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An Algorithm for Solving Nonlinear Equations Using Distributions

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Abstract

In this paper we will present a new algorithm to solve the nonlinear equation $f(x) = 0$ where x is a scalar, indeed using the theory of distributions, we will be able to construct a sequence with an explicit formula that converges to the solution of $f(x) = 0$.

Keywords: distributions, Dirac, nonlinear equation

1. Introduction

Many methods have been developed in order to solve a nonlinear equation ($f(x) = 0$) and find its solution, we can cite for example the Newton's method. Many of these methods lead to an iterative sequence that converges to the solution, the disadvantage of these methods is the dependency of the first value of the sequence, in other words the sequence converges to the solution if the first value is close enough to the solution.

The goal of this paper is to create a sequence with an explicit formula, which converges to the solution of a given nonlinear equation where the solution is a scalar. Therefore in the following we will present a brief recall of the theory of distributions, and by manipulating the distributions we will be able to create this sequence which converges to the solution, and we will also present an example to illustrate the algorithm.

In addition to the explicit formula of the sequence, the advantage of this method is that it requires only a few conditions for the function f in a neighborhood of the solution.

1.1 Brief Recall of the Theory of Distributions (Dreyfuss, 2012), (Schwartz, 1963), (Schwartz, 1966)

1.1.1 Test Functions

Let $\Omega \subseteq \mathbb{R}^d$ be an open set where $d \in \mathbb{N}^*$, and let ϕ an application $\phi : \Omega \to \mathbb{R}$, then ϕ is a test function if and only if:

1. The support of ϕ: $supp(\phi)$ is compact in Ω
2. The function ϕ is infinitely differentiable: $\phi \in C^\infty(\Omega)$

The space of the test functions on Ω is called $\mathcal{D}(\Omega)$.

Let ϕ_n be a sequence in $\mathcal{D}(\Omega)$ and let $\phi \in \mathcal{D}(\Omega)$ then we say that ϕ_n converges to ϕ in $\mathcal{D}(\Omega)$: $\phi_n \overset{\mathcal{D}(\Omega)}{\longrightarrow} \phi$ if and only if:

1. There exist a compact K of Ω such that $supp(\phi_n) \subset K$ $\forall n \in \mathbb{N}$.
2. $\sup_{x\in K}(|\partial^\alpha \phi_n(x) - \partial^\alpha \phi(x)|) \underset{n\to+\infty}{\longrightarrow} 0$ $\forall \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, where $\partial^\alpha \phi = \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_d$.

Example

The function ϕ defined by:

$$\phi(x) = \begin{cases} e^{\frac{-1}{1-x^2}} & if\ |x| < 1 \\ 0 & otherwise \end{cases}$$

belongs to $\mathcal{D}(\mathbb{R})$ and $supp(\phi) = [-1, 1]$.

1.1.2 Distributions

Let $\Omega \subseteq \mathbb{R}^d$ be an open set, a distribution is a linear application

$T : \mathcal{D}(\Omega) \to \mathbb{R}$ such that:

$$If\ \phi_n \in \mathcal{D}(\Omega)\ and\ \phi_n \overset{\mathcal{D}(\Omega)}{\longrightarrow} \phi \Rightarrow T(\phi_n) \underset{n\to+\infty}{\longrightarrow} T(\phi)$$

$< T, \phi >$ denotes $T(\phi)$, and $\mathcal{D}'(\Omega)$ denotes the space of the distributions on Ω.

Example

Let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let f a function $f : \Omega \to \mathbb{R}$ such that f is locally integrable on Ω: $f \in L^1_{loc}(\Omega)$. If T is an application such that:

$$< T, \phi >= \int_\Omega f\phi\, d\lambda \quad \forall \phi \in \mathcal{D}(\Omega)$$

Where λ is the Lebesgue measure, then T is a distribution: $T \in \mathcal{D}'(\Omega)$. Otherwise T is also noted by f, in other words $< f, \phi >=< T, \phi >$.

2. Creating a Sequence of Distributions that Converges to a Dirac

We consider in the following a function $g : \bar{I} \to \mathbb{R}$ where I is a non-empty, bounded and open interval $I \subset \mathbb{R}$. We suppose that g is a positive function $g(x) \geq 0\ \forall x \in \bar{I}$. Using some additional properties on g we will be able to create a sequence of distributions which converges to the Dirac distribution, this creation is done by the theorem 1.

The following lemma is used in the proof of the theorem 1.

