International Journal of Statistics and Probability; Vol. 6, No. 1; January 2017
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Limit Theorems for Negatively Dependent Fuzzy Set-Valued Random Variables

Li Guan[1]

[1] College of Applied Sciences, Beijing University of Technology, Beijing, China

Correspondence: Li Guan, College of Applied Sciences, Beijing University of Technology, 100 Pingleyuan, Chaoyang District, Beijing, 100124, P.R.China. E-mail: guanli@bjut.edu.cn

Received: November 4, 2016 Accepted: December 5, 2016 Online Published: December 13, 2016

doi:10.5539/ijsp.v6n1p40 URL: https://doi.org/10.5539/ijsp.v6n1p40

Abstract

In this paper, we shall discuss negatively dependent fuzzy set-valued random variables. And at last, we shall prove the limit theorems for rowwise negatively dependent fuzzy set-valued random variables in the sense of d_H^∞, which is the extension of (Guan & Sun, 2014) and (Guan & Wan, 2016).

Keywords: laws of large number, fuzzy set-valued random variables, negatively dependent

1. Introduction

We all know that strong laws of large numbers are one of the most important theories in probability. For independent set-valued random variables, many limit results have been obtained (cf. (Artstein & Vitale, 1975), (Cressie, 1978), (Hiai, 1984), (Taylor & Inoue, 1985), (Puri & Ralescu, 1983)). About the convergence theorems of fuzzy set-valued random variables, Klement *et al* proved the strong laws of large numbers (SLLN) for independent identically distributed (i.i.d) fuzzy set-valued random variables in the sense of d_H^1. Colubi *et al.* obtained the SLLN for i.i.d fuzzy set-valued random variables with respect to d_H^∞ in (Colubi, López-Díaz, Dominguez-Menchero & Gil, 1999), where the underlying space is $\mathbb{R}^d$. Li and Ogura proved the same SLLN as in (Colubi, López-Díaz, Dominguez-Menchero & Gil, 1999) by using a new embedding method (Li, Ogura & Kreinovich, 2002, Theorem 6.2.6). In 1991, Inoue (Inoue, 1991) extended the SLLN of set-valued case given by Taylor and Inoue (Taylor & Inoue, 1985) to the case of only independent fuzzy set-valued random variables in the sense of d_H^1. Li and Ogura (Li & Ogura, 2003) proved the SLLN of (Inoue, 1991) in the sense of d_H^∞.

In practice, the random variables are not always independent. So it is necessary to discuss the limit theorems for dependent random variables. Bozorgnia, Patterson and Taylor discussed the properties for negatively dependent random variables in (Bozorgnia, Patterson, & Taylor, 1993), and proved the laws of large number for negative dependence random variables in (Bozorgnia, Patterson & Taylor, 1992), where the random variables are single-valued. In (Guan, 2014), Guan and Sun discussed the property of set-valued random variables in real space $\mathbb{R}$, and proved the weak convergence theorem for dependent set-vallued random variables in the sense of Hausdorff metric. In (Guan, 2016), Guan and Wan proved the strong law of large number for set-valued dependent random variables in the sense of Hausdorff metric. In 2016, Shen and Guan (Shen & Guan, 2016) proved the strong laws of large numbers for independent fuzzy set-valued random variables, where the underlying space is G_α space. In this paper, we are concerned with the weak and strong limit theorems for dependent fuzzy set-valued random variables in the sense of d_H^∞.

This paper is organized as follows. In section 2, we shall briefly introduce some definitions and basic results of set-valued random variables and fuzzy set-valued random variables. In section 3, we shall give basic definition and discuss the properties of fuzzy set-valued negatively dependent random variables. And at last, we shall prove the weak and strong laws of large numbers for weighted sums of fuzzy set-valued negatively dependent random variables.

2. Preliminaries on Fuzzy Set-Valued Random Variables

Throughout this paper, we assume that $(\Omega, \mathcal{A}, \mu)$ is a complete probability space, $(\mathfrak{X}, \|\cdot\|)$ is a real separable Banach space, $\mathbf{K_k}(\mathfrak{X})$ is the family of all nonempty compact subsets of $\mathfrak{X}$, and $\mathbf{K_{kc}}(\mathfrak{X})$ is the family of all nonempty compact convex subsets of $\mathfrak{X}$.

