Limit Theorems for Negatively Dependent Fuzzy Set-Valued Random Variables

Li Guan

1 College of Applied Sciences, Beijing University of Technology, Beijing, China

Correspondence: Li Guan, College of Applied Sciences, Beijing University of Technology, 100 Pingleyuan, Chaoyang District, Beijing, 100124, P.R.China. E-mail: guanli@bjut.edu.cn

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Abstract

In this paper, we shall discuss negatively dependent fuzzy set-valued random variables. And at last, we shall prove the limit theorems for rowwise negatively dependent fuzzy set-valued random variables in the sense of $d^\infty_H$, which is the extension of (Guan & Sun, 2014) and (Guan & Wan, 2016).

Keywords: laws of large number, fuzzy set-valued random variables, negatively dependent

1. Introduction

We all know that strong laws of large numbers are one of the most important theories in probability. For independent set-valued random variables, many limit results have been obtained (cf. (Artstein & Vitale, 1975), (Cressie, 1978), (Hiai, 1984), (Taylor & Inoue, 1985), (Puri & Ralescu, 1983)). About the convergence theorems of fuzzy set-valued random variables, Klement et al proved the strong laws of large numbers (SLLN) for independent identically distributed (i.i.d) fuzzy set-valued random variables in the sense of $d^1_H$. Colubi et al. obtained the SLLN for i.i.d fuzzy set-valued random variables with respect to $d^1_H$ in (Colubi, López-Díaz, Domínguez-Menchero & Gil, 1999), where the underlying space is $\mathbb{R}^d$. Li and Ogura proved the same SLLN as in (Colubi, López-Díaz, Domínguez-Menchero & Gil, 1999) by using a new embedding method (Li, Ogura & Kreinovich, 2002, Theorem 6.2.6). In 1991, Inoue (Inoue, 1991) extended the SLLN of set-valued case given by Taylor and Inoue (Taylor & Inoue, 1985) to the case of only independent fuzzy set-valued random variables in the sense of $d^1_H$. Li and Ogura (Li & Ogura, 2003) proved the SLLN of (Inoue, 1991) in the sense of $d^\infty_H$.

In practice, the random variables are not always independent. So it is necessary to discuss the limit theorems for dependent random variables. Bozorgnia, Patterson and Taylor discussed the properties for negatively dependent random variables in (Bozorgnia, Patterson & Taylor, 1993), and proved the laws of large number for negative dependence random variables in (Bozorgnia, Patterson & Taylor, 1992), where the random variables are single-valued. In (Guan, 2014), Guan and Sun discussed the property of set-valued random variables in real space $\mathbb{R}$, and proved the weak convergence theorem for dependent set-valued random variables in the sense of Hausdorff metric. In (Guan, 2016), Guan and Wan proved the strong law of large number for set-valued dependent random variables in the sense of Hausdorff metric. In 2016, Shen and Guan (Shen & Guan, 2016) proved the strong laws of large numbers for independent fuzzy set-valued random variables, where the underlying space is $G_\alpha$ space. In this paper, we are concerned with the weak and strong limit theorems for dependent fuzzy set-valued random variables in the sense of $d^\infty_H$.

This paper is organized as follows. In section 2, we shall briefly introduce some definitions and basic results of set-valued random variables and fuzzy set-valued random variables. In section 3, we shall give basic definition and discuss the properties of fuzzy set-valued negatively dependent random variables. And at last, we shall prove the weak and strong laws of large numbers for weighted sums of fuzzy set-valued negatively dependent random variables.

2. Preliminaries on Fuzzy Set-Valued Random Variables

Throughout this paper, we assume that $(\Omega, \mathcal{A}, \mu)$ is a complete probability space, $(\mathcal{X}, \|\cdot\|)$ is a real separable Banach space, $\mathcal{K}_c(\mathcal{X})$ is the family of all nonempty compact subsets of $\mathcal{X}$, and $\mathcal{K}_{cc}(\mathcal{X})$ is the family of all nonempty compact convex subsets of $\mathcal{X}$.

