Statistical Distribution of Roots of a Polynomial 
Modulo Primes III

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Abstract

Let \( f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \) be a polynomial with complex roots \( \alpha_1, \ldots, \alpha_n \) and suppose \( \) that a linear relation over \( \mathbb{Q} \) among \( 1, \alpha_1, \ldots, \alpha_n \) is a multiple of \( \sum_i \alpha_i + a_{n-1} = 0 \) only. For a prime number \( p \) such that \( f(x) \mod p \) has \( n \) distinct integer roots \( 0 < r_1 < \cdots < r_n < p \), we proposed in a previous paper a conjecture that the sequence of points \( (r_1/p, \ldots, r_n/p) \) is equi-distributed in some sense. In this paper, we show that it implies the equi-distribution of the sequence of points \( (r_i/p, \ldots, r_n/p) \) in the ordinary sense and give the expected density of primes satisfying \( r_i/p < a \) for a fixed suffix \( i \) and \( 0 < a < 1 \).

Keywords: polynomial, equi-distribution

1. Introduction

Let

\[
f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0
\]

be a monic polynomial of degree \( n \) \((\geq 2)\) over the ring \( \mathbb{Z} \) of integers with complex roots \( \alpha_1, \ldots, \alpha_n \). We put

\[
Spl_X(f) := \{ p \leq X \mid f(x) \text{ is fully splitting modulo } p \}
\]

for a positive number \( X \) and \( Spl(f) := Spl_{\infty}(f) \). Here the letter \( p \) denotes a prime number, and a polynomial \( f(x) \) is fully splitting modulo \( p \) if and only if

\[
f(x) \equiv \prod_{i=1}^{n} (x - r_i) \mod p
\]

for some integers \( r_i \). We know that \( Spl(f) \) is an infinite set and that the density theorem due to Chebotarev holds; that is,

\[
\lim_{X \to \infty} \frac{\#Spl(f,X)}{\#\{ p \leq X \}} = \frac{1}{|Q(f) : Q|},
\]

where \( Q \) is the rational number field and \( Q(f) \) is a finite Galois extension field of \( Q \) generated by all roots of \( f(x) \). In this note, we require the following condition on the above local roots \( r_1, \ldots, r_n \):

\[
0 \leq r_1 \leq r_2 \leq \cdots \leq r_n < p.
\]

The condition \( (3) \) determines the \( i \)th local root \( r_i \) uniquely. As a basic assumption, we assume that there is no non-trivial linear relation over \( \mathbb{Q} \) among roots \( \alpha_1, \ldots, \alpha_n \) and \( 1 \) except for a trivial relation \( \sum_i \alpha_i + a_{n-1} = 0 \) in this paper. We know that any irreducible polynomial of prime degree, or a polynomial \( f \) of degree \( n \) with \( |Q(f) : Q| = n! \) has no non-trivial linear relation among roots and \( 1 \). An irreducible polynomial \( f \) of degree \( n \) has a non-trivial linear relation among roots and \( 1 \) if and only if \( f(x) \) is of the form \( g(h(x)) \) for quadratic polynomials \( g, h \) (Kitaoka, 2017). When the degree is greater than \( 5 \), there is no such a simple classification.

We consider the following two kinds of uniformity: Put

\[
\hat{D}_n := \{(x_1, \ldots, x_n) \in [0, 1)^n \mid 0 \leq x_1 \leq \cdots \leq x_n < 1, \sum_{i=1}^{n} x_i \in \mathbb{Z}\}
\]

which is on the union of hyper-planes defined by \( \sum x_i = k \in \mathbb{Z} \) in \( \mathbb{R}^n \) and for a set \( D \subset [0, 1)^n \) with \( D = \overline{D}^c \)

\[
Pr_D(f,X) := \frac{\#\{ p \in Spl_X(f) \mid (r_1/p, \ldots, r_n/p) \in D \}}{\#Spl_X(f)},
\]

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where local roots $r_i$ are supposed to satisfy properties (2), (3). We proposed (Kitaoka, 2017)

**Conjecture 1**

$$
\lim_{X \to \infty} Pr_D(f, X) = \frac{\text{vol}(D \cap \hat{\mathbb{F}}_n)}{\text{vol}(\hat{\mathbb{F}}_n)}.
$$

(5)

Here, “vol” is the volume on the hyper-plane in $\mathbb{R}^n$. On the other hand, the classical concept of the uniformity is

**Conjecture 2**

$$
\lim_{X \to \infty} \frac{\sum_{p \in \text{Spl}(f)} |i | r_i/p \leq a, 1 \leq i \leq n|}{n \cdot \#\text{Spl}_X(f)} = a
$$

for a real number $a \in [0, 1)$.

