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Estimating Three-way Latent Interaction Effects in Structural Equation Modeling

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Abstract

A Monte Carlo simulation was performed for estimating and testing hypotheses of three-way interaction effect in latent variable regression models. A considerable amount of research has been done on estimation of simple interaction and quadratic effect in nonlinear structural equation. The present study extended to three-way continuous latent interaction in structural equation model. The latent moderated structural equation (LMS) approach was used to estimate the parameters of the three-way interaction in structural equation model and investigate the properties of the method under different conditions though simulations. The approach showed least bias, standard error,and root mean square error as indicator reliability and sample size increased. The power to detect interaction effect and type I error control were also manipulated showing that power increased as interaction effect size, sample size and latent covariance increased.

Keywords: Interaction effect size, latent variable, latent interaction effect, LMS, nonlinear

1. Introduction

Structural Equation Modeling (SEM) is a statistical method used for building models, making inference and quantify the relationship among latent variables that are not observable or cannot be measured precisely. But, measurement on the indicator variable related to those unobservable variables are available. This relationship began its bases as a method for modeling linear relationship. However, because of many of the models for observable variables in the social and behavioral sciences involves nonlinearity, its unlikely that linear models are always enough to describe the relationship between latent variables.

Extending SEM to include nonlinear functions allows researchers meaningfully and accurately model the relationship underlying their data. (Kenny & Judd, 1984) introduced the first statistical method aimed at producing estimates of parameters in a nonlinear structural equation model (specifically a quadratic or cross-product structural model with a linear measurement model). Their method attracted methodological discussions and alterations by a number of papers. For instances, (Hayduck, 1987) demonstrated how the Kenny-Judd model could be implemented in LISREL.

Generally, most of the available literature were only for the specific quadratic and simple cross-product structural model. Hence, this study extended the simple cross-product to three-way interaction effect in nonlinear SEM and estimate its effects using latent moderated structural(LMS) equation method. By Montecarlo simulation,the statistical properties of the approach(LMS) were discussed.

In empirical research, models such as (1) can be very useful. It covers the situation in which there are two moderator variables which jointly influence the regression of the dependent variable on an independent variable. In other words, a regression model that has a significant three-way interaction of continuous variables. For instance, to study the moderating effect of social support, hardiness on the relationship between stress and depression,one hypothesizes that the effect of stress on depression was moderated by hardiness and social support. In such cases Model (1) gives a direct test of this hypothesis.

2. Model

A model with three latent variables with three observed indicators each for both endogenous and exogenous latent variables was used. For the identification purposes we chose to set a single factor loading to 1 for η, ξ_1, ξ_2 and ξ_3. By three-way interaction, we mean the interaction of three continuous exogenous latent variables (ξ_1, ξ_2, ξ_3). Following the LISREL specification, the structural equation considered was:

$$\eta = \gamma_1\xi_1 + \gamma_2\xi_2 + \gamma_3\xi_3 + \gamma_4\xi_1\xi_2 + \gamma_5\xi_1\xi_3 + \gamma_6\xi_2\xi_3 + \gamma_7\xi_1\xi_2\xi_3 + \zeta \quad (1)$$

The measurement equations for each models were given by

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \lambda_{2y} & 0 & 0 & 0 \\ \lambda_{3y} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda_{x2} & 0 & 0 \\ 0 & \lambda_{x3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda_{x5} & 0 \\ 0 & 0 & \lambda_{x6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \lambda_{x8} \\ 0 & 0 & 0 & \lambda_{x9} \end{bmatrix} \begin{bmatrix} \eta \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \end{bmatrix}$$

Where x is a qx1 vector of independent indicator variables,and y is a px1 vector of dependent indicator variables. λ_x is a regression coefficients predicting x by ξ and λ_y is a regression coefficients predicting y by η. τ_x is vector of x-intercept and τ_y is a px1 vector of y-intercept. δ is a q x 1 vector of measurements errors of x and ε is a p x 1 vector of measurements error of y.

η is mx1 vector of latent endogenous variables and ξ is n x 1 vector of exogenous variables. Γ_i is regression coefficients predicting η by ξ and ζ is a vector of disturbance.

