frac{1}{2(1-s)} \right) = \lambda(s)$$

Again, we should avoid handling $\eta_{m_r}(s)$, $q_{k_r}(s)$ and $\lambda_{k_r}(s)$ in 5.6, 6.6 and 6.9 directly with the infinity. However again, we can handle them with the infinity after re-transforming them to $\eta_m(s)$, $q_k(s)$ and $\lambda_k(s)$.

6.1 Some Other Features of the Functions $q(s)$ and $\lambda(s)$

The relation between $q(s)$ and the Riemann zeta function looks:

$$q(s) = 2^{-s} \left(\sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s}\, dx \right) - \frac{1}{1-s} \right) \tag{6.10}$$

$$q(s) = 2^{-s}\zeta(s) \qquad \text{for } 0 < \Re(s) < 1 \tag{6.11}$$

Theorem 2. *The function* $\lambda(s)$ *satisfies the Riemann zeta function:*

$$\lambda(s) = (1 - 2^{-s})\zeta(s) \qquad \textit{for} \ 0 < \Re(s) < 1 \tag{6.12}$$

Proof. In fact, the proof can be seen spontaneously depending on equations 4.1, 4.2, 6.1, 6.4, 6.7 and 6.11. However, let us try to show it in the following way which will also be useful for incoming topics.

If we find the following equality in the region $0 < \Re(s) < 1$:

$$q(s) + \lambda(s) = \zeta(s)$$

The proof is obtained with the equation $q(s) = 2^{-s}\zeta(s)$:

$$\lambda(s) = (1 - 2^{-s})\zeta(s)$$

Thus, by the next four stages, let us try to see the sum "$q(s) + \lambda(s)$":

i-) Let us start with the functions $q_{k_r}(s)$ and $\lambda_{k_r}(s)$ as below:

$$q_{k_r}(s) + \lambda_{k_r}(s) = \left(\sum_{n=1}^{k}\frac{1}{(2n-1)^s} + \sum_{n=1}^{k}\frac{1}{(2n)^s}\right) - \left(\int_{1/2}^{k+1}\frac{1}{(2x-1)^s}\,dx + \int_{0}^{k+1}\frac{1}{(2x)^s}\,dx\right)$$

For the above-sums, let remember eq. 5.5 as:

$$\sum_{n=1}^{k}\frac{1}{(2n-1)^s} + \sum_{n=1}^{k}\frac{1}{(2n)^s} = \sum_{n=1}^{m}\frac{1}{n^s} \qquad \text{for } 0 < \Re(s) < 1 \text{ and } m = 2k \tag{6.13}$$

ii-) Before to handle the integral parts, let us try to check the following limit for $0 < \Re(s) < 1$. It means, can we perform it or not?

$$\lim_{x\to\infty}\left(\int_{1/2}^{x}\frac{1}{(2x-1)^s}dx - \int_{0}^{x}\frac{1}{(2x)^s}dx\right) \overset{?}{=} 0 \implies \lim_{x\to\infty}\left(\frac{(2x-1)^{1-s}}{2(1-s)} - \frac{(2x)^{1-s}}{2(1-s)}\right) \overset{?}{=} 0$$

With the form of the complex variable as $\mathbf{s} = \sigma + it$; we obtain:

$$\lim_{x\to\infty}\left((2x-1)^{-it}\left((2x-1)^{1-\sigma} - (2x)^{1-\sigma}\frac{(2x)^{-it}}{(2x-1)^{-it}}\right)\right) \overset{?}{=} 0$$

We know that $\lim_{x\to\infty}(2x-1)^{-it} \neq 0$ and $\lim_{x\to\infty}\left(\frac{2x}{2x-1}\right)^{-it} = 1$

Thus with these results:

$$\lim_{x\to\infty}\left((2x-1)^{1-\sigma} - (2x)^{1-\sigma}\right) \overset{?}{=} 0$$

Then, if we take the advantage of *the mean value theorem,* we achieve:

$$\lim_{x\to\infty}\left((2x-1)^{1-\sigma} - (2x)^{1-\sigma}\right) = 0 \qquad \text{by } 0 < \sigma < 1. \quad \text{Here, } \sigma = \Re(s)$$

Right after, we can also write the following one which we will need in eq. 6.17:

$$\lim_{k\to\infty}\left(\int_{1/2}^{k+1}\frac{1}{(2x-1)^s}dx - \int_{0}^{k+1}\frac{1}{(2x)^s}dx\right) = 0 \qquad \text{for } 0 < \Re(s) < 1 \tag{6.14}$$

iii-) Now by $0 < \Re(s) < 1$, let us handle the integral parts as below with a new function $T_k(s)$ which also defined in the region $0 < \Re(s) < 1$. And let us remember that the ´k´ was sufficiently large.