Due to (Duke, Friedlander & Iwaniec, 1995) and (Tóth, 2000), Conjecture 2 is true for a quadratic polynomial, however nothing is known if $n > 2$.

We stated in (Kitaoka, 2017) that Conjecture 2 follows from Conjecture 1 as far as we checked by the Monte Carlo method. We give the rigorous proof here, that is,

**Theorem 1.** Let $f(x)$ be a monic polynomial over $\mathbb{Z}$ of degree $n$. Under the assumption that there is no non-trivial linear relation over $\mathbb{Q}$ among roots of $f(x)$ and 1, Conjecture 1 implies Conjecture 2.

To prove this, putting $D_{ia} := \{(x_1, \ldots, x_n) \in [0, 1]^n \mid x_i \leq a\}$ for a given number $a \in [0, 1)$, we have only to show

$$
\sum_{i=1}^n \frac{\text{vol}(D_{ia} \cap \hat{\mathbb{F}}_n)}{n \cdot \text{vol}(\hat{\mathbb{F}}_n)} = a
$$

(7)

by (Kitaoka, 2017). To show it, we evaluate $\frac{\text{vol}(D_{ia} \cap \hat{\mathbb{F}}_n)}{\text{vol}(\hat{\mathbb{F}}_n)}$ (Proposition 1), which gives as a by-product the density of primes $p$ satisfying $r_i/p < a$ :

**Theorem 2.** Let $f(x)$ be a monic polynomial over $\mathbb{Z}$ of degree $n$. Under the assumption that there is no non-trivial linear relation over $\mathbb{Q}$ among roots of $f(x)$ and 1. Then Conjecture 1 implies for $1 \leq i \leq n$

$$
\lim_{X \to \infty} \frac{|\{p \in \text{Spl}_X(f) \mid r_i/p \leq a\}|}{\#\text{Spl}_X(f)} = \frac{1}{(n-1)!} \sum_{\substack{0 \leq k \leq n-1 \leq m \leq n \leq l \leq M(n-ha)^{n-1},}}
$$

where the binomial coefficient $\binom{d}{B}$ is supposed to vanish unless $0 \leq B \leq A$, and $M(x) := \max(x, 0)$.

When $i = 1$, a simpler formula is given in Proposition 1 in the next section. Let us give numerical data for a polynomial $f(x) = x^6 + x^5 + \cdots + 1 = (x^{10} - 1)/(x - 1)$. Put

$$
\text{Ex}(a, m, i) := \frac{|\{p \in \text{Spl}_X(f) \mid r_i/p \leq a\}|}{\#\text{Spl}_X(f)} (X = 10^{10} \cdot m)
$$

and denote the expected limit given by the above theorem by $T(a, i)$ and the error by

$$
er(m) := 10^5 \max_{1 \leq k \leq 100, 1 \leq i \leq 6} |\text{Ex}(k/100, m, i) - T(k/100, i)|.
$$

The graph of $er(m)$ ($m = 1, \ldots, 300$) is below.

Conjecture 1 is generalized to a polynomial with a non-trivial linear relation among roots (Kitaoka, 2017). To treat such a polynomial, a more intrinsic proof of Theorem 1 independent of evaluation is desirable.

**2. Proof**

Hereafter, a real number $a$ satisfies $0 \leq a < 1$. 