Let $A$ and $B$ be two nonempty subsets of $\mathcal{X}$ and let $\lambda \in \mathbb{R}$, the set of all real numbers. We define addition and scalar multiplication by

$$A + B = \{a + b : a \in A, b \in B\}$$

$$\lambda A = \{\lambda a : a \in A\}$$
The Hausdorff metric on $\mathbf{K}_k(\mathcal{X})$ is defined by

$$d_H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\}$$

for $A, B \in \mathbf{K}_k(\mathcal{X})$. For an $A$ in $\mathbf{K}_k(\mathcal{X})$, let $\|A\|_k = d_H([0], A)$.

The metric space $(\mathbf{K}_k(\mathcal{X}), \mathbf{d}_H)$ is complete and separable, and $\mathbf{K}_{ke}(\mathcal{X})$ is a closed subset of $(\mathbf{K}_k(\mathcal{X}), \mathbf{d}_H)$ (cf. (Li, Ogura & Kreinovich, 2002), Theorems 1.1.2 and 1.1.3). Concerning its equivalent definitions, please refer to (Castaign & Valadier, 1977) and (Hiai & Umegaki, 1977).

A set-valued mapping $F : \Omega \rightarrow \mathbf{K}_k(\mathcal{X})$ is called a set-valued random variable (or a random set) if, for each open subset $O$ of $\mathcal{X}$, $F^{-1}(O) = \{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in \mathcal{A}$. In fact, set-valued random variables can be defined as mappings from $\Omega$ to the family of all closed subsets of $\mathcal{X}$. Concerning its equivalent definitions, please refer to (Castaign & Valadier, 1977) and (Hiai & Umegaki, 1977).

A set-valued random variable $F$ is called integrably bounded (cf. (Hiai & Umegaki, 1977) or (Li, Ogura & Kreinovich, 2002)) if $\int_{\Omega} \|F(\omega)\|_k d\mu < \infty$. Let $L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_k(\mathcal{X})]$ denote the space of all integrably bounded random variables, and $L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_{ke}(\mathcal{X})]$ denote the space of all integrably bounded random variables taking values in $\mathbf{K}_{ke}(\mathcal{X})$. For $F, G \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{K}_k(\mathcal{X})]$, $F = G$ if and only if $F(\omega) = G(\omega)$ a.e.$(\mu)$.

For each set-valued random variable $F$, define $S_F = \{f \in L^1[\Omega, \mathcal{X}] : f(\omega) \in F(\omega), a.e.(\mu)\}$. The expectation of $F$, denoted by $E[F]$, is defined as

$$E[F] = \left\{ \int_{\Omega} f d\mu : f \in S_F \right\},$$

where $\int_{\Omega} f d\mu$ is the usual Bochner integral in $L^1[\Omega, \mathcal{X}]$, the family of integrable $\mathcal{X}$-valued random variables. This integral was first introduced by Aumann (Aumann, 1965), called Aumann integral in literature.

For each $A \in \mathbf{K}_k(\mathcal{X})$, define the support function by

$$s(x^*, A) = \sup_{a \in A} < x^*, a >, \quad x^* \in \mathcal{X}^*,$$

where $\mathcal{X}^*$ is the dual space of $\mathcal{X}$.

Let $\mathcal{S}^*$ denote the unit sphere of $\mathcal{X}^*$, $C(\mathcal{S}^*)$ the all continuous functions of $\mathcal{S}^*$, and the norm is defined as $\|v\|_C = \sup_{r \in \mathcal{S}^*} |v(r)|$.

From now on, we begin to introduce necessary concepts, notation and basic results on fuzzy set space and fuzzy set-valued random variables.

Let $\mathbf{F}_k(\mathcal{X})$ be the family of all special fuzzy sets: $v : \mathcal{X} \rightarrow [0, 1]$ satisfying the following conditions:

1. the 1-level set $v_1 = \{x \in \mathcal{X} : v(x) = 1\} \neq \emptyset$,
2. each $v$ is upper semicontinuous, i.e. for each $a \in [0, 1]$, the $a$ level set $v_a = \{x \in \mathcal{X} : v(x) \geq a\}$ is a closed subset of $\mathcal{X}$,
3. the support set $v_0^+ = \text{cl}\{x \in \mathcal{X} : v(x) > 0\}$ is compact.