$$\int_{1/2}^{k+1} \frac{1}{(2x-1)^s}\,dx + \int_0^{k+1} \frac{1}{(2x)^s}\,dx \;=2\int_0^{k+1/2} \frac{1}{(2x)^s}\,dx \;+\; T_k(s) \tag{6.15}$$

Thus by the $m = 2k$ feature:

$$\int_{1/2}^{k+1} \frac{1}{(2x-1)^s}\,dx + \int_0^{k+1} \frac{1}{(2x)^s}\,dx \;=\int_0^{m+1} \frac{1}{x^s}dx \;+\; T_k(s) \tag{6.16}$$

iv-) Let us return to the sum ´$q_{k_r}(s) + \lambda_{k_r}(s)$ ´ with equations 6.13 and 6.16:

$$q_{k_r}(s) + \lambda_{k_r}(s) \;= \sum_{n=1}^{m} \frac{1}{n^s} - \left(\int_0^{m+1} \frac{1}{x^s}dx \;+\; T_k(s)\right)$$

By equation 6.3:

$$q_{k_r}(s) + \lambda_{k_r}(s) \;= \zeta_{m_r}(s) - T_k(s)$$

Then let us try to consider it with the infinity. As you know that we have already expressed in **Remark 3** how we can handle the functions $q_{k_r}(s)$, $\lambda_{k_r}(s)$ and $\zeta_{m_r}(s)$ with the infinity for $0 < \Re(s) < 1$. First, we have to transform them to $q_k(s)$, $\lambda_k(s)$ and $\zeta_m(s)$. It means:

$$\sum_{n=1}^{k} \frac{1}{(2n)^s} - \int_0^{k+1} \frac{1}{(2x)^s}\,dx \;\Rightarrow\; \sum_{n=1}^{k} \left(\frac{1}{(2n)^s} - \int_n^{n+1} \frac{1}{(2x)^s}\,dx\right) - \frac{1}{2^s(1-s)}$$

$$\sum_{n=1}^{k} \frac{1}{(2n-1)^s} - \int_{1/2}^{k+1} \frac{1}{(2x-1)^s}\,dx \;\Rightarrow\; \sum_{n=1}^{k} \left(\frac{1}{(2n-1)^s} - \int_n^{n+1} \frac{1}{(2x-1)^s}\,dx\right) - \frac{1}{2(1-s)}$$

$$\sum_{n=1}^{m} \frac{1}{n^s} - \int_0^{m+1} \frac{1}{x^s}\,dx \;\Rightarrow\; \sum_{n=1}^{m} \left(\frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s}\,dx\right) - \frac{1}{1-s}$$

Thus after the transforming, we can write:

$$\lim_{k\to\infty} (q_k(s) + \lambda_k(s)) = \lim_{m\to\infty} (\zeta_m(s) - T_k(s)) \qquad \text{by } m = 2k \tag{6.17}$$

Beside, if we evaluate equations 6.14 and 6.15 together, we can easily see that $\lim_{k\to\infty} T_k(s) = 0$. Thus if we re-consider the statements in **Remark 3** about how we can handle $q_k(s)$, $\lambda_k(s)$ and $\zeta_m(s)$ with the infinity, we get:

$$q(s) + \lambda(s) = \zeta(s)$$

□

Theorem 3. *In the region* $0 < \Re(s) < 1$, *we can write by the* $m = 2k$ *feature:*

$$\eta_m(s) = \lambda_k(s) - q_k(s) + \int_{1/2}^{k+1} \frac{1}{(2x-1)^s}dx - \int_0^{k+1} \frac{1}{(2x)^s}\,dx$$

Proof. The proof will be very similar to the proof of Theorem 2. Let us call equation 5.6 here with the $m = 2k$ feature:

$$\eta_{m_r}(s) = \sum_{n=1}^{k} \frac{1}{(2n-1)^s} - \sum_{n=1}^{k} \frac{1}{(2n)^s} \qquad \text{for } 0 < \Re(s) < 1$$

Then, let us handle it with the equations 6.6 and 6.9 as below:

$$\eta_{m_r}(s) = \lambda_{k_r}(s) - q_{k_r}(s) + \int_{1/2}^{k+1} \frac{1}{(2x-1)^s}dx \;- \int_0^{k+1} \frac{1}{(2x)^s}\,dx \tag{6.18}$$

Meanwhile, we have already showed how we can transform the functions "$\eta_{m_r}(s)$, $\lambda_{k_r}(s)$ and $q_{k_r}(s)$" to "$\eta_m(s)$, $\lambda_k(s)$ and $q_k(s)$" in the proof of Theorem 2.

$$\eta_m(s) = \lambda_k(s) - q_k(s) + \int_{1/2}^{k+1} \frac{1}{(2x-1)^s} dx - \int_0^{k+1} \frac{1}{(2x)^s}\, dx$$