It was assumed that

- $\xi_1, \xi_2, \xi_3, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \varepsilon_1, \varepsilon_2$, and ε_3 are multivariate normally distributed.
- $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \varepsilon_1, \varepsilon_2$, and ε_3 have expected values of zero and are uncorrelated with ξ_1, ξ_2 and ξ_3.
- Finally, ζ has an expected value of zero and assumed to be uncorrelated with $\xi_1, \xi_2, \xi_3, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \varepsilon_1$, ε_2, and ε_3.

Based on these assumptions the mean vector and covariance matrix of $(\xi_1, \xi_2, \xi_3, \xi_1\xi_2, \xi_1\xi_3, \xi_2\xi_3, \xi_1\xi_2\xi_3)$ were derived as follows:

$cov(\xi_1, \xi_1\xi_2) = E(\xi_1)cov(\xi_1, \xi_2) + E(\xi_2)cov(\xi_1, \xi_1) + E[(\xi_1 - E(\xi_1))(\xi_1 - E(\xi_1))(\xi_2 - E(\xi_2))]$

By centering ξ_1 and ξ_2, the expected value of both becomes ,$E(\xi_1) = 0$ and $E(\xi_2) = 0$. Hence $cov(\xi_1, \xi_1\xi_2) = E[(\xi_1 - E(\xi_1))(\xi_1 - E(\xi_1))(\xi_2 - E(\xi_2))]$. Under multivariate normality all third moments vanish (see Bohnstedt and Goldberger 1969). This indicates that $cov(\xi_1, \xi_1\xi_2) = E(\xi_1\xi_1\xi_2) = 0$. Accordingly, all the covariance of the main effects with their two-way interaction is zero under normality condition and the given assumptions.

Following the same procedure, $cov(\xi_1\xi_2, \xi_1\xi_3) = E(\xi_1)E(\xi_1)cov(\xi_2, \xi_3) + E(\xi_1)E(\xi_3)cov(\xi_2, \xi_1) + E(\xi_2)E(\xi_1)cov(\xi_1, \xi_3) + E(\xi_2)E(\xi_3)cov(\xi_1, \xi_1) + cov(\xi_1, \xi_1)cov(\xi_2, \xi_3) + cov(\xi_1, \xi_3)cov(\xi_2, \xi_1)$.

Centering ξ_1, ξ_2, ξ_3 the first four terms are zero and we have

$cov(\xi_1\xi_2, \xi_1\xi_3) = cov(\xi_1, \xi_1)cov(\xi_2, \xi_3) + cov(\xi_1, \xi_3)cov(\xi_2, \xi_1) = \phi_{11}\phi_{23} + \phi_{13}\phi_{12}$ and

$cov(\xi_1\xi_2, \xi_2\xi_3) = cov(\xi_1, \xi_2)cov(\xi_2, \xi_3) + cov(\xi_1, \xi_3)cov(\xi_2, \xi_2) = \phi_{12}\phi_{23} + \phi_{13}\phi_{22}$

The covariance of the main effects with the product of the three exogenous variables can be found in the similar manner

$$\begin{aligned} cov(\xi_1, \xi_1\xi_2\xi_3) &= cov(\xi_1, \xi_1)cov(\xi_2, \xi_3) + cov(\xi_1, \xi_2)cov(\xi_1, \xi_3) + cov(\xi_1, \xi_3)cov(\xi_1\xi_2) \\ &= \phi_{11}\phi_{23} + \phi_{12}\phi_{13} + \phi_{13}\phi_{12} \end{aligned}$$

$$\begin{aligned} cov(\xi_2, \xi_1\xi_2\xi_3) &= cov(\xi_2, \xi_1)cov(\xi_2, \xi_3) + cov(\xi_2, \xi_2)cov(\xi_1, \xi_3) + cov(\xi_2, \xi_3)cov(\xi_1, \xi_2) \\ &= \phi_{21}\phi_{23} + \phi_{22}\phi_{13} + \phi_{23}\phi_{12} \end{aligned}$$

$$\begin{aligned} cov(\xi_3, \xi_1\xi_2\xi_3) &= cov(\xi_3, \xi_1)cov(\xi_2, \xi_3) + cov(\xi_3, \xi_2)cov(\xi_1, \xi_3) + cov(\xi_3, \xi_3)cov(\xi_1, \xi_2) \\ &= \phi_{31}\phi_{23} + \phi_{32}\phi_{13} + \phi_{33}\phi_{12} \end{aligned}$$

The covariance between two and three product is zero since the covariances involving five variables, for example $cov(\xi_1\xi_2, \xi_1\xi_2\xi_3) = 0$ under normality.