A fuzzy set $v$ in $\mathbf{F}_k(\mathcal{X})$ is called convex if it satisfies

$$v(\lambda x + (1 - \lambda)y) \geq \min\{v(x), v(y)\}, \text{ for any } x, y \in \mathcal{X}, \lambda \in [0, 1].$$

It is known that $v$ is convex if and only if each level set $v_a$ $(a \in (0, 1))$ of $v$ is a closed convex subset of $\mathcal{X}$. Let $\mathbf{F}_{ke}(\mathcal{X})$ be the subset of all convex fuzzy sets in $\mathbf{F}_k(\mathcal{X})$.

The uniform metric in $\mathbf{F}_k(\mathcal{X})$, which is an extension of the Hausdorff metric $d_H$, is often used (cf. Puri & Ralescu, 1986): for $v^1, v^2 \in \mathbf{F}_k(\mathcal{X})$,

$$d_{H}^{\infty}(v^1, v^2) = \sup_{a \in (0, 1]} d_H(v^1_a, v^2_a).$$

Let $\|v\|_F = d_H^{\infty}(v, I_0) = \sup_{a > 0} \|v_a\|_k$, where $I_0$ is the function taking value one at 0 and zero for all $x \neq 0$. The space $(\mathbf{F}_k(\mathcal{X}), \mathbf{d}_H)$ is a complete metric space (cf. Li, & Ogura, 1996) but not separable (Li, Ogura & Kreinovich, 2002), Remark 5.1.7). Completeness was first proved by Puri and Ralescu (Puri & Ralescu, 1986) in the case of $\mathcal{X} = \mathbb{R}^d$, the $d$-dimensional Euclidean space.
A fuzzy set-valued random variable (or $F_k(\mathcal{X})$-valued random variable) is a measurable mapping $X$ from the space $(\Omega, \mathcal{A})$ to the space $(F_k(\mathcal{X}), \mathcal{B}(F_k(\mathcal{X})))$, where $\mathcal{B}(F_k(\mathcal{X}))$ is the Borel $\sigma$-field of $F_k(\mathcal{X})$ with respect to $d_H^\infty$.

It is well known that for any fuzzy set $v$, $\nu_\alpha = \bigcap_{\beta \geq \alpha} \nu_\beta$, for every $\alpha \in (0, 1)$, but usually $\nu_\alpha \neq \bigcup_{\beta > \alpha} \nu_\beta$. We denote $\nu_\alpha = cl(\bigcup_{\beta > \alpha} \nu_\beta)$, for $\alpha \in [0, 1)$, which will be used later. Obviously, $\nu_0$ is the support set of $v$. Due to the completeness of $(F_k(\mathcal{X}), d_H^\infty)$, every Cauchy sequence $\{v^n : n \in \mathbb{N}\}$ has a limit $v$ in $F_k(\mathcal{X})$.

A sequence of set-valued random variables $\{F_n : n \in \mathbb{N}\}$ is called to be stochastically dominated by a set-valued random variable $F$ if

$$\mu(\|F_n\|_K > t) \leq \mu(\|F\|_K > t), \quad t \geq 0, \quad n \geq 1.$$  

A sequence of fuzzy set-valued random variables $\{X^n : n \geq 1\}$ is called to be stochastically dominated by a fuzzy set-valued random variable $X$ if

$$\mu(\|X^n\|_K > t) \leq \mu(\|X\|_K > t), \quad t \geq 0, \quad n \geq 1, \alpha \in (0, 1).$$

It is obvious that if $\{X^n : n \geq 1\}$ is stochastically dominated by a fuzzy set-valued random variable $X$, then $\forall \alpha \in (0, 1)$, $\{X^n_\alpha : n \geq 1\}$ is stochastically dominated by the set-valued random variable $X_\alpha$. And also for any $\alpha \in (0, 1)$, $\{X^n_\alpha : n \geq 1\}$ is stochastically dominated by the set-valued random variable $X_\alpha$ (Guan & Li, 2004).