On the other hand, if we go a little further, we can also see some important results. As you know that we have already expressed in **Remark 3** how we can handle the functions $\eta_m(s)$, $q_k(s)$ and $\lambda_k(s)$ with the infinity.

$$\lim_{m\to\infty} \eta_m(s) = \lim_{k\to\infty} \left(\lambda_k(s) - q_k(s) + \int_{1/2}^{k+1} \frac{1}{(2x-1)^s} dx - \int_0^{k+1} \frac{1}{(2x)^s}\, dx \right)$$

$$\eta(s) = \lambda(s) - q(s) + \lim_{k\to\infty} \left(\int_{1/2}^{k+1} \frac{1}{(2x-1)^s} dx - \int_0^{k+1} \frac{1}{(2x)^s}\, dx \right)$$

Meanwhile, we know by equation 6.14 as:

$$\lim_{k\to\infty} \left(\int_{1/2}^{k+1} \frac{1}{(2x-1)^s} dx - \int_0^{k+1} \frac{1}{(2x)^s} dx \right) = 0 \qquad \text{for } 0 < \Re(s) < 1$$

Thus after the simplification, we can obtain **a nice new formula** as:

$$\eta(s) = \lambda(s) - q(s) \qquad \text{for } 0 < \Re(s) < 1 \tag{6.19}$$

Then if we handle it by equations 6.11 and 6.12; we obtain the well-known Riemann zeta function in 5.1.

$$\eta(s) = (1 - 2^{-s})\zeta(s) - 2^{-s}\zeta(s)$$

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}} \qquad \text{for } 0 < \Re(s) < 1$$

□

7. The Third Work

Let us try to write one of the integral representatives of the Riemann zeta function:

$$\zeta(s) + \frac{1}{1-s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s}\, dx \quad \text{for } 1 < \Re(s)$$

Then, we can write for $0 < \Re(s) < 1$ (see also the work of Saidak & Zvengrowski, 2003, p146):

$$\zeta(s) = \frac{1}{s-1} - s\int_1^{\infty} \frac{x - \lfloor x \rfloor - 1}{x^{s+1}} dx \quad \Rightarrow \quad \zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \frac{x - \lfloor x \rfloor}{x^{s+1}} dx \tag{7.1}$$

$\lfloor x \rfloor$ is the floor function which is also expressed as *floor*(x) which refers the largest integer less than or equal to ´x´.

Then, let us consider it as partial with a sufficiently large value ´u´ as $u \in \mathbb{R}$:

$$\zeta_u(s) = \frac{s}{s-1} - s\int_1^{u} \frac{x - \lfloor x \rfloor}{x^{s+1}} dx \tag{7.2}$$

Then up to $2u$:

$$\zeta_{2u}(s) = \frac{s}{s-1} - s\int_1^{2u} \frac{x - \lfloor x \rfloor}{x^{s+1}} dx$$

Naturally, we see as reversing:

$$\lim_{u\to\infty} \zeta_u(s) = \zeta(s) \qquad \text{and} \qquad \lim_{u\to\infty} \zeta_{2u}(s) = \zeta(s)$$

Then, we can write the integral representatives of the functions $q(s)$ and $\lambda(s)$. Let us recall equation 6.11, then let us handle it with equation 7.1:

$$\begin{aligned} q(s) &= 2^{-s}\zeta(s) \qquad \text{for } 0<\Re(s)<1 \\ q(s) &= 2^{-s}\left(\frac{s}{s-1}-s\int_1^{\infty}\frac{x-\lfloor x\rfloor}{x^{s+1}}dx\right) \end{aligned}$$

By the same approach for $\lambda(s)$ in 6.12:

$$\begin{aligned} \lambda(s) &= (1-2^{-s})\zeta(s) \qquad \text{for } 0<\Re(s)<1 \\ \lambda(s) &= (1-2^{-s})\left(\frac{s}{s-1}-s\int_1^{\infty}\frac{x-\lfloor x\rfloor}{x^{s+1}}dx\right) \end{aligned}$$

7.1 The Function $q_u(s)$ and $\lambda_u(s)$

To comprehend the function $q_u(s)$ and $\lambda_u(s)$ by an easy way, let us write the following partial sums with $k=5$ and the $m=2k$ feature first:

$$\begin{aligned} \zeta_k(s) &= \sum_{n=1}^{k}\frac{1}{n^s} = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s} \qquad \text{for } 1<\Re(s) \\ \zeta_m(s) &= \sum_{n=1}^{m}\frac{1}{n^s} = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}\cdots+\frac{1}{10^s} \qquad \text{for } 1<\Re(s) \\ q_k(s) &= \sum_{n=1}^{k}\frac{1}{(2n)^s} = \frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{10^s} \qquad \text{for } 1<\Re(s) \\ \lambda_k(s) &= \sum_{n=1}^{k}\frac{1}{(2n-1)^s} = \frac{1}{1^s}+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s} \qquad \text{for } 1<\Re(s) \end{aligned}$$

Then if we consider the ´u´ as 5 as the same; we can write $\zeta_u(s)=\zeta_k(s)$, $\zeta_{2u}(s)=\zeta_m(s)$, $q_u(s)=q_k(s)$ and $\lambda_u(s)=\lambda_k(s)$.