Following (Bohnstedt & Goldberger, 1969) and under normality, the variance of the latent product is

$$var(\xi_1\xi_2) = E^2(\xi_1)var(\xi_2) + E^2(\xi_2)var(\xi_1) + 2E(\xi_1)E(\xi_2)cov(\xi_1, \xi_2) + var(\xi_1)var(\xi_2) + cov(\xi_1, \xi_2)^2$$

And under the given assumptions it reduces to

$$\begin{aligned} var(\xi_1\xi_2) &= var(\xi_1)var(\xi_2) + cov(\xi_1, \xi_2)^2 \\ &= \phi_{11}\phi_{22} + \phi_{12}^2 \end{aligned}$$

Similarly,

$$\begin{aligned} var(\xi_1\xi_3) &= var(\xi_1)var(\xi_3) + cov(\xi_1, \xi_3)^2 \\ &= \phi_{11}\phi_{33} + \phi_{13}^2 \end{aligned}$$

$$\begin{aligned} var(\xi_2\xi_3) &= var(\xi_2)var(\xi_3) + cov(\xi_2, \xi_3)^2 \\ &= \phi_{22}\phi_{33} + \phi_{23}^2 \end{aligned}$$

For a normally distributed random variables $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$ with mean zero, the fourth and six moment is $E(\xi_1\xi_2\xi_3\xi_4) = cov(\xi_1\xi_2, \xi_3\xi_4) + cov(\xi_1\xi_3, \xi_2\xi_4) + cov(\xi_1\xi_4, \xi_2\xi_3)$ and

$E(\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6) = cov(\xi_1, \xi_2)E(\xi_3\xi_4\xi_5\xi_6) + cov(\xi_1, \xi_3)E(\xi_2\xi_4\xi_5\xi_6) + cov(\xi_1, \xi_4)E(\xi_2\xi_3\xi_4\xi_5) + cov(\xi_1, \xi_5)E(\xi_2\xi_3\xi_4\xi_6) + cov(\xi_1, \xi_6)E(\xi_2\xi_3\xi_4\xi_5)$ (Kendall & Stuart, 1958).

Then we can find $var(\xi_1\xi_2\xi_3)$. That is ;

$$\begin{aligned} var(\xi_1\xi_2\xi_3) &= E(\xi_1^2\xi_2^2\xi_3^2) - E(\xi_1\xi_2\xi_3) \\ &= E(\xi_1^2\xi_2^2\xi_3^2) \\ &= var(\xi_1)E(\xi_2^2\xi_3^2) + cov(\xi_1, \xi_2)E(\xi_1\xi_2\xi_3^2) + cov(\xi_1, \xi_2)E(\xi_1\xi_2\xi_3^2) + cov(\xi_1, \xi_3)E(\xi_1\xi_2^2\xi_3) + cov(\xi_1, \xi_3)E(\xi_1\xi_2^2\xi_3) \\ &= var(\xi_1)E(\xi_2^2\xi_3^2) + 2cov(\xi_1, \xi_2)E(\xi_1\xi_2\xi_3^2) + 2cov(\xi_1, \xi_3)E(\xi_1\xi_2^2\xi_3) \end{aligned}$$

Using the fourth moment, it can be shown that;

$$\begin{aligned} E(\xi_2^2\xi_3^2) &= var(\xi_2)var(\xi_3) + 2cov((\xi_2, \xi_3) \\ &= \phi_{22}\phi_{33} + 2\phi_{23}\phi_{23} \end{aligned}$$

$$\begin{aligned} E(\xi_1\xi_2\xi_3^2) &= cov(\xi_1, \xi_2)var(\xi_3) + 2cov(\xi_1, \xi_3)cov(\xi_2, \xi_3) \\ &= \phi_{12}\phi_{33} + 2\phi_{13}\phi_{23} \end{aligned}$$

and

$$E(\xi_1\xi_2^2\xi_3) = 2\phi_{12}\phi_{23} + \phi_{13}\phi_{22}$$

Hence,

$$var(\xi_1\xi_2\xi_3) = \phi_{11}(\phi_{22}\phi_{33} + 2\phi_{23}\phi_{23}) + 2\phi_{12}(\phi_{12}\phi_{33} + 2\phi_{13}\phi_{23}) + 2\phi_{13}(2\phi_{12}\phi_{23} + \phi_{13}\phi_{22})$$