An $F_k(\mathcal{X})$-valued random variable $X$ is called integrably bounded if the real-valued random variable $\|X_\alpha(\omega)\|_K$ is integrable. Let $L^1[\Omega, \mathcal{A}, \mu; F_k(\mathcal{X})]$ be the set of all integrably bounded $F_k(\mathcal{X})$-valued random variables and $L^1[\Omega, \mathcal{A}, \mu; \mathcal{F}_k(\mathcal{X})]$ be the set of all integrably bounded $F_k(\mathcal{X})$-valued random variables. Two $F_k(\mathcal{X})$-valued random variables $X, Y \in L^1[\Omega, \mathcal{A}, \mu; F_k(\mathcal{X})]$ are considered to be identical if for any $\alpha \in [0, 1], X_\alpha(\omega) = Y_\alpha(\omega)$ a.e.($\mu$).

A sequence of $F_k(\mathcal{X})$-valued random variables $\{X^n : n \in \mathbb{N}\}$ is called to converge to an $F_k(\mathcal{X})$-valued random variable $X$ in the sense of $d_H^\infty$, if $d_H^\infty(X^n(\omega), X(\omega)) \to 0$ a.e.($\mu$) as $n \to \infty$.

The expectation of an $F_k(\mathcal{X})$-valued random variables $X$, denoted by $E[X]$, is an element in $F_k(\mathcal{X})$ such that for every $\alpha \in (0, 1)$,

$$E[X]_\alpha = cl \int_{\Omega} X_\alpha d\mu = cl\{E(f) : f \in S_{X_\alpha}\}$$

where the closure is taken in $\mathcal{X}$ and $S_{X_\alpha} = \{f \in L^1[\Omega, \mathcal{X}] : f(\omega) \in X_\alpha(\omega), a.e.(\mu)\}$. By virtue of the existence theorem (cf. (cf. Li, & Ogura, 1996), (Li, Ogura & Kreinovich, 2002)), we have an equivalent definition as follows:

$$E[X](x) = sup\{\alpha \in (0, 1) : x \in E[X_\alpha]\}.$$

Furthermore, $(E[\overline{\alpha X}])_\alpha = E[\overline{\alpha X_\alpha}]$ for any $\alpha \in (0, 1]$.

### 3. Negatively Dependent and Main Results

In this section, we will give the definition of negatively dependent for fuzzy set-valued random variables and discuss the properties. Then we will prove the weak and strong limit theorem of weighted sums for fuzzy set-valued negatively dependent random variables in the sense of $d_H^\infty$.

The following definition is Toeplitz sequence, which will be used later.

**Definition 3.1** A double array $\{a_{nk} : n, k = 1, 2, \cdots\}$ of real numbers is said to be a Toeplitz sequence, if

(i) $\lim_{n \to \infty} a_{nk} = 0$ for each $k$;

(ii) $\sum_{k=1}^\infty |a_{nk}| \leq c$ for each $n$.

Now we will recall some concepts of negatively dependent random variables.

**Definition 3.2** (cf. Bozorgnia, Patterson & Taylor, 1993) A finite family of real-valued random variables $X_1, \cdots, X_n$ is said to be negatively dependent if for all real $x_1, x_2, \cdots, x_n$,

$$\mu\{X_1 > x_1, \cdots, X_n > x_n\} \leq \prod_{i=1}^n \mu\{X_i > x_i\}$$
and
\[ \mu[X_1 \leq x_1, \cdots, X_n \leq x_n] \leq \prod_{i=1}^{n} \mu[X_i \leq x_i]. \]

An infinite family of random variables is negatively dependent if every finite subfamily is negatively dependent. The following results are from (Bozorgnia, Patterson & Taylor, 1993) of Bozorgnia et. al., and we will use them in the later.

**Lemma 3.3** (cf. Bozorgnia, Patterson & Taylor, 1993) Let real-valued random variables \( X_i : 1 \leq i \leq n \) be negatively dependent. Then the following are true:

(i) \( E[\prod_{i=1}^{n} X_i] \leq \prod_{i=1}^{n} E[X_i] \);

(ii) \( \text{Cov}(X_i, X_j) \leq 0, \quad i \neq j \);

(iii) If \( g_i : 1 \leq i \leq n \) be all nondecreasing (or all nonincreasing) Borel functions, then random variables \( g_1(X_1), g_2(X_2), \cdots, g_n(X_n) \) are negatively dependent random variables.