Meanwhile, we have already known that $\zeta_u(s)$, $\zeta_{2u}(s)$, $q_u(s)$ and $\lambda_u(s)$ are valid in the $0<\Re(s)<1$ region by analytic continuation. Thus now, we can write the function $q_u(s)$ with the sufficiently large value ´u´ depending on the above-partial sums::

$$q_u(s) = 2^{-s}\zeta_u(s) = 2^{-s}\left(\frac{s}{s-1}-s\int_1^{u}\frac{x-\lfloor x\rfloor}{x^{s+1}}dx\right) \tag{7.3}$$

We can also see this relation on equation 6.5 as $q_k(s)=2^{-s}\zeta_k(s)$:

$$q_k(s) = 2^{-s}\zeta_k(s) = 2^{-s}\left(\sum_{n=1}^{k}\left(\frac{1}{n^s}-\int_n^{n+1}\frac{1}{x^s}\,dx\right)-\frac{1}{1-s}\right)$$

Anyway, let us try to show $\lambda_u(s)$:

$$\zeta_{2u}(s) = q_u(s)+\lambda_u(s)$$
$$\lambda_u(s) = \zeta_{2u}(s)-2^{-s}\zeta_u(s)$$

$$\lambda_u(s) = \left(\frac{s}{s-1}-s\int_1^{2u}\frac{x-\lfloor x\rfloor}{x^{s+1}}dx\right)-2^{-s}\left(\frac{s}{s-1}-s\int_1^{u}\frac{x-\lfloor x\rfloor}{x^{s+1}}dx\right) \tag{7.4}$$

Then, if we handle them with infinity:

$$\lim_{u\to\infty} q_u(s) = \lim_{u\to\infty} 2^{-s}\zeta_u(s) = 2^{-s}\zeta(s) = q(s) \tag{7.5}$$

$$\lim_{u\to\infty} \lambda_u(s) = \lim_{u\to\infty} \left(\zeta_{2u}(s) - 2^{-s}\zeta_u(s)\right) = \zeta(s) - 2^{-s}\zeta(s) = \lambda(s) \tag{7.6}$$

Thus, depending on Remark 3 and equations 7.5 & 7.6, we can consider:

$$\lim_{u\to\infty} q_u(s) \text{ and } \lim_{u\to\infty} \lambda_u(s) \quad \text{instead of} \quad \lim_{k\to\infty} q_k(s) \text{ and } \lim_{k\to\infty} \lambda_k(s) \tag{7.7}$$

7.2 The $q_u(s)$ and $\lambda_u(s)$ with the Infinity and the Non-trivial Zeros

$$\lim_{s\to s_\circ} \left(\lim_{u\to\infty} q_u(s)\right) = \lim_{s\to s_\circ} q(s) = \lim_{s\to s_\circ} 2^{-s}\zeta(s) = 2^{-s_\circ}\zeta(s_\circ) = 0 \tag{7.8}$$

$$\lim_{s\to s_\circ} \left(\lim_{u\to\infty} \lambda_u(s)\right) = \lim_{s\to s_\circ} \lambda(s) = \lim_{s\to s_\circ} (1 - 2^{-s})\zeta(s) = (1 - 2^{-s_\circ})\zeta(s_\circ) = 0 \tag{7.9}$$

7.3 The Derivative of the Function $q_u(s)$ and $\lambda_u(s)$

First, let us take the derivative of $\zeta_u(s)$ in 7.2 with respect to the parameter ´u´.

$$\lim_{u\to\infty} \frac{d}{du}\zeta_u(s) = \lim_{u\to\infty} \frac{d}{du}\left(\frac{s}{s-1} - s\int_1^u \frac{x - \lfloor x \rfloor}{x^{s+1}}dx\right)$$

Thus, we see that it will be equal to zero:

$$\lim_{u\to\infty} \frac{d}{du}\zeta_u(s) = \lim_{u\to\infty} -s\frac{u - \lfloor u \rfloor}{u^{s+1}} = 0 \qquad \text{for } 0 < \Re(s) < 1 \tag{7.10}$$

Then, the derivative of $\zeta_{2u}(s)$

$$\lim_{u\to\infty} \frac{d}{du}\zeta_{2u}(s) = \lim_{u\to\infty} -2s\frac{2u - \lfloor 2u \rfloor}{(2u)^{s+1}} = 0 \qquad \text{for } 0 < \Re(s) < 1 \tag{7.11}$$