The mean vectors for $(\xi_1\xi_2, \xi_1\xi_3, \xi_2\xi_3, \xi_1\xi_2\xi_3)$

Centering ξ_1, ξ_2, ξ_3, the mean for the two products is

$$\begin{aligned} E(\xi_1\xi_2) &= cov(\xi_1, \xi_2) = \phi_{12} \\ E(\xi_1\xi_3) &= cov(\xi_1, \xi_3) = \phi_{13} \\ E(\xi_2\xi_3) &= cov(\xi_2, \xi_3) = \phi_{23} \\ E(\xi_1\xi_2\xi_3) &= 0 \end{aligned}$$

Then the mean and variance of the endogenous latent variable in equation (1) is

$$E(\eta) = \gamma_4\phi_{12} + \gamma_5\phi_{13} + \gamma_6\phi_{23}$$

and

$$\begin{aligned} var(\eta) = \gamma_1^2 var(\xi_1) + \gamma_2^2 var(\xi_2) + \gamma_3^2 var(\xi_3) + \gamma_4^2 var(\xi_1\xi_2) + \gamma_5^2 var(\xi_1\xi_3) + \gamma_6^2 var(\xi_2\xi_3) \\ + \gamma_7^2 var(\xi_1\xi_2\xi_3) + 2[cov(\gamma_1\xi_1, \gamma_2\xi_2) + cov(\gamma_1\xi_1, \gamma_3\xi_3) \\ + cov(\gamma_2\xi_2, \gamma_3\xi_3) + cov(\gamma_1\xi_1, \gamma_7\xi_1\xi_2\xi_3) \\ + cov(\gamma_2\xi_2, \gamma_7\xi_1\xi_2\xi_3) + cov(\gamma_3\xi_3, \gamma_7\xi_1\xi_2\xi_3)] + var(\zeta) \end{aligned}$$

That is

$$\begin{aligned} var(\eta) = \sigma_\eta^2 = \gamma_1^2\phi_{11} + \gamma_2^2\phi_{22} + \gamma_3^2\phi_{33} + \gamma_4^2(\phi_{11}\phi_{22} + \phi_{12}^2) + \gamma_5^2(\phi_{11}\phi_{33} + \phi_{13}^2) \\ + \gamma_6^2(\phi_{22}\phi_{33} + \phi_{23}^2) + \gamma_7^2[\phi_{11}(\phi_{22}\phi_{33} + 2\phi_{23}\phi_{23}) \\ + 2\phi_{12}(\phi_{12}\phi_{33} + 2\phi_{13}\phi_{23}) + 2\phi_{13}(2\phi_{12}\phi_{23} \\ + \phi_{13}\phi_{22})] + 2[\gamma_1\gamma_2\phi_{12} + \gamma_1\gamma_3\phi_{13} + \gamma_2\gamma_3\phi_{23} \\ + \gamma_1\gamma_7(\phi_{11}\phi_{23} + \phi_{12}\phi_{13} + \phi_{13}\phi_{12}) \\ + \gamma_2\gamma_7(\phi_{21}\phi_{23} + \phi_{22}\phi_{13} + \phi_{23}\phi_{12}) \\ + \gamma_3\gamma_7(\phi_{31}\phi_{23} + \phi_{23}\phi_{13} + \phi_{33}\phi_{12})] + \psi \end{aligned}$$

see (Gerry Gray, 1999)