**Lemma 1.** For an integer $k$ with $1 \leq k \leq n$, let

$$ V(k) := \text{vol} \left( \left\{ x \in [0, 1]^n \mid x_1, \ldots, x_k \leq a < x_{k+1}, \ldots, x_n, \sum_{j=1}^{n} x_j \in \mathbb{Z} \right\} \right) \cos \theta, $$

for the angle $\theta$ of two hyper-planes defined by $x_j = 0$ and by $x_1 + \cdots + x_n = 0$ in $\mathbb{R}^n$. Then we have

$$ \frac{\text{vol}(D_{ia} \cap \hat{D}_n)}{\text{vol}(\hat{D}_n)} = \sum_{k=1}^{n} \binom{n}{k} V(k). $$

**Proof.** It is easy to see

$$ \text{vol}(D_{ia} \cap \hat{D}_n) $$

$$ = \sum_{k=i}^{n} \text{vol} \left( \left\{ x \mid 0 \leq x_1 \leq \cdots \leq x_k \leq a < x_{k+1} \leq \cdots \leq x_n \leq 1, \sum x_j \in \mathbb{Z} \right\} \right) $$

$$ = \sum_{k=i}^{n} \frac{1}{k!(n-k)!} \text{vol} \left( \left\{ x \mid 0 \leq x_1, \ldots, x_k \leq a < x_{k+1}, \ldots, x_n \leq 1, \sum x_j \in \mathbb{Z} \right\} \right) $$

$$ = \frac{1}{n!} \sum_{k=i}^{n} \binom{n}{k} \text{vol} \left( \left\{ x \mid 0 \leq x_1, \ldots, x_k \leq a < x_{k+1}, \ldots, x_n \leq 1, \sum x_j \in \mathbb{Z} \right\} \right) $$

$$ = \text{vol}(\hat{D}_n) \sum_{k=i}^{n} \binom{n}{k} V(k). $$

using $\text{vol}(\hat{D}_n) = \frac{1}{n! \cos \theta}$.

To evaluate $V(k)$, we quote the following (Feller, 1966):

**Lemma 2.** For a natural number $k$, the volume of a subset of the unit cube $[0, 1]^k$ defined by $\{(x_1, \ldots, x_k) \mid x_1 + \cdots + x_k \leq x\}$ is given by

$$ U_k(x) := \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} M(x-i)^k. $$

**Lemma 3.** For $k = n$, we have

$$ V(n) = \frac{1}{(n-1)!} \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} M(k-ia)^{n-1}. $$

**Proof.** It is easy to see that

$$ V(n) = \text{vol}(\{ x \in \mathbb{R}^n \mid 0 \leq x_1, \ldots, x_n \leq a, \sum x_i \in \mathbb{Z} \}) \cos \theta $$

$$ = \sum_{k=1}^{n-1} \text{vol}(\{ x \in \mathbb{R}^n \mid 0 \leq x_1, \ldots, x_n \leq a, \sum_{i=1}^{n} x_i = k \}) \cos \theta $$

$$ = \sum_{k=1}^{n-1} \binom{n}{k} \text{vol}(\{ x \in \mathbb{R}^n \mid 0 \leq x_1, \ldots, x_n \leq a, \sum x_i = k \}) \cos \theta $$

$$ = \sum_{k=1}^{n-1} \binom{n}{k} U_k(a) \cos \theta $$

$$ = \sum_{k=1}^{n-1} \binom{n}{k} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} M(a-i)^k \cos \theta $$

$$ = \frac{1}{(n-1)!} \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} M(k-ia)^{n-1} \cos \theta $$

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The volume of the set \( \{ x \in \mathbb{R}^n \mid 0 \leq x_1, \ldots, x_{n-1} \leq a, 0 \leq k - \sum_{j=1}^{n-1} x_j \leq a \} \) is equal to

\[
\begin{align*}
&= \sum_{k=1}^{n-1} \text{vol}\left( \{ x \in \mathbb{R}^n \mid 0 \leq x_1, \ldots, x_{n-1} \leq a, 0 \leq k - \sum_{j=1}^{n-1} x_j \leq a \} \right) \\
&= \sum_{k=1}^{n-1} \text{vol}\left( \{ x \in \mathbb{R}^n \mid 0 \leq x_1, \ldots, x_{n-1} \leq a, \sum_{j=1}^{n-1} x_j \leq k \} \right) \\
&\quad - \sum_{k=1}^{n-1} \text{vol}\left( \{ x \in \mathbb{R}^n \mid 0 \leq x_1, \ldots, x_{n-1} \leq a, \sum_{j=1}^{n-1} x_j \leq k - a \} \right). 
\end{align*}
\]