Lemma 1 *Let $I_0 \subset I$ be an open interval. We suppose that $g \in C^2(I_0)$, $g(x) \geq 0\ \forall x \in I_0$ and there exists a zero $x^* \in I_0$ for the function g: $g(x^*) = 0$, let g' denotes the derivative of the function g, we suppose in addition that there exists a finite number of zeros for the function g' in a neighborhood of x^*. Then there exists a neighborhood of x^*: $[x_1, x_2]$ such that $g'(x) \leq 0\ \forall x \in [x_1, x^*]$ and $g'(x) \geq 0\ \forall x \in [x^*, x_2]$.*

Proof. First we will prove that $\exists\ r > 0$ such that $\forall x \in [x^* - r, x^*[\Rightarrow g'(x) \leq 0$. In order to prove this we will proceed by contradiction, in other words we suppose that: $\forall r > 0\ \exists\ x \in [x^* - r, x^*[\Rightarrow g'(x) > 0$.

Let the interval $J \subset I_0$ be the neighborhood of x^* which contains a finite number of zeros of the function g', and let N be the number of these zeros. we take $r^* > 0$ such that $[x^* - r^*, x^*[\subset J$ and such that $[x^* - r^*, x^*[$ contains no zeros of g'.

Therefore $\exists X \in [x^* - r^*, x^*[\Rightarrow g'(X) > 0$, if we suppose that $g'(x) > 0\ \forall x \in [X, x^*[$ then g is a strictly increasing function on $[X, x^*[$ and $g(x^*) > 0$ which is absurd. Thus we conclude that $\exists\ y \in]X, x^*[$ such that $g'(y) = 0$ because g' is a continuous function.

Therefore there exist a $N + 1$ zeros of g' in the neighborhood J of x^*, which contradicts our hypothesis.

With the same manner we prove that $\exists\ r_2 > 0$ such that $\forall x \in]x^*, x^* + r_2] \Rightarrow g'(x) \geq 0$.

We conclude that there exist a neighborhood of x^*: $[x_1, x_2]$ such that $g'(x) \leq 0\ \forall x \in [x_1, x^*]$ and $g'(x) \geq 0\ \forall x \in [x^*, x_2]$.

Theorem 1 *Let the function g have only one zero x^* in $\bar{I}$ and $x^* \in I$ such that g is C^2 in a neighborhood of x^* and $g''(x^*) > 0$, we suppose in addition that there exists a finite number of zeros for the function g' in a neighborhood of x^*, if we consider the sequence of functions g_n defined by: $g_n(x) = \boldsymbol{1}_I(x)\frac{n}{\sqrt{\pi}}e^{-n^2 g(x)}$ then:*

$$g_n \underset{n\to+\infty}{\longrightarrow} \frac{1}{\sqrt{c}}\ \delta_{x^*} \quad \textit{In the sense of distributions}$$

Where $c = \frac{g''(x^*)}{2}$, δ_{x^*} the Dirac distribution at x^*. The convergence in the sense of distributions means that:

$$< g_n, \phi >= \int_\mathbb{R} g_n(x)\phi(x)\, d\lambda(x) \underset{n\to+\infty}{\longrightarrow} \frac{1}{\sqrt{c}}\ \phi(x^*) \quad \forall \phi \in \mathcal{D}(\mathbb{R})$$

.

Proof. First the function g reaches its minimum at x^* because $g \geq 0$ thus $g'(x^*) = 0$. Using the Taylor's theorem we can express g by:

$$g(x) = c(x - x^*)^2 + (x - x^*)^2\epsilon(x - x^*)$$

Where $c = \frac{g''(x^*)}{2}$ and ϵ a function $\epsilon : \mathbb{R} \to \mathbb{R}$ such that: $\epsilon(x) \underset{x\to 0}{\longrightarrow} 0$.

Let f_n denotes the sequence of functions defined by: $f_n(x) = \frac{n}{\sqrt{\pi}} e^{-cn^2(x-x^*)^2}$. In the following of this proof we will prove that:

$$\|f_n - g_n\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f_n - g_n| \, d\lambda \underset{n\to+\infty}{\longrightarrow} 0$$

In order to prove this convergence we will study it on a different intervals, thus we begin by the interval $I_n = [x^* - \frac{1}{\sqrt{n}}, x^* + \frac{1}{\sqrt{n}}]$:

$$\int_{I_n} |f_n - g_n| \, d\lambda = \frac{n}{\sqrt{\pi}} \int_{x^*-\frac{1}{\sqrt{n}}}^{x^*+\frac{1}{\sqrt{n}}} e^{-cn^2(x-x^*)^2} |1 - e^{-n^2(x-x^*)^2 \epsilon(x-x^*)}| \, d\lambda(x)$$