Let A and B be two nonempty subsets of $\mathfrak{X}$ and let $\lambda \in \mathbb{R}$, the set of all real numbers. We define addition and scalar multiplication by

$$A + B = \{a + b : a \in A, b \in B\}$$

$$\lambda A = \{\lambda a : a \in A\}$$

The Hausdorff metric on $\mathbf{K_k}(\mathfrak{X})$ is defined by

$$d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\}$$

for $A,\ B \in \mathbf{K_k}(\mathfrak{X})$. For an A in $\mathbf{K_k}(\mathfrak{X})$, let $\|A\|_{\mathbf{K}} = d_H(\{0\},A)$.

The metric space $(\mathbf{K_k}(\mathfrak{X}),\mathbf{d_H})$ is complete and separable, and $\mathbf{K_{kc}}(\mathfrak{X})$ is a closed subset of $(\mathbf{K_k}(\mathfrak{X}),\mathbf{d_H})$ (cf. (Li, Ogura & Kreinovich, 2002), Theorems 1.1.2 and 1.1.3). Concerning the concepts and results of set-valued random variables, readers may refer to the book (Li, Ogura & Kreinovich, 2002).

A set-valued mapping $F:\Omega\to K_k(\mathfrak{X})$ is called a set-valued random variable (or a random set) if, for each open subset O of $\mathfrak{X}$, $F^{-1}(O)=\{\omega\in\Omega : F(\omega)\cap O\neq\emptyset\}\in\mathcal{A}$. In fact, set-valued random variables can be defined as mappings from Ω to the family of all closed subsets of $\mathfrak{X}$. Concerning its equivalent definitions, please refer to (Castaing & Valadier, 1977) and (Hiai & Umegaki, 1977).

A set-valued random variable F is called integrably bounded (cf. (Hiai & Umegaki, 1977) or (Li, Ogura & Kreinovich, 2002)) if $\int_\Omega \|F(\omega)\|_{\mathbf{K}} d\mu < \infty$. Let $L^1[\Omega,\mathcal{A},\mu;\mathbf{K_k}(\mathfrak{X})]$ denote the space of all integrably bounded random variables, and $L^1[\Omega,\mathcal{A},\mu;\mathbf{K_{kc}}(\mathfrak{X})]$ denote the space of all integrably bounded random variables taking values in $\mathbf{K_{kc}}(\mathfrak{X})$. For $F,G\in L^1[\Omega,\mathcal{A},\mu;\mathbf{K_k}(\mathfrak{X})]$, $F=G$ if and only if $F(\omega)=G(\omega)\ a.e.(\mu)$.

For each set-valued random variable F, define $S_F=\{f\in L^1[\Omega;\mathfrak{X}] : f(\omega)\in F(\omega), a.e.(\mu)\}$. The expectation of F, denoted by $E[F]$, is defined as

$$E[F]=\Big\{\int_\Omega f d\mu : f\in S_F\Big\},$$

where $\int_\Omega f d\mu$ is the usual Bochner integral in $L^1[\Omega,\mathfrak{X}]$, the family of integrable $\mathfrak{X}$-valued random variables. This integral was first introduced by Aumann (Aumann, 1965), called Aumann integral in literature.

For each $A\in\mathbf{K}(\mathfrak{X})$, define the support function by

$$s(x^*,A)=\sup_{a\in A} < x^*,a >,\quad x^*\in\mathfrak{X}^*,$$

where $\mathfrak{X}^*$ is the dual space of $\mathfrak{X}$.

Let $\mathbf{S}^*$ denote the unit sphere of $\mathfrak{X}^*$, $C(\mathbf{S}^*)$ the all continuous functions of $\mathbf{S}^*$, and the norm is defined as $\|v\|_C=\sup_{x^*\in S^*}$

From now on, we begin to introduce necessary concepts, notation and basic results on fuzzy set space and fuzzy set-valued random variables.

Let $\mathbf{F_k}(\mathfrak{X})$ be the family of all special fuzzy sets: $v:\mathfrak{X}\to[0,1]$ satisfying the following conditions:
(1) the 1-level set $v_1=\{x\in\mathfrak{X} : v(x)=1\}\neq\emptyset$,
(2) each v is upper semicontinuous, i.e. for each $\alpha\in[0,1]$, the α level set $v_\alpha=\{x\in\mathfrak{X} : v(x)\geq\alpha\}$ is a closed subset of $\mathfrak{X}$,
(3) the support set $v_{0+}=\mathrm{cl}\{x\in\mathfrak{X} : v(x)>0\}$ is compact.