Then, the derivative of the functions $q_u(s)$ and $\lambda_u(s)$ depending on equations 7.3, 7.4, 7.10 and 7.11 by the same conditions:

$$\lim_{u\to\infty} \frac{d}{du}q_u(s) = \lim_{u\to\infty} \frac{d}{du}\left(2^{-s}\zeta_u(s)\right) = 0 \quad \text{for } 0 < \Re(s) < 1 \tag{7.12}$$

$$\lim_{u\to\infty} \frac{d}{du}\lambda_u(s) = \lim_{u\to\infty} \frac{d}{du}\left(\zeta_{2u}(s) - 2^{-s}\zeta_u(s)\right) = 0 \qquad \text{for } 0 < \Re(s) < 1 \tag{7.13}$$

PART III: In this part, we will present the proof directly.

8. The Proof of the Riemann Hypothesis

Let us call equation 1.3 as $\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$.

We know from Remark 1 that $\Gamma(1-s)$ has no zeros and poles in **the critical strip** $(0 < \Re(s) < 1)$ and it is finite. Now let us denote any complex value in this critical strip as $\mathbf{s_c}$. We see $s_\circ \subset s_c$ easily. Thus, depending on Theorem 1, we can put forth the following conclusion:

$$\lim_{s\to s_c} \frac{\zeta(s)}{\zeta(1-s)} = 2^{s_c}\pi^{s_c-1} \sin(\frac{\pi s_c}{2})\Gamma(1-s_c) \tag{8.1}$$

$$\lim_{s\to s_c} \frac{\zeta(s)}{\zeta(1-s)} \quad \text{is analytic/continuous and it is finite \& nonzero} \tag{8.2}$$

If you want to see it with any non-trivial zeros of the Riemann zeta function which $s_\circ = \sigma_\circ + it_\circ$ with the feature $0 < \sigma_\circ < 1$ (even, with equation 1.2):

$$\lim_{s\to s_\circ} \frac{\zeta(s)}{\zeta(1-s)} = 2^{s_\circ}\pi^{s_\circ-1} \sin(\frac{\pi s_\circ}{2})\Gamma(1-s_\circ) = \pi^{s_\circ-1/2} * \frac{\Gamma\left(\frac{1-s_\circ}{2}\right)}{\Gamma\left(\frac{s_\circ}{2}\right)}$$

Now, let us handle eq. 8.1 with eq. 5.1 by noticing the result in 5.2:

$$\lim_{s\to s_c} \frac{\zeta(s)}{\zeta(1-s)} = \frac{1-2^{s_c}}{1-2^{1-s_c}} * \lim_{s\to s_c} \frac{\eta(s)}{\eta(1-s)} \tag{8.3}$$

You can also see the similar equations/conclusions in Schipani's work (see p100 and p103).

Then, let us call equation 5.3 here by considering Remark 3 as:

$$\lim_{m\to\infty} \eta_m(s) = \lim_{m\to\infty} \left(\sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^s} \right) = \eta(s)$$

Then, let us consider equation 8.3 along with it **by proceeding in the absolute value** as below:

$$\left|\lim_{s\to s_c} \frac{\zeta(s)}{\zeta(1-s)}\right| = \left|\frac{1-2^{s_c}}{1-2^{1-s_c}}\right| * \left|\lim_{s\to s_c} \left(\lim_{m\to\infty} \frac{\eta_m(s)}{\eta_m(1-s)}\right)\right| \quad \text{for } 0 < \Re(s_c) < 1 \tag{8.4}$$

Remark 4. *On the other hand, we have presented another possible proof* ***on last page****. There, we have used eq. 8.12 instead of the above equation.*

Anyway, we had already showed with Theorem 3 that the following equation was valid in the region $0 < \Re(s) < 1$ by the feature $m = 2k$:

$$\eta_m(s) = \lambda_k(s) - q_k(s) + \int_{1/2}^{k+1} \frac{1}{(2x-1)^s} dx - \int_0^{k+1} \frac{1}{(2x)^s} dx$$

Thus, we can place $\eta_m(s)$ and $\eta_m(1-s)$ into equation 8.4 accordingly.

$$\left|\lim_{s\to s_c} \frac{\zeta(s)}{\zeta(1-s)}\right| = \left|\frac{1-2^{s_c}}{1-2^{1-s_c}}\right| * \left|\lim_{s\to s_c} \left(\lim_{k\to\infty} \frac{\lambda_k(s) - q_k(s) + \dfrac{(2k+1)^{1-s}}{2(1-s)} - \dfrac{(2k+2)^{1-s}}{2(1-s)}}{\lambda_k(1-s) - q_k(1-s) + \dfrac{(2k+1)^{s}}{2s} - \dfrac{(2k+2)^{s}}{2s}}\right)\right| \tag{8.5}$$