We make a substitution: $z = x - x^*$:

$$\int_{I_n} |f_n - g_n| \, d\lambda = \frac{n}{\sqrt{\pi}} \int_{-\frac{1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} e^{-cn^2 z^2} |1 - e^{-n^2 z^2 \epsilon(z)}| \, d\lambda(z)$$

Using another substitution: $y = nz$ we obtain:

$$\int_{I_n} |f_n - g_n| \, d\lambda = \frac{1}{\sqrt{\pi}} \int_{-\sqrt{n}}^{\sqrt{n}} e^{-cy^2} |1 - e^{-y^2 \epsilon(\frac{y}{n})}| \, d\lambda(y) \tag{1}$$

Otherwise we know that $\epsilon(x) \underset{x\to 0}{\longrightarrow} 0$ therefore:

$$\text{Let } 0 < \eta < c \Rightarrow \exists A > 0 \, / \, \forall x \in]-A, A[\Rightarrow |\epsilon(x)| \leq \eta$$

From the integral we have: $-\sqrt{n} \leq y \leq \sqrt{n} \Rightarrow \frac{1}{-\sqrt{n}} \leq \frac{y}{n} \leq \frac{1}{\sqrt{n}}$ thus:

$$\exists N_1 \in \mathbb{N} \, / \, \forall n \geq N_1 \ \Rightarrow \frac{y}{n} \in]-A, A[\quad \forall y \in [-\sqrt{n}, \sqrt{n}]$$

Therefore:

$$\forall n \geq N_1 \quad |\epsilon(\frac{y}{n})| \leq \eta \quad \forall y \in [-\sqrt{n}, \sqrt{n}]$$

For $n \geq N_1$ we have two cases to treat:

1. If $0 \leq \epsilon(\frac{y}{n}) \leq \eta \Rightarrow 1 - e^{-y^2\epsilon(\frac{y}{n})} \leq 1 - e^{-y^2\eta}$, and because $y^2\epsilon(\frac{y}{n}) \geq 0$:

$$|1 - e^{-y^2\epsilon(\frac{y}{n})}| = 1 - e^{-y^2\epsilon(\frac{y}{n})} \leq 1 - e^{-y^2\eta}$$

2. If $-\eta \leq \epsilon(\frac{y}{n}) \leq 0 \Rightarrow e^{-y^2\epsilon(\frac{y}{n})} - 1 \leq e^{y^2\eta} - 1$, and because $-y^2\epsilon(\frac{y}{n}) \geq 0$:

$$|1 - e^{-y^2\epsilon(\frac{y}{n})}| = e^{-y^2\epsilon(\frac{y}{n})} - 1 \leq e^{y^2\eta} - 1$$

We conclude that:

$$\forall n \geq N_1 \Rightarrow |1 - e^{-y^2\epsilon(\frac{y}{n})}| \leq Max(1 - e^{-y^2\eta}, e^{y^2\eta} - 1) \leq (1 - e^{-y^2\eta}) + (e^{y^2\eta} - 1)$$

$$\Rightarrow \forall n \geq N_1 \quad |1 - e^{-y^2\epsilon(\frac{y}{n})}| \leq e^{y^2\eta} - e^{-y^2\eta}$$

The term in integral in the equation (1) is majored by:

$$\frac{1}{\sqrt{\pi}} \mathbf{1}_{[-\sqrt{n}, \sqrt{n}]}(y) e^{-cy^2} |1 - e^{-y^2\epsilon(\frac{y}{n})}| \leq \frac{1}{\sqrt{\pi}} e^{-cy^2} (e^{y^2\eta} - e^{-y^2\eta}) \ \ \forall y \in \mathbb{R}, \forall n \geq N_1$$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \mathbf{1}_{[-\sqrt{n}, \sqrt{n}]}(y) e^{-cy^2} |1 - e^{-y^2\epsilon(\frac{y}{n})}| \leq \frac{1}{\sqrt{\pi}} (e^{-y^2(c-\eta)} - e^{-y^2(c+\eta)}) \ \ \forall y \in \mathbb{R}, \forall n \geq N_1$$