A fuzzy set v in $\mathbf{F_k}(\mathfrak{X})$ is called convex if it satisfies

$$v(\lambda x+(1-\lambda)y)\geq\min\{v(x),v(y)\},\ \text{for any } x,y\in\mathfrak{X},\lambda\in[0,1].$$

It is known that v is convex if and only if each level set v_α ($\alpha\in(0,1]$) of v is a convex subset of $\mathfrak{X}$. Let $\mathbf{F_{kc}}(\mathfrak{X})$ be the subset of all convex fuzzy sets in $\mathbf{F_k}(\mathfrak{X})$.

The uniform metric in $\mathbf{F_k}(\mathfrak{X})$, which is an extension of the Hausdorff metric d_H, is often used (cf. Puri & Ralescu, 1986): for $v^1,v^2\in\mathbf{F_k}(\mathfrak{X})$,

$$d_H^\infty(v^1,v^2)=\sup_{\alpha\in(0,1]} d_H(v^1_\alpha,v^2_\alpha).$$

Let $\|v\|_{\mathbf{F}} =: d_H^\infty(v,I_0)=\sup_{\alpha>0}\|v_\alpha\|_{\mathbf{K}}$, where I_0 is the function taking value one at 0 and zero for all $x\neq 0$. The space $(\mathbf{F_k}(\mathfrak{X}),\mathbf{d_H^\infty})$ is a complete metric space (cf. Li, & Ogura, 1996) but not separable (Li, Ogura & Kreinovich, 2002), Remark 5.1.7). Completeness was first proved by Puri and Ralescu (Puri & Ralescu, 1986) in the case of $\mathfrak{X}=\mathbb{R}^d$, the d-dimensional Euclidean space.

A fuzzy set-valued random variable (or $\mathbf{F_k}(\mathfrak{X})$-valued random variable) is a measurable mapping X from the space $(\Omega, \mathcal{A})$ to the space $(\mathbf{F_k}(\mathfrak{X}), \mathcal{B}(\mathbf{F_k}(\mathfrak{X})))$, where $\mathcal{B}(\mathbf{F_k}(\mathfrak{X}))$ is the Borel σ-field of $\mathbf{F_k}(\mathfrak{X})$ with respect to d_H^∞.

It is well known that for any fuzzy set v, $v_\alpha = \bigcap_{\beta<\alpha} v_\beta$, for every $\alpha \in (0,1]$, but usually $v_\alpha \neq \bigcup_{\beta>\alpha} v_\beta$. We denote $v_{\alpha+} = \mathrm{cl}(\bigcup_{\beta>\alpha} v_\beta)$, for $\alpha \in [0,1)$, which will be used later. Obviously, v_{0+} is the support set of v. Due to the completeness of $(\mathbf{F_k}(\mathfrak{X}), \mathbf{d_H^\infty})$, every Cauchy sequence $\{v^n : n \in \mathbb{N}\}$ has a limit v in $\mathbf{F_k}(\mathfrak{X})$.

A sequence of set-valued random variables $\{F_n : n \in \mathbb{N}\}$ is called to be stochastically dominated by a set-valued random variable F if

$$\mu\{\|F_n\|_{\mathbf{K}} > t\} \leq \mu\{\|F\|_{\mathbf{K}} > t\}, \;\; t \geq 0, \; n \geq 1.$$

A sequence of fuzzy set-valued random variables $\{X^n : n \geq 1\}$ is called to be stochastically dominated by a fuzzy set-valued random variable X if

$$\mu\{\|X_\alpha^n\|_{\mathbf{K}} > t\} \leq \mu\{\|X_\alpha\|_{\mathbf{K}} > t\}, \quad t \geq 0, \; n \geq 1, \alpha \in (0,1].$$

It is obvious that if $\{X^n : n \geq 1\}$ is stochastically dominated by a fuzzy set-valued random variable X, then $\forall\ \alpha \in (0,1]$, $\{X_\alpha^n : n \geq 1\}$ is stochastically dominated by the set-valued random variable X_α. And also for any $\alpha \in (0,1]$, $\{X_{\alpha+}^n : n \geq 1\}$ is stochastically dominated by the set-valued random variable $X_{\alpha+}$ (Guan & Li, 2004).