**Definition 3.4** (cf. Guan & Wan, 2016) A finite family of set-valued random variables \( F_1, F_2, \cdots, F_n \) is said to be negatively dependent if \( s(\cdot, F_1), \cdots, s(\cdot, F_n) \) is single-valued negatively dependent random variables.

Now we give the definition of fuzzy set-valued random variables and discuss the property.

**Definition 3.5** A finite family of fuzzy set-valued random variables \( X_1, X_2, \cdots, X^n \) be said to be negatively dependent if for any \( \alpha \in (0, 1) \), the set-valued random variables \( X_{\alpha_1}, \cdots, X_{\alpha_n} \) are negatively dependent.

From the definition 3.5, we can easily obtain the following result.

**Theorem 3.6** Fuzzy set-valued random variables \( X_1, X_2, \cdots, X^n \) are negatively dependent, then for any \( \alpha \in (0, 1) \), \( X_{\alpha_1}, \cdots, X_{\alpha_n} \) are negatively dependent set-valued random variables.

**Proof.** By definition 3.5, we know that for any \( \alpha \in (0, 1) \), \( X_{\alpha_1}^1, X_{\alpha_2}^2, \cdots, X_{\alpha_n}^n \) are negatively dependent set-valued random variables. That means for any \( x^* \in X^* \), \( \alpha \in (0, 1) \), \( s(x^*, X_{\alpha_1}^1), s(x^*, X_{\alpha_2}^2), \cdots, s(x^*, X_{\alpha_n}^n) \) are negatively dependent single-valued random variables. Since \( X_{\alpha_1}^n = \text{cl}(\bigcup_{\beta \geq \alpha} X_{\beta}^n) \), take a decreasing sequence \( \alpha_j \) which converges to \( \alpha \). Then
\[ \lim_{j \to \infty} d_\mu(X_{\alpha_j}^n, X_{\alpha_1}^n) = 0. \]

So we have
\[ s(x^*, X_{\alpha_j}^n) \to s(x^*, X_{\alpha_1}^n), \quad \text{as} \quad j \to \infty. \]

Then by the continuous of probability and lemma 3.3, we have
\[
\mu \left\{ s(x^*, X_{\alpha_1}^1) \leq x_1, \cdots, s(x^*, X_{\alpha_n}^n) \leq x_n \right\} = \\
\mu \left\{ \lim_j s(x^*, X_{\alpha_1}^1) \leq x_1, \cdots, \lim_j s(x^*, X_{\alpha_n}^n) \leq x_n \right\} = \\
\leq \lim_j \prod_{i=1}^{n} \mu\{s(x^*, X_{\alpha_i}^i) \leq x_i\} = \\
\prod_{i=1}^{n} \lim_j \mu\{s(x^*, X_{\alpha_i}^i) \leq x_i\} = \\
\prod_{i=1}^{n} \mu\{s(x^*, X_{\alpha_i}^i) \leq x_i\} = \\
\prod_{i=1}^{n} \mu\{s(x^*, X_{\alpha_i}^i) \leq x_i\}
\]

Similarly, we can prove
Then the result was proved. □

The following limit theorem is a weak convergence result for fuzzy set-valued negatively dependent random variables, which is the extension of (Guan & Sun, 2014). Here we assume the Banach space $X = \mathbb{R}$.

**Theorem 3.7** Let $\{X^nk : k \geq 1, n \geq 1\}$ be an array of fuzzy set-valued random variables in $F_k(\mathbb{R}^r)$ which are stochastically dominated by a fuzzy set-valued random variable $X$, and are pairwise negatively dependent in each row. Let $E[X^nk] = I_0$ for all $n$ and $k$. Let $\{a_{nk}\}$ be an array of nonnegative real numbers such that $\sum_{n=1}^{\infty} a_{nk}^r \leq M$ for all $n$ where $0 < r \leq 1$ and $\max_k a_{nk} \to 0$ as $n \to \infty$. If $E[\|X\|_\rho] < \infty$, then

$$\lim_{n \to \infty} \mu\left[ d_H^\infty \left( \sum_{k=1}^{\infty} a_{nk} X^nk, \{0\} \right) > \varepsilon \right] = 0.$$  