The volume of the set \( \{ x \in \mathbb{R}^n \mid 0 \leq x_1, \ldots, x_{n-1} \leq a, \sum_{j=1}^{n-1} x_j \leq K \} \) is equal to

\[
\begin{align*}
a^{n-1} \text{vol}\left( \{ x \in \mathbb{R}^n \mid 0 \leq t_1, \ldots, t_{n-1} \leq 1, \sum_{j=1}^{n-1} t_j \leq K/a \} \right) \\
&= a^{n-1} U_{n-1}(K/a) \\
&= \frac{a^{n-1}}{(n-1)!} \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} M(K/a - i)^{n-1} \\
&= \frac{1}{(n-1)!} \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} M(K - ia)^{n-1}.
\end{align*}
\]

Therefore we have

\[
V(n) = \frac{1}{(n-1)!} \sum_{k=1}^{n-1} \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} [M(k - ia)^{n-1} - M(k - (i+1)a)^{n-1}] \\
= \frac{1}{(n-1)!} \sum_{k=1}^{n-1} \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} M(k - ia)^{n-1} \\
+ \frac{1}{(n-1)!} \sum_{k=1}^{n-1} \sum_{i=1}^{n} (-1)^i \binom{n-1}{i-1} M(k - ia)^{n-1} \\
= \frac{1}{(n-1)!} \sum_{k=1}^{n-1} \sum_{i=0}^{n} (-1)^i \binom{n}{i} M(k - ia)^{n-1}.
\]

**Lemma 4.** In case of \( 1 \leq k \leq n - 1 \), we have

\[
V(k) = \sum_{m=1}^{n-1} (U_{k,n-1}(m - a) - U_{k,n-1}(m - 1)), 
\]

where

\[
U_{k,r}(t) := \text{vol}\{ x \in [0,1)^r \mid x_1, \ldots, x_k \leq a < x_{k+1}, \ldots, x_r, \sum_{j=1}^{r} x_j < t \}. 
\]

**Proof.** We see that

\[
V(k) = \sum_{m=1}^{n-1} \text{vol}\left\{ x \in [0,1)^n \left| x_1, \ldots, x_k \leq a < x_{k+1}, \ldots, x_n, \sum_{j=1}^{n} x_j = m \right. \right\} \cos \theta \\
= \sum_{m=1}^{n-1} \text{vol}\left\{ x \in [0,1)^{n-1} \left| x_1, \ldots, x_k \leq a < x_{k+1}, \ldots, x_{n-1}, a < m - \sum_{j=1}^{n-1} x_j < 1 \right. \right\} \\
= \sum_{m=1}^{n-1} (U_{k,n-1}(m - a) - U_{k,n-1}(m - 1)).
\]
Lemma 5. For integers \( r, k \) with \( 1 \leq k \leq r \), we have

\[
U_{k,r+1}(t) = \int_a^1 U_{k,r}(t - w)dw.
\]

Proof. This follows from the equation

\[
U_{k,r+1}(t) = \int_{x_{r+1}=a}^1 (\int_D dx_1 \ldots dx_r)dx_{r+1}
\]

where the domain \( D \) is given by the conditions \( 0 \leq x_1, \ldots, x_k \leq a < x_{k+1}, \ldots, x_r, \sum_{j=1}^r x_j < t - x_{r+1} \).