An $\mathbf{F_k}(\mathfrak{X})$-valued random variable X is called integrably bounded if the real-valued random variable $\|X_{0+}(\omega)\|_{\mathbf{K}}$ is integrable. Let $L^1[\Omega, \mathcal{A}, \mu; \mathbf{F_k}(\mathfrak{X})]$ be the set of all integrably bounded $\mathbf{F_k}(\mathfrak{X})$-valued random variables and $L^1[\Omega, \mathcal{A}, \mu; \mathbf{F_{kc}}(\mathfrak{X})]$ be the set of all integrably bounded $\mathbf{F_{kc}}(\mathfrak{X})$-valued random variables. Two $\mathbf{F_k}(\mathfrak{X})$-valued random variables $X, Y \in L^1[\Omega, \mathcal{A}, \mu; \mathbf{F_k}(\mathfrak{X})]$ are considered to be identical if for any $\alpha \in [0,1], X_\alpha(\omega) = Y_\alpha(\omega)\ a.e.(\mu)$.

A sequence of $\mathbf{F_k}(\mathfrak{X})$-valued random variables $\{X^n : n \in \mathbb{N}\}$ is called to converge to an $\mathbf{F_k}(\mathfrak{X})$-valued random variable X in the sense of d_H^∞, if $d_H^\infty(X^n(\omega), X(\omega)) \to 0\ a.e.(\mu)\ as\ n \to \infty$.

The expectation of an $\mathbf{F_k}(\mathfrak{X})$-valued random variables X, denoted by $E[X]$, is an element in $\mathbf{F_k}(\mathfrak{X})$ such that for every $\alpha \in (0,1]$,

$$(E[X])_\alpha = cl \int_\Omega X_\alpha d\mu = cl\{E(f) : f \in S_{X_\alpha}\}$$

where the closure is taken in $\mathfrak{X}$ and $S_{X_\alpha} = \{f \in L^1[\Omega; \mathfrak{X}] : f(\omega) \in X_\alpha(\omega), a.e.(\mu)\}$. By virtue of the existence theorem (cf. (cf. Li, & Ogura, 1996), (Li, Ogura & Kreinovich, 2002)), we have an equivalent definition as follows:

$$E[X](x) = \sup\{\alpha \in (0,1] : x \in E[X_\alpha]\}.$$

Furthermore, $(E[\overline{co}X])_\alpha = E[\overline{co}X_\alpha]$ for any $\alpha \in (0,1]$.

3. Negatively Dependent and Main Results

In this section, we will give the definition of negatively dependent for fuzzy set-valued random variables and discuss the properties. Then we will prove the weak and strong limit theorem of weighted sums for fuzzy set-valued negatively dependent random variables in the sense of d_H^∞.

The following definition is Toeplitz sequence, which will be used later.

Definition 3.1 *A double array $\{a_{nk} : n, k = 1, 2, \cdots\}$ of real numbers is said to be a Toeplitz sequence, if*

(i) $\lim_{n\to\infty} a_{nk} = 0$ *for each k;*

(ii) $\sum_{k=1}^{\infty} |a_{nk}| \leq c$ *for each n.*

Now we will recall some concepts of negatively dependent random variables.

Definition 3.2 (cf. Bozorgnia, Patterson & Taylor, 1993)) *A finite family of real-valued random variables $X_1, \cdots, X_n$ is said to be negatively dependent if for all real $x_1, x_2, \cdots, x_n$,*

$$\mu\{X_1 > x_1, \cdots, X_n > x_n\} \leq \prod_{i=1}^{n} \mu\{X_i > x_i\}$$

and

$$\mu\{X_1 \le x_1, \cdots, X_n \le x_n\} \le \prod_{i=1}^{n} \mu\{X_i \le x_i\}.$$

An infinite family of random variables is negatively dependent if every finite subfamily is negatively dependent. The following results are from (Bozorgnia, Patterson & Taylor, 1993) of Bozorgnia et. al., and we will use them in the later.

Lemma 3.3 (cf. Bozorgnia, Patterson & Taylor, 1993) *Let real-valued random variables* $\{X_i : 1 \le i \le n\}$ *be negatively dependent. Then the following are true:*

(i)$E[\prod_{i=1}^{n} X_i] \le \prod_{i=1}^{n} E[X_i]$;

(ii)$Cov(X_i, X_j) \le 0, \quad i \ne j$;

(iii) If $\{g_i : 1 \le i \le n\}$ *be all nondecreasing (or all nonincreasing) Borel functions, then random variables* $g_1(X_1), g_2(X_2), \cdots, g_n(X_n)$ *are negatively dependent random variables.*

Definition 3.4 (cf. (Guan & Wan, 2016) *A finite family of set-valued random variables* $F_1, F_2, \cdots, F_n$ *is said to be negatively dependent if* $s(\cdot, F_1), \cdots, s(\cdot, F_n)$ *is single-valued negatively dependent random variables.*

Now we give the definition of fuzzy set-valued random variables and discuss the property.