**Proof.** From the definition 3.5 and theorem 3.6, we know that $\{X^nk\}, \{X^nk\} \in K_k(\mathbb{R}^r)$ are all rowwise negatively dependent set-valued random variables. Then by theorem 4.1 of (Guan & Sun, 2014), for any $\alpha \in (0, 1]$, $\varepsilon > 0$, we have

$$\lim_{n \to \infty} \mu\left( d_H\left( \sum_{k=1}^{\infty} a_{nk} X^nk, \{0\} \right) > \varepsilon \right) = 0,$$

and

$$\lim_{n \to \infty} \mu\left( d_H\left( \sum_{k=1}^{\infty} a_{nk} X^nk, \{0\} \right) > \varepsilon \right) = 0.$$

Take a finite partition $0 = a_0 < a_1 < \cdots < a_M = 1$, for $a_{k-1} < \alpha < a_k$, by virtue of monotone property of level sets and the formula

$$\left( \sum_{k=1}^{\infty} a_{nk} X^nk \right)_\alpha = \sum_{k=1}^{\infty} a_{nk} X^nk_\alpha,$$

it holds that

$$d_H\left( \sum_{k=1}^{\infty} a_{nk} X^nk, \{0\} \right) \leq d_H\left( \sum_{k=1}^{\infty} a_{nk} X^nk_\alpha, \{0\} \right) + d_H\left( \sum_{k=1}^{\infty} a_{nk} X^nk_{\alpha_{k-1}+}, \{0\} \right)$$

$$= d_H\left( \sum_{k=1}^{\infty} a_{nk} X^nk_\alpha, \{0\} \right) + d_H\left( \sum_{k=1}^{\infty} a_{nk} X^nk_{\alpha_{k-1}+}, \{0\} \right).$$

Consequently, we have

$$d_H^\infty\left( \sum_{k=1}^{\infty} a_{nk} X^nk, \{0\} \right) = \sup_{\alpha \in (0, 1]} d_H\left( \sum_{k=1}^{\infty} a_{nk} X^nk_\alpha, \{0\} \right)$$

$$\leq \max_{1 \leq k \leq M} d_H\left( \sum_{k=1}^{\infty} a_{nk} X^nk_{\alpha_{k-1}+}, \{0\} \right)$$

Then we have

$$\lim_{n \to \infty} \mu\left[ d_H^\infty \left( \sum_{k=1}^{\infty} a_{nk} X^nk, \{0\} \right) > \varepsilon \right] \leq \lim_{n \to \infty} \mu\left( \max_{1 \leq k \leq M} d_H\left( \sum_{k=1}^{\infty} a_{nk} X^nk_{\alpha_{k-1}+}, \{0\} \right) > \varepsilon \right)$$

$$+ \lim_{n \to \infty} \mu\left( \max_{1 \leq k \leq M} d_H\left( \sum_{k=1}^{\infty} a_{nk} X^nk_\alpha, \{0\} \right) > \varepsilon \right)$$

$$= 0.$$
The result was proved. □

Next, we shall prove the strong convergence theorem, which is the extend of theorem 4.1 of (Guan & Wan, 2016). In (Guan & Wan, 2016), the authors proved the convergence theorem of set-valued random variables in the sense of $d_H$. Here we extend their result to fuzzy set-valued random variables, and the metric is $d_H^*$.

**Theorem 3.8** Let $\{X^n_k\} \in F_1(\mathbb{R}^+)$ be an array of rowwise negatively dependent fuzzy set-valued random variables with $E[X^n_k] = I_0$ and stochastic dominated by a fuzzy set-valued random variable $X$. If $\max_k a_{nk} = O(n^{-r}), \ r > 0$, then $E[||X||^{1+1/r}_F] < \infty$ implies that

$$
\sum_{k=1}^{\infty} a_{nk} X^n_k \to 0 \ a.e.
$$

with respect to the Hausdorff metric $d_H^*$.