Now, let us inspect its terms one by one:

i-) First, if we handle the term in the left side, we have already known by the statement in 8.2:

$$\left|\lim_{s\to s_c}\frac{\zeta(s)}{\zeta(1-s)}\right| \quad \text{is finite and nonzero.}$$

This criteria is very important when we evaluate equation 8.5**.** Because, the limit of the right side will be bounded and orientated to a value which can not be equal to zero or to infinity by this criteria. It means that the limit of the right side has to be also finite and nonzero.

ii-) Then, let us handle the first term in the right side. We have already known that its value can be only finite and nonzero:

$$\left|\frac{1-2^{s_c}}{1-2^{1-s_c}}\right| \quad \text{is finite and nonzero} \quad \text{for} \quad 0<\Re(s_c)<1$$

iii-) Finally, let us handle the other limit term (which is also **finite and nonzero naturally**) by denoting it as F :

$$F=\left|\lim_{s\to s_c}\left(\lim_{k\to\infty}\frac{\lambda_k(s)-q_k(s)+\dfrac{(2k+1)^{1-s}}{2(1-s)}-\dfrac{(2k+2)^{1-s}}{2(1-s)}}{\lambda_k(1-s)-q_k(1-s)+\dfrac{(2k+1)^{s}}{2s}-\dfrac{(2k+2)^{s}}{2s}}\right)\right|$$

Let us arrange it as following:

$$F=\left|\lim_{s\to s_c}\left(\lim_{k\to\infty}\left[\frac{\dfrac{(2k+1)^{1-s}}{2(1-s)}}{\dfrac{(2k+1)^{s}}{2s}}*\frac{2(1-s)\dfrac{\lambda_k(s)-q_k(s)}{(2k+1)^{1-s}}+1-\left(\dfrac{2k+2}{2k+1}\right)^{1-s}}{2s\dfrac{\lambda_k(1-s)-q_k(1-s)}{(2k+1)^{s}}+1-\left(\dfrac{2k+2}{2k+1}\right)^{s}}\right]\right)\right| \tag{8.6}$$

If we denote the multipliers with two new functions as $G_k(s)$ and $H_k(s)$:

$$F=\left|\lim_{s\to s_c}\left(\lim_{k\to\infty}\left[G_k(s)*H_k(s)\right]\right)\right|$$

Now, after this critical stage, let us work **ONLY with** the non-trivial zeros of the Riemann zeta function (which $s_\circ=\sigma_\circ+it_\circ$ and we know that $s_\circ\subset s_c$).

$$F=\left|\lim_{s\to s_\circ}\left(\lim_{k\to\infty}\left[G_k(s)*H_k(s)\right]\right)\right| \tag{8.7}$$

Remark 5. *Here let us pay attention that; since F is finite and nonzero, so if one of* $\lim_{s\to s_\circ}\left(\lim_{k\to\infty}G_k(s)\right)$ *or* $\lim_{s\to s_\circ}\left(\lim_{k\to\infty}H_k(s)\right)$ *is finite and nonzero, we can split it as follows:*

$$F=\left|\lim_{s\to s_\circ}\left(\lim_{k\to\infty}G_k(s)\right)*\lim_{s\to s_\circ}\left(\lim_{k\to\infty}H_k(s)\right)\right| \tag{8.8}$$

iii.a-) Thus, let us start with $H_k(s)$ to see if its limit is finite&nonzero or not. Let us change something on it. First, let take ´u´ instead of the ´k´ as $u\in\mathbb{R}$. Also as we stated in statement 7.7, we could take $\lim_{u\to\infty}q_u(s)$ and $\lim_{u\to\infty}\lambda_u(s)$ instead of $\lim_{k\to\infty}q_k(s)$ and $\lim_{k\to\infty}\lambda_k(s)$. Thus, it will be as:

$$H=\lim_{s\to s_\circ}\left(\lim_{k\to\infty}H_k(s)\right) \qquad \Rightarrow \qquad H=\lim_{s\to s_\circ}\left(\lim_{u\to\infty}H_u(s)\right)$$

$$H=\lim_{s\to s_\circ}\left(\lim_{u\to\infty}\frac{2(1-s)\dfrac{\lambda_u(s)-q_u(s)}{(2u+1)^{1-s}}+1-\left(\dfrac{2u+2}{2u+1}\right)^{1-s}}{2s\dfrac{\lambda_u(1-s)-q_u(1-s)}{(2u+1)^{s}}+1-\left(\dfrac{2u+2}{2u+1}\right)^{s}}\right)$$