Definition 3.5 *A finite family of fuzzy set-valued random variables* $X^1, X^2, \cdots, X^n$ *be said to be negatively dependent if for any* $\alpha \in (0, 1]$, *the set-valued random variables* $X^1_\alpha, X^2_\alpha, \cdots, X^n_\alpha$ *are negatively dependent.*

From the definition 3.5, we can easily obtain the following result.

Theorem 3.6 *Fuzzy set-valued random variables* $X^1, X^2, \cdots, X^n$ *are negatively dependent, then for any* $\alpha \in (0, 1]$, $X^1_{\alpha+}, X^2_{\alpha+}, \cdots, X^n_{\alpha+}$ *are negatively dependent set-valued random variables.*

Proof. By definition 3.5, we know that for any $\alpha \in (0, 1]$, $X^1_\alpha, X^2_\alpha, \cdots, X^n_\alpha$ are negatively dependent set-valued random variables. That means for any $x^* \in \mathfrak{X}^*, \alpha \in (0, 1]$, $s(x^*, X^1_\alpha), s(x^*, X^2_\alpha), \cdots, s(x^*, X^n_\alpha)$ are negatively dependent single-valued random variables. Since $X^n_{\alpha+} = cl(\bigcup_{\beta>\alpha} X^n_\beta)$, take a decreasing sequence α_j which converges to α. Then

$$\lim_{j\to\infty} d_H(X^n_{\alpha_j}, X^n_{\alpha+}) = 0.$$

So we have

$$s(x^*, X^n_{\alpha_j}) \longrightarrow s(x^*, X^n_{\alpha+}), \quad \text{as} \quad j \to \infty.$$

Then by the continuous of probability and lemma 3.3, we have

$$\begin{aligned}
&\mu\Big\{s(x^*, X^1_{\alpha+}) \le x_1, \cdots, s(x^*, X^n_{\alpha+}) \le x_n\Big\}\\
&= \mu\Big\{\lim_j s(x^*, X^1_{\alpha_j}) \le x_1, \cdots, \lim_j s(x^*, X^n_{\alpha_j}) \le x_n\Big\}\\
&\le \lim_j \mu\Big\{s(x^*, X^1_{\alpha_j}) \le x_1, \cdots, s(x^*, X^n_{\alpha_j}) \le x_n\Big\}\\
&\le \lim_j \prod_{i=1}^{n} \mu\Big\{s(x^*, X^i_{\alpha_j}) \le x_i\Big\}\\
&= \prod_{i=1}^{n} \lim_j \mu\Big\{s(x^*, X^i_{\alpha_j}) \le x_i\Big\}\\
&= \prod_{i=1}^{n} \mu\Big\{\lim_j s(x^*, X^i_{\alpha_j}) \le x_i\Big\}\\
&= \prod_{i=1}^{n} \mu\Big\{s(x^*, X^i_{\alpha+}) \le x_i\Big\}
\end{aligned}$$

Similarly, we can prove

$$\mu\Big\{s(x^*, X^1_{\alpha+}) > x_1, \cdots, s(x^*, X^n_{\alpha+}) > x_n\Big\} \le \prod_{i=1}^{n} \mu\Big\{s(x^*, X^i_{\alpha+}) > x_i\Big\}.$$

Then the result was proved. □

The following limit theorem is a weak convergence result for fuzzy set-valued negatively dependent random variables, which is the extension of (Guan & Sun, 2014). Here we assume the Banach space $\mathfrak{X} = \mathbb{R}$.

Theorem 3.7 *Let $\{X^{nk} : k \ge 1, n \ge 1\}$ be an array of fuzzy set-valued random variables in $\mathbf{F}_k(\mathbb{R}^+)$ which are stochastically dominated by a fuzzy set-valued random variable X, and are pairwise negatively dependent in each row. Let $E[X^{nk}] = I_0$ for all n and k. Let $\{a_{nk}\}$ be an array of nonnegative real numbers such that $\sum_{n=1}^{\infty} a^r_{nk} \le M$ for all n where $0 < r \le 1$ and $\max_k a_{nk} \to 0$ as $n \to \infty$. If $E[\|X\|_{\mathbf{F}}] < \infty$, then*

$$\lim_{n\to\infty} \mu\Big\{d^{\infty}_H\Big(\sum_{k=1}^{\infty} a_{nk} X^{nk}, \{0\}\Big) > \varepsilon\Big\} = 0.$$