Let $h_n(y) = \frac{1}{\sqrt{\pi}} \mathbf{1}_{[-\sqrt{n}, \sqrt{n}]}(y) e^{-cy^2} |1 - e^{-y^2\epsilon(\frac{y}{n})}| \ \forall y \in \mathbb{R}$, by the choice that we made: $c - \eta > 0$ and $c + \eta > 0$ then:

$$\int_{\mathbb{R}} |\frac{1}{\sqrt{\pi}} (e^{-y^2(c-\eta)} - e^{-y^2(c+\eta)})| \, d\lambda(y) \leq \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2(c-\eta)} \, d\lambda(y) + \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2(c+\eta)} \, d\lambda(y)$$

$$\Rightarrow \int_{\mathbb{R}} |\frac{1}{\sqrt{\pi}}(e^{-y^2(c-\eta)} - e^{-y^2(c+\eta)})| \, d\lambda(y) \leq \sqrt{\frac{1}{c-\eta}} + \sqrt{\frac{1}{c+\eta}} < \infty$$

In the other hand if we fix y then $\epsilon(\frac{y}{n}) \underset{n\to+\infty}{\longrightarrow} 0 \Rightarrow h_n(y) \underset{n\to+\infty}{\longrightarrow} 0$. Therefore using the dominated convergence theorem we obtain:

$$\int_{\mathbb{R}} h_n(y) \, d\lambda(y) \underset{n\to+\infty}{\longrightarrow} 0$$

As $\int_{I_n} |f_n - g_n| \, d\lambda = \int_{\mathbb{R}} h_n(y) \, d\lambda(y)$ thus:

$$\int_{I_n} |f_n - g_n| \, d\lambda \underset{n\to+\infty}{\longrightarrow} 0$$

From lemma 1 we conclude that there exist a neighborhood of x^*: $[D, E]$ such that $g'(x) \leq 0 \ \forall x \in [D, x^*]$ and $g'(x) \geq 0 \ \forall x \in [x^*, E]$. In addition $\exists N_2 \geq 0$ such that $\forall n \geq N_2 \Rightarrow x^* - \frac{1}{\sqrt{n}} \in]D, x^*[$ and $x^* + \frac{1}{\sqrt{n}} \in]x^*, E[$.

Let $J_n = [D, x^* - \frac{1}{\sqrt{n}}]$ and $K_n = [x^* + \frac{1}{\sqrt{n}}, E]$ then g_n is an increasing function on J_n and a decreasing function on K_n, same thing for f_n.

Now we study the quantity:

$$\int_{J_n} |f_n - g_n| \, d\lambda = \int_{D}^{x^*-\frac{1}{\sqrt{n}}} |f_n - g_n| \, d\lambda(x)$$

Therefore:

$$\int_{J_n} |f_n - g_n| \, d\lambda \leq \int_{D}^{x^*-\frac{1}{\sqrt{n}}} f_n \, d\lambda(x) + \int_{D}^{x^*-\frac{1}{\sqrt{n}}} g_n \, d\lambda(x)$$

Because g_n and f_n are an increasing functions on J_n then:

$$\int_{J_n} |f_n - g_n| \, d\lambda \leq (x^* - \frac{1}{\sqrt{n}} - D)(\frac{n}{\sqrt{\pi}} e^{-cn^2(-\frac{1}{\sqrt{n}})^2} + \frac{n}{\sqrt{\pi}} e^{-cn^2(-\frac{1}{\sqrt{n}})^2 - n^2(-\frac{1}{\sqrt{n}})^2 \epsilon(-\frac{1}{\sqrt{n}})})$$

$$\Rightarrow \int_{J_n} |f_n - g_n| \, d\lambda \leq \frac{n}{\sqrt{\pi}}(x^* - \frac{1}{\sqrt{n}} - D)e^{-cn}(1 + e^{-n\epsilon(-\frac{1}{\sqrt{n}})})$$

We have $\epsilon(-\frac{1}{\sqrt{n}}) \underset{n\to+\infty}{\longrightarrow} 0$, let $0 < \eta_1 < c$ thus $\exists N_3 \geq 0$ such that $\forall n \geq N_3 \Rightarrow |\epsilon(-\frac{1}{\sqrt{n}})| < \eta_1 < c$. Therefore $\forall n \geq N_3 \Rightarrow e^{-n\epsilon(-\frac{1}{\sqrt{n}})} \leq e^{\eta_1 n}$

$$\Rightarrow \int_{J_n} |f_n - g_n| \, d\lambda \leq \frac{n}{\sqrt{\pi}}(x^* - \frac{1}{\sqrt{n}} - D)(e^{-cn} + e^{-(c-\eta_1)n}) \quad \forall n \geq Max(N_2, N_3)$$

As $c - \eta_1 > 0$ we conclude that:

$$\int_{J_n} |f_n - g_n| \, d\lambda \underset{n\to+\infty}{\longrightarrow} 0$$

The same procedure is used for the quantity:

$$\int_{K_n} |f_n - g_n| \, d\lambda = \int_{x^*+\frac{1}{\sqrt{n}}}^{E} |f_n - g_n| \, d\lambda(x)$$