Lemma 6.

\[
\int_a^1 M(t - w)^m dw = \frac{1}{m + 1} (M(t - a)^{m+1} - M(t - 1)^{m+1}).
\]

Proof. The left-hand side is equal to

\[
\int_a^1 \max(t - w, 0)^m dw = \int_{t-a}^{t-a} \max(W,0)^m(-dW)
\]

\[
= - \int_{-\infty}^{t-a} \max(W,0)^m dW + \int_{t-a}^{-\infty} \max(W,0)^m dW
\]

\[
= - \frac{1}{m + 1} M(t - 1)^{m+1} + \frac{1}{m + 1} M(t - a)^{m+1}.
\]

Lemma 7. For integers \( j, k \) with \( j \geq 0, k \geq 1 \), \( U_{k,k+j}(t) \) is equal to

\[
\frac{1}{(k + j)!} \sum_{i=0}^k (-1)^i \left( \begin{array}{c} k \\ i \\end{array} \right) \sum_{h=0}^j (-1)^{i+h} \left( \begin{array}{c} j \\ h \end{array} \right) M(t + h - j - (i + h)a)^{k+j}.
\]

Proof. Suppose that \( j = 0 \); then \( U_{k,k}(t) \) equals

\[
\text{vol}\{x \in [0,1]^k \mid x_1, \ldots, x_k < a, \sum_{j=1}^k x_j < t \}
\]

\[
= a^k U_k(t/a)
\]

\[
= \frac{1}{k!} \sum_{i=0}^k (-1)^i \left( \begin{array}{c} k \\ i \end{array} \right) M(t - ia)^k.
\]

Second, suppose that the equation (13) is true; then we see that \( U_{k,k+j+1}(t) \) equals

\[
\int_a^1 U_{k,k+j+1}(t - w)dw
\]

\[
= \frac{1}{(k + j)!} \sum_{i=0}^k (-1)^i \left( \begin{array}{c} k \\ i \end{array} \right) \sum_{h=0}^j (-1)^{i+h} \left( \begin{array}{c} j \\ h \end{array} \right) \int_a^1 M(t - w + h - j - (i + h)a)^{k+j}/dw
\]

\[
= \frac{1}{(k + j)!} \sum_{i=0}^k (-1)^i \left( \begin{array}{c} k \\ i \end{array} \right) \sum_{h=0}^j (-1)^{i+h} \left( \begin{array}{c} j \\ h \end{array} \right) \times
\]
\[
\frac{1}{k + j + 1}\{M(t + h - j - (i + h) a)^{k+j+1} - M(t + h - j - (i + h) a)^{k+j+1}\}
\]
\[
= \frac{1}{(k + j + 1)!} \sum_{i=0}^{j+1} (-1)^i \binom{k}{i} \times 
\left\{ \sum_{h=1}^{j+1} (-1)^{i+h+1} \binom{j}{h-1} M(t + h - j - (i + h) a)^{k+j+1} - \sum_{h=0}^{j} (-1)^{j+h} \binom{j}{h} M(t + h - j - (i + h) a)^{k+j+1} \right\}
\]
\[
= \frac{1}{(k + j + 1)!} \sum_{i=0}^{j} (-1)^i \binom{k}{i} \times \left\{ \sum_{h=1}^{j+1} (-1)^{i+h+1} \left( \binom{j}{h-1} + \binom{j}{h} \right) M(t + h - j - (i + h) a)^{k+j+1} + M(t - (i + j + 1) a)^{k+j+1} - (-1)^{j} M(t - (j - 1) a)^{k+j+1} \right\},
\]
which completes the induction. \(\square\)

**Lemma 8.** For 1 \(\leq k \leq n - 1\), we have

\[
V(k) = \frac{1}{(n - 1)!} \sum_{i,j \in \mathbb{Z}} (-1)^{n+k+h} \sum_{m=1}^{n-1} \left( \begin{array}{c} k \\ n-h-m+i \end{array} \right) \left( \begin{array}{c} n-k \\ m-1 \end{array} \right) M(l-ha)^{n-1}.
\]