Proof. From the definition 3.5 and theorem 3.6, we know that $\{X^{nk}_{\alpha}\}, \{X^{nk}_{\alpha+}\} \in \mathbf{K}_k(\mathbb{R}^+)$ are all rowwise negatively dependent set-valued random variables. Then by theorem 4.1 of (Guan & Sun, 2014), for any $\alpha \in (0, 1]$, $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \mu\Big\{d_H\Big(\sum_{k=1}^{\infty} a_{nk} X^{nk}_{\alpha}, \{0\}\Big) > \frac{\varepsilon}{2}\Big\} = 0,$$

and

$$\lim_{n\to\infty} \mu\Big\{d_H\Big(\sum_{k=1}^{\infty} a_{nk} X^{nk}_{\alpha+}, \{0\}\Big) > \frac{\varepsilon}{2}\Big\} = 0.$$

Take a finite partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_M = 1$, for $\alpha_{k-1} < \alpha < \alpha_k$, by virtue of monotone property of level sets and the formula

$$(\sum_{k=1}^{\infty} a_{nk} X^{nk})_{\alpha} = \sum_{k=1}^{\infty} a_{nk} X^{nk}_{\alpha},$$

it holds that

$$\begin{aligned} d_H\Big((\sum_{k=1}^{\infty} a_{nk} X^{nk})_{\alpha}, \{0\}\Big) &\le d_H\Big((\sum_{k=1}^{\infty} a_{nk} X^{nk})_{\alpha_k}, \{0\}\Big) + d_H\Big((\sum_{k=1}^{\infty} a_{nk} X^{nk})_{\alpha_{k-1}+}, \{0\}\Big) \\ &= d_H\Big(\sum_{k=1}^{\infty} a_{nk} X^{nk}_{\alpha_k}, \{0\}\Big) + d_H\Big(\sum_{k=1}^{\infty} a_{nk} X^{nk}_{\alpha_{k-1}+}, \{0\}\Big). \end{aligned}$$

Consequently, we have

$$\begin{aligned} d^{\infty}_H\Big(\sum_{k=1}^{\infty} a_{nk} X^{nk}, \{0\}\Big) &= \sup_{\alpha\in(0,1]} d_H\Big((\sum_{k=1}^{\infty} a_{nk} X^{nk})_{\alpha}, \{0\}\Big) \\ &\le \max_{1\le k\le M} d_H\Big(\sum_{k=1}^{\infty} a_{nk} X^{nk}_{\alpha_k}, \{0\}\Big) \\ &\quad + \max_{1\le k\le M} d_H\Big(\sum_{k=1}^{\infty} a_{nk} X^{nk}_{\alpha_{k-1}+}, \{0\}\Big). \end{aligned}$$

Then we have

$$\begin{aligned} \lim_{n\to\infty} \mu\Big\{d^{\infty}_H\Big(\sum_{k=1}^{\infty} a_{nk} X^{nk}, \{0\}\Big) > \varepsilon\Big\} &\le \lim_{n\to\infty} \mu\Big\{\max_{1\le k\le M} d_H\Big(\sum_{k=1}^{\infty} a_{nk} X^{nk}_{\alpha_k}, \{0\}\Big) > \frac{\varepsilon}{2}\Big\} \\ &\quad + \lim_{n\to\infty} \mu\Big\{\max_{1\le k\le M} d_H\Big(\sum_{k=1}^{\infty} a_{nk} X^{nk}_{\alpha_{k-1}+}, \{0\}\Big) > \frac{\varepsilon}{2}\Big\} \\ &= 0. \end{aligned}$$

The result was proved. □

Next, we shall prove the strong convergence theorem, which is the extend of theorem 4.1 of (Guan & Wan, 2016). In (Guan & Wan, 2016), the authors proved the convergence theorem of set-valued random variables in the sense of d_H. Here we extend their result to fuzzy set-valued random variables, and the metric is d_H^∞.

Theorem 3.8 *Let $\{X^{nk}\} \in \mathbf{F}_k(\mathbb{R}^+)$ be an array of rowwise negatively dependent fuzzy set-valued random variables with $E[X^{nk}] = I_0$ and stochastic dominated by a fuzzy set-valued random variable X. If $\max_k a_{nk} = O(n^{-r})$, $r > 0$, then $E[\|X\|_{\mathbf{F}}^{1+1/r}] < \infty$ implies that*

$$\sum_{k=1}^{\infty} a_{nk} X^{nk} \to 0 \ \ a.e.$$

with respect to the Hausdorff metric d_H^∞.