**Proof.** From the definition 3.5 and theorem 3.6, we know that $\{X^n_k\}, \{X^n_{nk}\} \in K_1(\mathbb{R}^+)$ are all rowwise negatively dependent set-valued random variables. Then by theorem 4.1 of (Guan & Wan, 2016), we have

$$
\sum_{k=1}^{\infty} a_{nk} X^n_k \to 0 \ a.e.
$$

with respect to the Hausdorff metric $d_H$. And

$$
\sum_{k=1}^{\infty} a_{nk} X^n_{nk} \to 0 \ a.e.
$$

with respect to the Hausdorff metric $d_H$.

We can find a finite partition $0 = a_0 < a_1 < \cdots < a_M = 1$, for $a_{L-1} < a < a_L$. By virtue of monotone property of level sets and the formula

$$(\sum_{k=1}^{\infty} a_{nk} X^n_k)_\alpha = \sum_{k=1}^{\infty} a_{nk} X^n_k;$$

it holds that

$$
d_H((\sum_{k=1}^{\infty} a_{nk} X^n_k)_\alpha, \{0\}) \leq d_H((\sum_{k=1}^{\infty} a_{nk} X^n_k)_{a_1}, \{0\}) + d_H((\sum_{k=1}^{\infty} a_{nk} X^n_k)_{a_{L-1}+}, \{0\})
$$

$$
= d_H((\sum_{k=1}^{\infty} a_{nk} X^n_{nk})_\alpha, \{0\}) + d_H((\sum_{k=1}^{\infty} a_{nk} X^n_{nk+})_\alpha, \{0\}).
$$

Consequently, we have

$$
\sup_{\alpha \in (0,1]} d_H((\sum_{k=1}^{\infty} a_{nk} X^n_k)_\alpha, \{0\}) \leq \max_{1 \leq L \leq M} d_H((\sum_{k=1}^{\infty} a_{nk} X^n_k)_{a_L}, \{0\})
$$

$$
+ \max_{1 \leq L \leq M} d_H((\sum_{k=1}^{\infty} a_{nk} X^n_k)_{a_L+}, \{0\}).
$$

Since the two terms on the right hand converge to 0 in the sense of $d_H$, then we can have

$$
\limsup_{n \to \infty} \sup_{\alpha \in (0,1]} d_H((\sum_{k=1}^{\infty} a_{nk} X^n_k)_\alpha, \{0\}) = 0
$$

The result was proved. □

From theorem 3.8, we can easily get the following two corollaries.

**Corollary 3.9** If $\{X^n\}$ is a sequence of independent identical distributed fuzzy set-valued random variables in $F_1(\mathbb{R}^+)$ with $E[X^1] = I_0$ and $\max_k |a_{nk}| = O(n^{-r}), \ r > 0$, then $E[||X^1||^{1+1/\gamma}_F] < \infty$ implies that

$$
\sum_{k=1}^{\infty} a_{nk} X^n_k \to 0 \ a.e.
$$
with respect to the Hausdorff metric $d_H^\infty$.

**Corollary 3.10** If $\{X^n\}$ is a sequence of independent fuzzy set-valued random variables in $F_k(\mathbb{R}^+)$ with $E[X^k] = I_0$ for $k = 1, 2, \cdots$ and stochastically dominated by a fuzzy set-valued random variable $X$. If $\max_k |a_{nk}| = O(n^{-\gamma}), \gamma > 0$, then $E[|X|^1_k] < \infty$ implies that

$$
\sum_{k=1}^\infty a_{nk}X_{nk} \to 0 \ a.e.
$$

with respect to the Hausdorff metric $d_H^\infty$.

**4. Discussion**

In this paper, we mainly proved the limit theorems for negatively dependent fuzzy set-valued random variables, where underlying space is $\mathbb{R}^+$. When the underlying space is $\{x \leq 0 : x \in \mathbb{R}\}$, then the above results are also true. Since the $K(\mathcal{X})$ are not linear space, even the $K_{kc}(\mathcal{X})$ are not linear space, the difference of sets do not have good properties. So it is not easy to discuss the properties of negatively dependent and obtain the convergence theorems in general Banach space.

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**References**


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