**Proof.** For 1 \(\leq k, m \leq n - 1\), we have

\[
(n-1)![U_{k,n-1}(m-a) - U_{k,n-1}(m-1)]
\]
\[
= \sum_{i,j \in \mathbb{Z}} (-1)^i \binom{k}{i} (-1)^{n-1-k} \binom{n-1-k}{h} M(m-a+h-(n-1-k)-(i+h)a)^{n-1}
\]
\[
- \sum_{i,j \in \mathbb{Z}} (-1)^i \binom{k}{i} (-1)^{n-1-k} \binom{n-1-k}{h} M(m+1+h-(n-1-k)-(i+h)a)^{n-1}
\]
\[
= \sum_{h \in \mathbb{Z}} (-1)^{n+k+h} \binom{k}{n+l-h-m} \left( \binom{n-1-k}{m-l} + \binom{n-1-k}{m-1} \right) M(l-ha)^{n-1}
\]
\[
= \sum_{h \in \mathbb{Z}} (-1)^{n+k+h} \binom{k}{n+l-h-m} \left( \binom{n-1-k}{m-l} \right) M(l-ha)^{n-1},
\]
where the restrictions on \(h, l\) follow from conditions 1 \(\leq k, m \leq n - 1, 0 \leq n + l - h - m \leq k, 0 \leq m - l \leq n - k\). Lemma 4 completes the proof. \(\square\)

**Lemma 9.** Let \(m, n\) be integers satisfying 0 \(\leq m \leq n - 1\). Then we have

\[
\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^{n} \binom{n-1}{m}.
\]

For a polynomial \(P(x) = c_n x^n + \cdots + c_0\), we have

\[
\sum_{k=0}^{n} (-1)^k P(k) \binom{n}{k} = c_n (-1)^n n!.
\]

These are well-known and we omit the proof.
Proposition 1. For an integer \( i \) with \( 1 \leq i \leq n \) and a real number \( a \in [0, 1) \), we have

\[
(n-1)! \frac{\text{vol}(D_{l,a} \cap \hat{\mathcal{D}}_n)/\text{vol}(\hat{\mathcal{D}}_n)}{	ext{vol}(D_{l,a} \cap \hat{\mathcal{D}}_n)/\text{vol}(\hat{\mathcal{D}}_n)}
\]

\[
= \sum_{k=i}^{n} \sum_{k \geq 0 \leq m} (-1)^{k+n} \binom{n}{k} \frac{n}{k} \binom{n-k}{q-h} M(l-ha)_{n-1}.
\]

In particular, we have for \( i = 1 \)

\[
\text{vol}(D_{l,a} \cap \hat{\mathcal{D}}_n)/\text{vol}(\hat{\mathcal{D}}_n) = \sum_{k=0}^{n} C_1(l,h) M(l-ha)_{n-1},
\]

where

\[
C_1(l,h) = \frac{1}{(n-1)!} \begin{cases} 
(-1)^{n+h+1}\binom{n}{h} & \text{if } h \geq l + 1, \\
0 & \text{if } 1 \leq h \leq l, \\
(-1)^{n+l+1}\binom{n-1}{l} & \text{if } h = 0.
\end{cases}
\]

Proof. By Lemma 1, we have

\[
(n-1)! \frac{\text{vol}(D_{l,a} \cap \hat{\mathcal{D}}_n)/\text{vol}(\hat{\mathcal{D}}_n)}{	ext{vol}(D_{l,a} \cap \hat{\mathcal{D}}_n)/\text{vol}(\hat{\mathcal{D}}_n)}
\]

\[
= (n-1)! \sum_{k=i}^{n} \binom{n}{k} V(k)
\]

\[
= (n-1)! V(n) + (n-1)! \sum_{k=i}^{n-1} \binom{n}{k} V(k)
\]

\[
= \sum_{k=0}^{n-1} (-1)^{k} \binom{n}{k} M(l-ha)_{n-1}
\]

\[
+ \sum_{k=i}^{n} \binom{n}{k} \sum_{k \geq 0 \leq m} (-1)^{k+n} \binom{n-k}{m} \binom{n-k}{m-l} M(l-ha)_{n-1}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \sum_{k \geq 0 \leq m} (-1)^{k+n} \binom{n-k}{m} \binom{n-k}{m-l} M(l-ha)_{n-1},
\]

since the binomial coefficient \( \binom{n}{m} \) vanishes unless \( m = l \). The partial sum \( \sum_{m=1}^{n-1} \binom{k}{m} \binom{n-k}{n-m} \) is equal to

\[
\sum_{1 \leq q \leq n-1} \binom{k}{n-q} \binom{n-k}{q} = \sum_{0 \leq q \leq \min(n-1-(n-h))} \binom{k}{n-h-q} \binom{n-k}{q} = \binom{n}{n-h} - \sum_{\min(n-1-(n-h)) \leq q \leq n-h-q} \binom{k}{n-h-q} \binom{n-k}{q} = \binom{n}{n-h} - \sum_{h \leq q \leq \max(h-1)} \binom{k}{q-h} \binom{n-k}{n-q}.
\]