Proof. From the definition 3.5 and theorem 3.6, we know that $\{X_\alpha^{nk}\}, \{X_{\alpha+}^{nk}\} \in \mathbf{K}_k(\mathbb{R}^+)$ are all rowwise negatively dependent set-valued random variables. Then by theorem 4.1 of (Guan & Wan, 2016), we have

$$\sum_{k=1}^{\infty} a_{nk} X_\alpha^{nk} \to 0 \ \ a.e.$$

with respect to the Hausdorff metric d_H. And

$$\sum_{k=1}^{\infty} a_{nk} X_{\alpha+}^{nk} \to 0 \ \ a.e.$$

with respect to the Hausdorff metric d_H.

We can find a finite partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_M = 1$, for $\alpha_{k-1} < \alpha < \alpha_k$. By virtue of monotone property of level sets and the formula

$$(\sum_{k=1}^{\infty} a_{nk} X^{nk})_\alpha = \sum_{k=1}^{\infty} a_{nk} X_\alpha^{nk},$$

it holds that

$$\begin{aligned} d_H\Big((\sum_{k=1}^{\infty} a_{nk} X^{nk})_\alpha, \{0\}\Big) &\le d_H\Big((\sum_{k=1}^{\infty} a_{nk} X^{nk})_{\alpha_k}, \{0\}\Big) + d_H\Big((\sum_{k=1}^{\infty} a_{nk} X^{nk})_{\alpha_{k-1}+}, \{0\}\Big) \\ &= d_H\Big(\sum_{k=1}^{\infty} a_{nk} X_{\alpha_k}^{nk}, \{0\}\Big) + d_H\Big(\sum_{k=1}^{\infty} a_{nk} X_{\alpha_{k-1}+}^{nk}, \{0\}\Big). \end{aligned}$$

Consequently, we have

$$\begin{aligned} \sup_{\alpha\in(0,1]} d_H\Big((\sum_{k=1}^{\infty} a_{nk} X^{nk})_\alpha, \{0\}\Big) &\le \max_{1\le k\le M} d_H\Big(\sum_{k=1}^{\infty} a_{nk} X_{\alpha_k}^{nk}, \{0\}\Big) \\ &\quad + \max_{1\le k\le M} d_H\Big(\sum_{k=1}^{\infty} a_{nk} X_{\alpha_{k-1}+}^{nk}, \{0\}\Big). \end{aligned}$$

Since the two terms on the right hand converge to 0 in the sense of d_H, then we can have

$$\limsup_{n\to\infty} \sup_{\alpha\in(0,1]} d_H\Big((\sum_{k=1}^{\infty} a_{nk} X^{nk})_\alpha, \{0\}\Big) = 0$$

The result was proved. □

From theorem 3.8, we can easily get the following two corollaries.

Corollary 3.9 If $\{X^n\}$ is a sequence of independent identical distributed fuzzy set-valued random variables in $\mathbf{F}_k(\mathbb{R}^+)$ with $E[X^1] = I_0$ and $\max_k |a_{nk}| = O(n^{-\gamma}), \gamma > 0$, then $E[\|X^1\|_{\mathbf{F}}^{1+1/\gamma}] < \infty$ implies that

$$\sum_{k=1}^{\infty} a_{nk} X^{nk} \to 0 \ \ a.e.$$

with respect to the Hausdorff metric d_H^∞.

Corollary 3.10 If $\{X^n\}$ is a sequence of independent fuzzy set-valued random variables in $\mathbf{F}_k(\mathbb{R}^+)$ with $E[X^k] = I_0$ for $k = 1, 2, \cdots$ and stochastically dominated by a fuzzy set-valued random variable X. If $\max_k |a_{nk}| = O(n^{-\gamma}), \gamma > 0$, then $E[\|X\|_{\mathbf{F}}^{1+1/\gamma}] < \infty$ implies that

$$\sum_{k=1}^{\infty} a_{nk} X^{nk} \to 0 \ \ a.e.$$

with respect to the Hausdorff metric d_H^∞.

4. Discussion

In this paper, we mainly proved the limit theorems for negatively dependent fuzzy set-valued random variables, where underlying space is $\mathbb{R}+$. When the underlying space is $\{x \leq 0 : x \in \mathbb{R}\}$, then the above results are also true. Since the $\mathbf{K}(\mathfrak{X})$ are not linear space, even the $\mathbf{K}_{kc}(\mathfrak{X})$ are not linear space, the difference of sets do not have good properties. So it is not easy to discuss the properties of negatively dependent and obtain the convergence theorems in general Banach space.