Meanwhile at this critical stage, we know the following results by equations 7.8, 7.9 and Theorem 1:

$$\lim_{s\to s_\circ}\left(\lim_{u\to\infty} q_u(s)\right) = 0\,, \qquad \text{and} \qquad \lim_{s\to s_\circ}\left(\lim_{u\to\infty} \lambda_u(s)\right) = 0\,,$$

$$\lim_{s\to s_\circ}\left(\lim_{u\to\infty} q_u(1-s)\right) = 0 \qquad \text{and} \qquad \lim_{s\to s_\circ}\left(\lim_{u\to\infty} \lambda_u(1-s)\right) = 0$$

Then immediately, we can see that the limit of $H_u(s)$ will appear as ´$0/0$´ with the non-trivial zeros ´$s_\circ$´. (Here, at moment; let us draw your attention that we will start to use the non-trivial zeros of the Riemann zeta function in the operations actively)**.** Thus by this advantage, if we apply L'Hospital rule with respect to the parameter ´u´ :

$$H = \lim_{s\to s_\circ}\left(\lim_{u\to\infty}\frac{\dfrac{\mathbf{d}}{\mathbf{du}}\left(2(1-s)\dfrac{\lambda_u(s)-q_u(s)}{(2u+1)^{1-s}}\right) - \dfrac{2(1-s)\left(\dfrac{2u+2}{2u+1}\right)^{-s}}{(2u+1)^2}}{\dfrac{\mathbf{d}}{\mathbf{du}}\left(2s\dfrac{\lambda_u(1-s)-q_u(1-s)}{(2u+1)^{s}}\right) - \dfrac{2s\left(\dfrac{2u+2}{2u+1}\right)^{s-1}}{(2u+1)^2}}\right)$$

Then, let us call equations 7.12 and 7.13 here by considering Theorem 1:

$$\lim_{u\to\infty}\frac{d}{du}q_u(s) = 0 \quad \text{and} \quad \lim_{u\to\infty}\frac{d}{du}\lambda_u(s) = 0 \quad \text{for} \quad 0<\Re(s)<1$$

$$\lim_{u\to\infty}\frac{d}{du}q_u(1-s) = 0 \quad \text{and} \quad \lim_{u\to\infty}\frac{d}{du}\lambda_u(1-s) = 0 \quad \text{for} \quad 0<\Re(s)<1$$

Let us remember Remark 2 as $0<\sigma_\circ<1$ by $s_\circ = \sigma_\circ + it_\circ$. Thus, we can reach the following conclusions with eq. 7.8, 7.9, 7.12 and 7.13:

$$\lim_{s\to s_\circ}\left(\lim_{u\to\infty}\frac{\mathbf{d}}{\mathbf{du}}\left(2(1-s)\frac{\lambda_u(s)-q_u(s)}{(2u+1)^{1-s}}\right)\right) = 0$$

$$\lim_{s\to s_\circ}\left(\lim_{u\to\infty}\frac{\mathbf{d}}{\mathbf{du}}\left(2s\frac{\lambda_u(1-s)-q_u(1-s)}{(2u+1)^{s}}\right)\right) = 0$$

Consequently, after the simplification of H, we obtain:

$$H = \frac{1-s_\circ}{s_\circ} \qquad \text{which is finite and nonzero for} \quad 0<\sigma_\circ<1 \tag{8.9}$$

iii.b-) Thanks to L'Hospital rule here, because we have reached a solid limit result on the most complicated term in equation 8.6. Since we have showed that there are appropriate conditions for Remark 5, we can pass from equation 8.7 to 8.8 without any hesitation as below:

$$F = \left|\lim_{s\to s_\circ}\left(\lim_{k\to\infty}[G_k(s) * H_k(s)]\right)\right| \quad \Rightarrow \quad F = \left|\lim_{s\to s_\circ}\left(\lim_{k\to\infty}G_k(s)\right) * \lim_{s\to s_\circ}\left(\lim_{k\to\infty}H_k(s)\right)\right|$$

iii.c-) Since we obtained the outcome in 8.9 **by using the non-trivial zeros** of the Riemann zeta function; the results which can be found will be related **only** to these zeros (it mean the results which can be found will not refer to **the other values** in the critical strip $0<\Re(s)<1$ anymore)**.**

Now, let us write equation 8.8 with the result in 8.9 as below:

$$F = \left|\lim_{s\to s_\circ}\left(\lim_{k\to\infty}\frac{\dfrac{(2k+1)^{1-s}}{1-s}}{\dfrac{(2k+1)^{s}}{s}}\right) * \frac{1-s_\circ}{s_\circ}\right|$$

$$F = \left| \lim_{k\to\infty} \frac{(2k+1)^{1-s_\circ}}{(2k+1)^{s_\circ}} \right| \tag{8.10}$$

With the form of the non-trivial zeros as $s_\circ = \sigma_\circ + it_\circ$:

$$F = \left| \lim_{k\to\infty} (2k+1)^{1-2\sigma_\circ} \right| * \left| \lim_{k\to\infty} (2k+1)^{-i2t_\circ} \right| \tag{8.11}$$

Then, let us analyze it with $0 < \sigma_\circ < 1$**:**

- We have already known that F is finite and nonzero.