Let us assume that \( i = 1 \) to show (16). Putting

\[
T(l,h) := \sum_{k=1}^{n} (-1)^{k+n} \binom{n}{k} \frac{n}{k} \binom{n-k}{q-h} M(l-ha)_{n-1}
\]

\[
= - (-1)^{h+n} \binom{n}{h} - \sum_{k=1}^{n} (-1)^{k+n} \binom{n}{k} \sum_{q=0 \leq m} \binom{k}{q-h} \binom{n-k}{n-q}
\]

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we have only to prove $T(l, h) = (n - 1)! C_1(l, h)$. It is obviously true if $h \geq l + 1$, since the partial sum on $q$ is empty. In case of $h = 0$, we see that $T(l, 0)$ is equal to

\[ - (-1)^n - \sum_{k=1}^{n} (-1)^{k+n} \binom{n}{k} \sum_{q=0}^{l} \binom{k}{n-q} \]

\[ = - (-1)^n - \sum_{k=1}^{n} (-1)^{k+n} \binom{n}{k} \sum_{q=0}^{l} \delta_{k,q} \]

\[ = - (-1)^n - \sum_{k=1}^{n} (-1)^{k+n} \binom{n}{k} \]

\[ = - \sum_{l=0}^{n} (-1)^{k+n} \binom{n}{l} \]

Lastly assume that $1 \leq h \leq l$. The sum $T(l, h) + (-1)^{h+n} \binom{n}{h}$ is equal to

\[ - \sum_{k=1}^{n} (-1)^{h+k+n} \binom{n}{k} \sum_{q=h}^{n} \binom{k}{q-h} \binom{n-q}{n-q} \]

\[ = - \sum_{q=h}^{n} (-1)^{h+n} \binom{n}{q-h} \sum_{k=1}^{n} (-1)^{k+h} \binom{h}{q-k} \]

\[ = - \sum_{q=h}^{n} (-1)^{h+n} \binom{n}{q-h} (-1)^{q-1} \sum_{K=0}^{q-1} (-1)^{K} \binom{h}{K} \]

\[ = - \sum_{q=h}^{n} (-1)^{h+n} \binom{n}{q-h} (-1)^{q-1} \binom{h-1}{q-1} \]

\[ = \sum_{q=h}^{n} (-1)^{h+n} \binom{n}{q-h} \binom{h-1}{q-1} \]

\[ = (-1)^{h+n} \binom{n}{h} \]

which implies $T(l, h) = 0$. □

The proposition gives Theorem 2 by (17), and we see that the left-hand side of (7) is the sum of $C(l, h)M(l - ha)^{n-1}$ over integers $l, h$ satisfying

\[ 1 \leq l \leq n - 1, 0 \leq h \leq n, \]

\[ (18) \]

where

\[ C(l, h) := \frac{1}{n!} \sum_{1 \leq k \leq n} (-1)^{h+k+n} \binom{n}{k} \left( \binom{n}{h} - \sum_{h\leq s \leq \max(l, h-1)} (\binom{k}{s-h} \binom{n-k}{n-q}) \right) \]

\[ = \frac{1}{n!} \sum_{0 \leq k \leq n} (-1)^{h+k+n} \binom{n}{k} \left( \binom{n}{h} - \sum_{h\leq s \leq \max(l, h-1)} (\binom{k}{s-h} \binom{n-k}{n-q}) \right) \]

\[ = -\frac{1}{n!} \sum_{0 \leq k \leq n} (-1)^{h+k+n} \binom{n}{k} \sum_{h\leq s \leq \max(l, h-1)} (\binom{k}{s-h} \binom{n-k}{n-q}) \]  \[ (19) \]
To prove (7), we will show

\[
C(l, h) = \begin{cases} 
\frac{(-1)^{n-1}(n-2)}{(n-1)!} & \text{if } h = 0, \\
\frac{(-1)^{n-1}(n-2)}{(n-1)!} (l-1) & \text{if } h = 1, \\
0 & \text{if } h \geq 2.
\end{cases}
\] (20)