Acknowledgements

The author is extremely grateful to the referees for their valuable comments and suggestions on improvement of the paper. The research was supported by NSFC(11301015,11571024, 11401016).

References

Artstein, Z., & Vitale, R. A. (1975). A strong law of large numbers for random compact sets. *Ann. Probab. 3*, 879-882. http://dx.doi.org/10.1214/aop/1176996275

Aumann, R. (1965). Integrals of set-valued functions. *J. Math. Anal. Appl.*, *12*, 1-12. http://dx.doi.org/10.1016/0022-247X(65)90049-1

Bozorgnia, A., Patterson, R. F.,& Taylor, R. L., (1992). Limit theorems for dependent random variables. *WCNA'92 proceedings of the first world congress on world congress of nonlinear analysts'92*, *2*, 1639-1650.

Bozorgnia, A., Patterson, R. F., & Taylor, R.L. (1993). Limit theorems for ND random variables. *University of Georgia Technical Report*.

Castaing, C., & Valadier, M. (1977). Convex Analysis and Measurable Multifunctions. *Lect. Notes in Math.*, *580*, Springer–Verlag, Berlin, New York.

Colubi, A., López-Díaz, M., Dominguez-Menchero J. S, & Gil, M. A. (1999). A generalized strong law of large numbers. *Probab. Theory and Rel. Fields*, *114*, 401-417.

Cressie, N. (1978). A strong limit theorem for random sets. *Suppl. Adv. in Appl. Probab., 10*, 36-46.

Guan, L., & Li, S. (2004). Laws of large numbers for weighted sums of fuzzy set-valued random variables. *International J. Uncertainty , Fuzziness and Knowlidge-Based Systems*, *12*, 811-825. http://dx.doi.org/10.1142/S0218488504003223

Guan, L., & Sun, L. J. (2014). The laws of large numbers for set-valued negatively dependent random variables. *International Journal of Intelligent Technologies and Applied Statistics*, *7*(3), 99-112.

Guan, L., & Wan, Y. (2016). A strong law of large numbers for set-valued negatively dependent random variables. *International Journal of Statistics and Probability*, *5*(3), 102-112. http://dx.doi.org/10.5539/ijsp.v5n3p102

Hiai, F. (1984). Strong laws of large numbers for multivalued random variables, Multifunctions and Integrands (G. Salinetti, ed.). *Lecture Notes in Math., 1091*, Springer, Berlin, 160-172.

Hiai, F., & Umegaki, H. (1977). Integrals, conditional expectations and martingales of multivalued functions. *Jour. Multivar. Anal.*, *7*, 149-182.

Inoue, H. (1991). A strong law of large numbers for fuzzy random sets. *Fuzzy Sets and Syst.*, *41*, 285-291.

Li, S., & Ogura, Y. (1996). Fuzzy random variables, conditional expectations and fuzzy martingales. *J. Fuzzy Math., 4*, 905-927.

Li, S., Ogura, Y., & Kreinovich, V. (2002). Limit Theorems and Applications of Set-Valued and Fuzzy Sets-Valued Random Variables. *Kluwer Academic Publishers.* http://dx.doi.org/10.1007/978-94-015-9932-0

Li, S,& Ogura, Y. (2003). Strong laws of numbers for independent fuzzy set-valued random variables. *Fuzzy Sets and*

Systems, *157*, 2569-2578. http://dx.doi.org/10.1016/j.fss.2003.06.011

Puri, M. L, & Ralescu, D. A. (1983). Strong law of large numbers for Banach space valued random sets. *Ann. Probab.*, *11*, 222–224. http://dx.doi.org/10.1214/aop/1176993671

Puri, M. L, & Ralescu, D. A. (1986). Fuzzy random variables. *J. Math. Anal. Appl.*, *114*, 406-422.

Shen, L. M., & Guan, L. (2016). Strong laws of large numbers foe fuzzy set-valued random variables in G_α space. *Advances in Pure Mathematics*, *6*, 583-592. http://dx.doi.org/10.4236/apm.2016.69047

Taylor, R. L., Daffer, P. Z., & Patterson, R. F. (1985). Limit theorems for sums of exchangeable variables. *Rowman and Allanheld*, Totowa.

Taylor, R. L., & Inoue, H. (1985). A strong law of large numbers for random sets in Banach spaces. *Bull. Instit. Math. Academia Sinica*, *13*, 403-409.

Taylor, R. L., & Inoue, H. (1985) A strong law of large numbers for random sets in Banach spaces. *Bull. Instit. Math. Academia Sinica*, *13*, 403-409.