- The limit results of $\lim_{k\to\infty} (2k+1)^{1-2\sigma_\circ}$ for $0 < \sigma_\circ < 1$:

$$\begin{array}{lcll} \lim_{k\to\infty} (2k+1)^{1-2\sigma_\circ} & = & 1 & \text{If} \quad 1-2\sigma_\circ = 0 \\ \lim_{k\to\infty} (2k+1)^{1-2\sigma_\circ} & = & 0 & \text{If} \quad 1-2\sigma_\circ < 0 \\ \lim_{k\to\infty} (2k+1)^{1-2\sigma_\circ} & = & \infty & \text{If} \quad 1-2\sigma_\circ > 0 \end{array}$$

- Meanwhile with Euler's formula, we know that last term will equal to ´1´:

$$\left| \lim_{k\to\infty} (2k+1)^{-i2t_\circ} \right| = \left| \lim_{k\to\infty} [\cos(-2t_\circ \ln(2k+1)) + i\sin(-2t_\circ \ln(2k+1))] \right| = 1$$

Thus, according to these analyses, $\lim_{k\to\infty} (2k+1)^{1-2\sigma_\circ}$ has to equal to ´1´. It means that $1 - 2\sigma_\circ = 0$. Consequently, we find $\sigma_\circ = 1/2$. **So, the proof of the Riemann Hypothesis has been completed.**

8.1 The Second Way for the Proof

This new approach will have very little nuance differences when we compare to the first one. We know $|z| = |\overline{z}|$ for any ´z´ complex number**.** Now, let us call equation 1.3 and write it with absolute value as below:

$$|\zeta(\overline{s})| = |\zeta(s)| = \left| 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s) \right|$$

Then, depending on Theorem 1, we can write the following limit with $\mathbf{s_c}$ (and we know $s_\circ \subset s_c$):

$$\left| \lim_{s\to s_c} \frac{\zeta(\overline{s})}{\zeta(1-s)} \right| = \left| \lim_{s\to s_c} \frac{\zeta(s)}{\zeta(1-s)} \right| = \left| 2^{s_c} \pi^{s_c-1} \sin(\frac{\pi s_c}{2}) \Gamma(1-s_c) \right|$$

Thus again, we can say that $\left| \lim_{s\to s_c} \frac{\zeta(\overline{s})}{\zeta(1-s)} \right|$ is analytic/continuous and it is finite and nonzero.

Consequently, we can write the following one instead of equation 8.4*:*

$$\left| \lim_{s\to s_c} \frac{\zeta(\overline{s})}{\zeta(1-s)} \right| = \left| \frac{1-2^{s_c}}{1-2^{1-\overline{s}_c}} \right| * \left| \lim_{s\to s_c} \left(\lim_{m\to\infty} \frac{\eta_m(\overline{s})}{\eta_m(1-s)} \right) \right| \quad \text{for } 0 < \Re(s_c) < 1 \tag{8.12}$$

Thus if we go on with the same steps which we performed in the first proof, we obtain below one as instead of eq. 8.5:

$$\left| \lim_{s\to s_c} \frac{\zeta(\overline{s})}{\zeta(1-s)} \right| = \left| \frac{1-2^{s_c}}{1-2^{1-\overline{s}_c}} \right| * \left| \lim_{s\to s_c} \left(\lim_{k\to\infty} \frac{\lambda_k(\overline{s}) - q_k(\overline{s}) + \dfrac{(2k+1)^{1-\overline{s}}}{2(1-\overline{s})} - \dfrac{(2k+2)^{1-\overline{s}}}{2(1-\overline{s})}}{\lambda_k(1-s) - q_k(1-s) + \dfrac{(2k+1)^{s}}{2s} - \dfrac{(2k+2)^{s}}{2s}} \right) \right|$$

Then with the same steps, we see instead of equation 8.10:

$$F = \left| \lim_{k\to\infty} \frac{(2k+1)^{1-\bar{s}_\circ}}{(2k+1)^{s_\circ}} \right|$$

If we simplify it with the form of the nontrivial zeros as $s_\circ = \sigma_\circ + it_\circ$ with the feature $0 < \sigma_\circ < 1$, we obtain **a more simple appearance** compared to equation 8.11 in the first proof:

$$F = \left| \lim_{k\to\infty} (2k+1)^{1-2\sigma_\circ} \right|$$

Then, the next steps are considered the same as the first proof.

Acknowledgements

The author is grateful to Prof. Machiel van Frankenhuijsen who show courtesy by accepting to review the file and making useful suggestions about some points.

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