Under the equations (20), Theorem 1 is proved as follows: The left-hand side of (7) is equal to

\[
\sum_{l=1}^{n-1} \frac{(-1)^{n-1}(n-2)}{(n-1)!} (l-1)^{m-1} + \sum_{l=1}^{n-1} \frac{(-1)^{n-1}(n-2)}{(n-1)!} (l-1)^{m-1} (l-a)^{n-1}
\]

\[
= \sum_{l=0}^{n-2} \frac{(-1)^{n+1}}{(n-1)!} (n-2) ((n-1)a^{n-2} + O(n^{-3}))
\]

Suppose \( h = 0 \); we see that

\[
C(l, 0) = \frac{-1}{n!} \sum_{0 \leq k \leq n} (-1)^{k+n} k \binom{n}{k} \sum_{0 \leq q \leq l} \binom{k}{q} (n-q)
\]

\[
= \frac{-1}{n!} \sum_{0 \leq k \leq n} (-1)^{k+n} k \binom{n}{k} \sum_{0 \leq q \leq l} \delta_{k,q}
\]

\[
= \frac{-1}{n!} \sum_{0 \leq k \leq l} (-1)^{k+n} \binom{n}{k}
\]

\[
= \frac{-1}{(n-1)!} \sum_{0 \leq k \leq l-1} (-1)^{k+n} \binom{n-1}{k}
\]

\[
= \frac{(-1)^{n+1} (n-2)}{(n-1)!} \binom{l}{1},
\]

which is (20).

Second we see that

\[
C(l, 1) = \frac{-1}{n!} \sum_{0 \leq k \leq n} (-1)^{1+k+n} k \binom{n}{k} \sum_{1 \leq q \leq l} \binom{k}{q-1} (n-q)
\]

Unless \( q-1 \leq k \) and \( n-q \leq n-k \), binomial coefficients vanish, hence we may assume that \( q = k \) or \( q = k+1 \), and we see

\[
C(l, 1) = \frac{-1}{n!} \sum_{0 \leq k \leq n} (-1)^{1+k+n} k \binom{n}{k} \sum_{1 \leq q \leq l} (k \delta_{k,k} + (n-k) \delta_{q,k+1})
\]

\[
= \frac{-1}{n!} \sum_{0 \leq k \leq l} (-1)^{1+k+n} k^2 \binom{n}{k} + \frac{-1}{n!} \sum_{0 \leq k \leq l-1} (-1)^{1+k+n} k(n-k) \binom{n}{k}
\]

\[
= \frac{(-1)^n}{(n-1)!} \sum_{0 \leq k \leq l-1} (-1)^k \binom{n}{k} + \frac{1}{n!} (-1)^{n+1} \binom{n}{1}
\]

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Finally, assume that $h \geq 2$; hence $1 \leq l \leq n - 1, 2 \leq h \leq n$ are supposed. By (19), we have

$$- n! C(l, h) = \sum_{h \leq q \leq \max(l, h - 1)} (-1)^{n+1} \binom{n}{h} \binom{n - h}{q} \sum_{0 \leq k \leq n} (-1)^{h-k} \binom{h}{q - k}$$

$$= \sum_{h \leq q \leq \max(l, h - 1)} (-1)^{n+1} \binom{n}{h} \binom{n - h}{q} \sum_{0 \leq K \leq q} (-1)^{h+K}(q - K) \binom{h}{K}$$

$$= 0,$$

since

$$\sum_{0 \leq K \leq q} (-1)^{h+K}(q - K) \binom{h}{K} = (-1)^q \sum_{0 \leq K \leq h} (-1)^K(q - K) \binom{h}{K} = 0$$

by $h \geq 2$. Thus we have completed the proof.

References


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