Indeed g_n and f_n are an decreasing functions on K_n then:

$$\int_{K_n} |f_n - g_n| \, d\lambda \leq \frac{n}{\sqrt{\pi}}(E - x^* - \frac{1}{\sqrt{n}})e^{-cn}(1 + e^{-n\epsilon(\frac{1}{\sqrt{n}})})$$

We have $\epsilon(\frac{1}{\sqrt{n}}) \underset{n\to+\infty}{\longrightarrow} 0$, let $0 < \eta_2 < c$ thus $\exists N_4 \geq 0$ such that $\forall n \geq N_4 \Rightarrow |\epsilon(\frac{1}{\sqrt{n}})| < \eta_2 < c$. Therefore $\forall n \geq N_4 \Rightarrow e^{-n\epsilon(\frac{1}{\sqrt{n}})} \leq e^{\eta_2 n}$

$$\Rightarrow \int_{K_n} |f_n - g_n| \, d\lambda \leq \frac{n}{\sqrt{\pi}}(E - x^* - \frac{1}{\sqrt{n}})(e^{-cn} + e^{-(c-\eta_2)n}) \quad \forall n \geq Max(N_2, N_4)$$

As $c - \eta_2 > 0$ we conclude that:

$$\int_{K_n} |f_n - g_n| \, d\lambda \underset{n\to+\infty}{\longrightarrow} 0$$

In this final part of the proof we know from the hypothesis of this theorem that $g > 0$ on $\bar{I}\backslash\{x^*\}$, in order to finish the proof it remains to compute the 2 quantities: $\int_{-\infty}^{D} |f_n - g_n| \, d\lambda(x)$ and $\int_{E}^{\infty} |f_n - g_n| \, d\lambda(x)$. Let $[x_1, x_2]$ denotes $\bar{I}$ then:

$$\int_{-\infty}^{D} |f_n - g_n| \, d\lambda(x) = \int_{-\infty}^{x_1} f_n \, d\lambda(x) + \int_{x_1}^{D} |f_n - g_n| \, d\lambda(x)$$

$$\Rightarrow \int_{-\infty}^{D} |f_n - g_n| \, d\lambda(x) \leq \int_{-\infty}^{D} f_n \, d\lambda(x) + \int_{x_1}^{D} g_n \, d\lambda(x)$$

And:

$$\int_{E}^{\infty} |f_n - g_n| \, d\lambda(x) \leq \int_{E}^{\infty} f_n \, d\lambda(x) + \int_{E}^{x_2} g_n \, d\lambda(x)$$

Let L_1 be the minimum of g on $[x_1, D]$ then $L_1 > 0$ and:

$$\int_{x_1}^{D} g_n \, d\lambda(x) \leq \frac{n}{\sqrt{\pi}}(D - x_1)e^{-n^2 L_1}$$

$$\Rightarrow \int_{x_1}^{D} g_n \, d\lambda(x) \underset{n\to+\infty}{\longrightarrow} 0$$

Let L_2 be the minimum of g on $[E, x_2]$ then $L_2 > 0$ and:

$$\int_{E}^{x_2} g_n \, d\lambda(x) \leq \frac{n}{\sqrt{\pi}}(x_2 - E)e^{-n^2 L_2}$$

$$\Rightarrow \int_{E}^{x_2} g_n \, d\lambda(x) \underset{n\to+\infty}{\longrightarrow} 0$$

Otherwise:

$$\int_{-\infty}^{D} f_n \, d\lambda(x) = \int_{-\infty}^{D} \frac{n}{\sqrt{\pi}} e^{-cn^2(x-x^*)^2} \, d\lambda(x)$$

$$\Rightarrow \int_{-\infty}^{D} f_n \, d\lambda(x) = \int_{-\infty}^{D-x^*} \frac{n}{\sqrt{\pi}} e^{-cn^2 z^2} \, d\lambda(z)$$

$$\Rightarrow \int_{-\infty}^{D} f_n \, d\lambda(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{n(D-x^*)} e^{-cy^2} \, d\lambda(y)$$

As $D - x^* < 0$ thus:

$$\int_{-\infty}^{D} f_n \, d\lambda(x) \underset{n\to+\infty}{\longrightarrow} 0$$

As before $E - x^* > 0$ then:

$$\int_{E}^{\infty} f_n \, d\lambda(x) \underset{n\to+\infty}{\longrightarrow} 0$$

Finally we conclude that:

$$\|f_n - g_n\|_{L^1(\mathbb{R})} \underset{n\to+\infty}{\longrightarrow} 0$$

We know that f_n converges in the sense of distributions to $\frac{1}{\sqrt{c}} \delta_{x^*}$ then:

$$\forall \phi \in \mathcal{D}(\mathbb{R}) \quad < g_n, \phi >= \int_{\mathbb{R}} g_n \phi \, d\lambda = \int_{\mathbb{R}} (g_n - f_n)\phi \, d\lambda + < f_n, \phi >$$

Otherwise if $\phi \in \mathcal{D}(\mathbb{R})$ then ϕ is bounded:

$$\exists K \geq 0 \Rightarrow \|\phi\|_\infty \leq K$$

Therefore:

$$\left| \int_{\mathbb{R}} (g_n - f_n)\phi \, d\lambda \right| \leq K \int_{\mathbb{R}} |g_n - f_n| \, d\lambda = K\|f_n - g_n\|_{L^1(\mathbb{R})}$$

$$\Rightarrow \int_{\mathbb{R}} (g_n - f_n)\phi \, d\lambda \underset{n\to+\infty}{\longrightarrow} 0$$

In conclusion:

$$\underset{n\to+\infty}{lim} < g_n, \phi >= \frac{1}{\sqrt{c}}\,\phi(x^*) \quad \forall\phi \in \mathcal{D}(\mathbb{R})$$

In general case if g has several zeros in I we have the following theorem:

Theorem 2 *If the function g has only k zeros* $x_1^*, \ldots, x_k^*$ *in* $\bar{I}$ *and* $x_1^*, \ldots, x_k^* \in I$ *such that g satisfies the hypotheses of the theorem 1 at* x_i $\forall i = 1, \ldots, k$ *, and if we consider the sequence of functions* g_n *defined by:* $g_n(x) = \mathbf{1}_I(x)\frac{n}{\sqrt{\pi}}e^{-n^2 g(x)}$ *then:*

$$g_n \underset{n\to+\infty}{\longrightarrow} \sum_{i=1}^{k} \frac{1}{\sqrt{c_i}}\,\delta_{x_i^*} \quad \textit{In the sense of distributions}$$

Where $c_i = \frac{g''(x_i^*)}{2}$.

Proof. It's enough to write $I = \bigcup_{i=1}^{k} I_i$ and $g_n(x) = \sum_{i=1}^{k} \mathbf{1}_{I_i}(x)\frac{n}{\sqrt{\pi}}e^{-n^2 g(x)}$ where I_i are disjoints and $x_i^* \in \mathring{I}_i$.

3. Application

Let a function $f : \bar{I} \to \mathbb{R}$ where $I \subset \mathbb{R}$ is an open and bounded interval. We suppose that the function f has only one zero x^* in $\bar{I}$ and $x^* \in I$ such that f is C^2 in a neighborhood of x^* and $f'(x^*) \neq 0$, let $g(x) = f^2(x)$ we suppose in addition that there exists a finite number of zeros for the function g' in a neighborhood of x^* , then we can apply the theorem 1.

Indeed $f(x) = 0 \iff g(x) = 0$, $g'(x^*) = 2f'(x^*)f(x^*) = 0$ and finally:

$$g''(x^*) = 2(f''(x^*)f(x^*) + (f'(x^*))^2) = 2(f'(x^*))^2 = 2c > 0$$

4. Algorithm to Find the Solution

In this section we will present an algorithm to find the zero x^* of the function g which satisfies all conditions presented in the theorem 1.

We consider then the sequence of functions $g_n(x) = \mathbf{1}_I(x)\frac{n}{\sqrt{\pi}}e^{-n^2 g(x)}$ where $I =]X_1, X_2[$ with $X_1, X_2 \in \mathbb{R}$. From theorem 1 we know that:

$$\lim_{n\to+\infty} \int_{\mathbb{R}} g_n(x)\phi(x)\, d\lambda(x) = \frac{1}{\sqrt{c}}\,\phi(x^*) \quad \forall\phi \in \mathcal{D}(\mathbb{R})$$

Let X_0 be the midpoint of I: $X_0 = \frac{1}{2}(X_1 + X_2)$, and R the radius of I:$R = \frac{1}{2}(X_2 - X_1)$, let $y_1 = X_2 - X.R$ and $y_2 = X_2 + X.R$ where X is a positive real value such that $X \geq 2$, we can choose for example $X = 3$.

From the space $\mathcal{D}(\mathbb{R})$ we will consider the function:

$$\phi(x) = \begin{cases} e^{\frac{-1}{\left(\frac{y_1-y_2}{2}\right)^2-\left(x-\frac{y_1+y_2}{2}\right)^2}} & if\ y_1 < x < y_2 \\ 0 \quad otherwise & \end{cases}$$

Because ϕ and ϕ^2 belong to $\mathcal{D}(\mathbb{R})$ and because $x^* \in I \subset]y_1, y_2[$ therefore:

$$\lim_{n\to+\infty} \frac{\int_{\mathbb{R}} g_n\phi^2\, d\lambda}{\int_{\mathbb{R}} g_n\phi\, d\lambda} = \frac{\phi^2(x^*)}{\phi(x^*)} = \phi(x^*)$$

Considering the 2 sequences $y_n = \frac{\int_{\mathbb{R}} g_n\phi^2\, d\lambda}{\int_{\mathbb{R}} g_n\phi\, d\lambda}$ and $\rho_n = \sqrt{\left(\frac{y_1-y_2}{2}\right)^2 + \frac{1}{Ln(y_n)}}$, the sequence x_n^* will be defined by: $x_n^* = X_2 - \rho_n$. The important result is that:

$$\lim_{n\to+\infty} x_n^* = x^*$$

Proof. First we have :

$$\lim_{n\to+\infty} y_n = \phi(x^*) = e^{\frac{-1}{\left(\frac{y_1-y_2}{2}\right)^2-\left(x^*-\frac{y_1+y_2}{2}\right)^2}}$$

then

$$\lim_{n\to+\infty} \rho_n = \sqrt{\left(x^* - \frac{y_1+y_2}{2}\right)^2}$$

Otherwise $\frac{y_1+y_2}{2} = X_2$ therefore $x^* - \frac{y_1+y_2}{2} < 0$ we obtain then:

$$\lim_{n\to+\infty} \rho_n = -x^* + \frac{y_1 + y_2}{2} = -x^* + X_2$$

We conclude that:

$$\lim_{n\to+\infty} x_n^* = x^*$$

In other words we only have to compute y_n for a big value of n in order to have an approximation of the solution x^*.

This approximation can be used for another algorithm for solving nonlinear equation.

Remark 1 If we don't know the properties of the function g at the solution x^*, in other words if we don't know if g is C^2 in the neighborhood of x^* and $g''(x^*) > 0$, then we apply the algorithm for a big value of n and we verify these properties at the neighborhood of the approximated solution x_n^*.

4.1 Example

Let us take a simple equation:

$$ln(x) - 1 = 0$$

The exact solution of this equation is: $x = 2.7182$.

In order to apply our algorithm we take the function $g(x) = (ln(x) - 1)^2$ and its corresponding sequence of functions $g_n(x) = \mathbf{1}_I(x)\frac{n}{\sqrt{\pi}}e^{-n^2 g(x)}$ where $I =]2, 4[$, and for the test function ϕ we take $y_1 = 4 - 3 = 1$ and $y_2 = 4 + 3 = 7$. The function g satisfies all conditions presented in the theorem 1 therefore by taking $n = 10$ we obtain:

$$x_{10}^* = 2.7138$$

5. Algorithm to Localize the Solutions

In the case of the existence of several zeros $x_1^*, \ldots, x_k^*$ of a function g which satisfies all conditions of the theorem 2, then the following algorithm helps to localize the zeros of this function.

As before we consider the sequence of functions $g_n(x) = \mathbf{1}_I(x)\frac{n}{\sqrt{\pi}}e^{-n^2 g(x)}$ where $I =]X_1, X_2[$ with $X_1, X_2 \in \mathbb{R}$. We will take a test function $\phi \in \mathcal{D}(\mathbb{R})$ centered on X_2, in other words $\frac{y_1+y_2}{2} = X_2$, and we will also suppose that $\bar{I} = [X_1, X_2] \subset]y_1, y_2[$. We suppose in addition that:

$$Max(g"(x_1^*), \ldots, g"(x_k^*)) \leq S \quad where\ S > 0$$

Then using the same notation in the theorem 2 we have $\frac{1}{\sqrt{c_i}} \geq \sqrt{\frac{2}{S}} \quad \forall i = 1, \ldots, k.$

The test function ϕ is chosen such that $\phi \geq 0$ and $\phi(X_1) = \eta\sqrt{\frac{S}{2}}$ where η is a strictly positive value which helps for precision, for example we can choose $\eta = 10$. A candidate for the test function is:

$$\phi(x) = \begin{cases} \eta\sqrt{\frac{S}{2}}\Phi_1 e^{\frac{-1}{(\frac{y_1-y_2}{2})^2-(x-\frac{y_1+y_2}{2})^2}} & if\ y_1 < x < y_2 \\ 0 \quad otherwise \end{cases}$$

Where $\Phi_1 = e^{\frac{1}{(\frac{y_1-y_2}{2})^2-(X_1-\frac{y_1+y_2}{2})^2}}$. We obtain then that $\phi(x) \geq \eta\sqrt{\frac{S}{2}} \quad \forall\ x \in I$. From theorem 2 we obtain:

$$\lim_{n\to+\infty} \int_{X_1}^{X_2} g_n(x)\phi(x)\, d\lambda(x) = \sum_{i=1}^{k} \frac{1}{\sqrt{c_i}}\ \phi(x_i^*) \geq k\eta$$

Let the sequence z_m such that $z_0 = X_1$ and $z_{m+1} \in]z_m, X_2]$, then we define the quantity $\mathcal{H}_m^n$ by:

$$\mathcal{H}_m^n = \int_{z_m}^{X_2} g_n(x)\phi(x)\, d\lambda(x)$$

$$\Rightarrow \mathcal{H}_m^n - \mathcal{H}_{m+1}^n = \int_{z_m}^{z_{m+1}} g_n(x)\phi(x)\, d\lambda(x)$$

If $\lim_{n\to+\infty}(\mathcal{H}_m^n - \mathcal{H}_{m+1}^n) = 0$ then there is no zero in $[z_m, z_{m+1}]$.

If $\lim_{n\to+\infty}(\mathcal{H}_m^n - \mathcal{H}_{m+1}^n) \geq \frac{\eta}{2}$ then there is minimum a zero in $[z_m, z_{m+1}]$. Indeed the value $\frac{\eta}{2}$ is reached when we have only one zero in this interval and this zero is equal to z_m or z_{m+1}.

The algorithm consists to take a big value for n and if $\mathcal{H}_m^n - \mathcal{H}_{m+1}^n$ is close to 0 then there is no zero in $[z_m, z_{m+1}]$, if not, there is at least a zero in $[z_m, z_{m+1}]$. The η guarantees that $\mathcal{H}_m^n - \mathcal{H}_{m+1}^n$ is not close to 0 when a zero exists in $[z_m, z_{m+1}]$. After we can take $z_{m+2} \in]z_{m+1}, X_2]$ to localize the other zeros or we can take $z_{m+2} \in]z_m, z_{m+1}[$ to more localize the zeros in this interval if they exist.

5.1 Example

Let the function f be defined on $I =]-2\pi, 0[$ by: $f(x) = cos(x)$, theoretically f has 2 zeros $x_1^* = -3\frac{\pi}{2}$ and $x_2^* = -\frac{\pi}{2}$.

Using the previous algorithm we want to localize the zeros of the function f. Let the function g be defined on $I =]-2\pi, 0[$ by: $g(x) = f(x)^2 = cos(x)^2$, its corresponding sequence of functions is defined by:

$$g_n(x) = \mathbf{1}_{]-2\pi,0[}(x)\frac{n}{\sqrt{\pi}}e^{-n^2 g(x)}$$

Otherwise if x_i^* is a zero of g then $g''(x_i^*) = 2$. We choose $\eta = 20$ and we consider the test function $\phi \in \mathcal{D}(\mathbb{R})$:

$$\phi(x) = \begin{cases} 20\Phi_1 e^{\frac{-1}{(3\pi)^2-(x)^2}} & if \ -3\pi < x < 3\pi \\ 0 \quad otherwise \end{cases}$$

Where $\Phi_1 = e^{\frac{1}{(3\pi)^2-(2\pi)^2}}$.

For n big ($n = 10$):

By choosing $z_0 = -2\pi$ and $z_1 = -\pi$ we obtain:

$$\mathcal{H}_0^n - \mathcal{H}_1^n = 20.15$$

By choosing $z_2 = -\frac{\pi}{4}$ we obtain:

$$\mathcal{H}_1^n - \mathcal{H}_2^n = 20.22$$

By choosing $z_3 = 0$ we obtain:

$$\mathcal{H}_2^n - \mathcal{H}_3^n = 0$$

Therefore we conclude that there exist zeros in $]-2\pi, -\pi]$ and in $[-\pi, -\frac{\pi}{4}]$, and there is no zeros in $[-\frac{\pi}{4}, 0[$.

6. Conclusion

The method and the algorithm developed above request only to do an integration in order to find the solution of the nonlinear equation, and it presents a new advantages regarding some other methods. The generalization of this method for big dimensions was developed and will be published.

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