Hence the variance covarince matrix for the latent variables in the model become:

$$\Phi = \begin{pmatrix} \phi 11 & & & & & & \\ \phi_{21} & \phi_{22} & & & & & \\ \phi_{31} & \phi_{32} & \phi_{33} & & & & \\ 0 & 0 & 0 & \phi_{11}\phi_{22} + \phi_{12}^2 & & & \\ 0 & 0 & 0 & \phi_{11}\phi_{23} + \phi_{13}\phi_{12} & \phi_{11}\phi_{33} + \phi_{13}^2 & & \\ 0 & 0 & 0 & \phi_{12}\phi_{23} + \phi_{13}\phi_{22} & \phi_{12}\phi_{33} + \phi_{13}\phi_{23} & \phi_{22}\phi_{33} + \phi_{23}^2 & \\ \phi_{11}\phi_{23} + 2\phi_{12}\phi_{13} & \phi_{22}\phi_{13} + 2\phi_{23}\phi_{12} & \phi_{33}\phi_{12} + 2\phi_{32}\phi_{13} & 0 & 0 & 0 & \omega \end{pmatrix}$$

where ω stands for

$$var(\xi_1\xi_2\xi_3) = \phi_{11}(\phi_{22}\phi_{33} + 2\phi_{23}\phi_{23}) + 2\phi_{12}(\phi_{12}\phi_{33} + 2\phi_{13}\phi_{23}) + 2\phi_{13}(2\phi_{12}\phi_{23} + \phi_{13}\phi_{22})$$

2.1 Estimation Method

Let $f_i = (\eta, \xi_1, \xi_2, \xi_3)'$, $Z_i = (y_1, y_2, y_3, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)'$, $\epsilon_i = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9)'$. Then the full nonlinear structural equation model can be specified as follows

$$Z_i = \Lambda f_i + \epsilon_i \tag{2}$$

Following the notation in (Wall, 2009), let θ_m represent the measurement model parameters (i.e., parameters in Λ, Θ and θ_s denote the nonlinear structural parameters (i.e.,γ_1 to γ_7, Ψ). Where Θ is varance- covariance matrix for ϵ_i in equation

(2). Note that $\theta = ((\theta m)^{'}, (\theta_s)^{'})^{'}$.

For individual i, the joint distribution of the observed data and the latent variables conditional on the parameter vector θ can be written under the nonlinear structural equation model in equation (1) and (2) as follows.

$$\begin{aligned} P(Z_i, f_i; \theta) &= P(Z_i|f_i, \theta_m)P(f_i, \theta_s) \\ &= P(Z_i|\eta_i, \xi_{1i}, \xi_{2i}, \xi_{3i}; \theta_m)P(\eta_i, \xi_{1i}, \xi_{2i}, \xi_{3i}; \theta_s) \\ &= P(Z_i|\eta_i, \xi_{1i}, \xi_{2i}, \xi_{3i}; \theta_m)P(\eta_i|\xi_{1i}, \xi_{2i}, \xi_{3i}; \theta_s)P(\xi_{1i}, \xi_{2i}, \xi_{3i}; \theta_\xi) \end{aligned} \tag{3}$$

Where θ_ξ is describing the distribution of ξ_i. However, the latent variables are not observable. Therefore, one must integrate the latent variables out of the joint distribution to obtain the marginal density of Z_i. That is:

$P(Z_i; \theta_m, \theta_s, \theta_\xi) = \int P(Z_i|\eta_i, \xi_{1i}, \xi_{2i}, \xi_{3i}; \theta_m)P(\eta_i|\xi_{1i}, \xi_{2i}, \xi_{3i}; \theta_s)P(\xi_{1i}, \xi_{2i}, \xi_{3i}; \theta_\xi)d\xi_i$

Hence, the likelihood function is

$$L(\Theta) = \prod \int P(Z_i|\eta_i, \xi_{1i}, \xi_{2i}, \xi_{3i}; \theta_m)P(\eta_i|\xi_{1i}, \xi_{2i}, \xi_{3i}; \theta_s)P(\xi_{1i}, \xi_{2i}, \xi_{3i}; \theta_\xi)d\xi_i \tag{4}$$

Rather than directly approximate the integral in equation (4) (Klein & Moosbrugger, 2000) proposed the latent moderated structural equation method, which does not require the creation of indicators for the interaction of latent variable. LMS uses numerical integration methods for approximating the integrals in Equation (4) and uses a finite mixture of normal distributions to approximate the nonnormal distribution.Then they develop an EM algorithm to find the MLEs of this distribution(see Klein & Moosbrugger, 2000).

2.2 Simulation Design

The simulation study was designed to examine the performance of the estimation method in terms of parameter bias, root-mean-square error (RMSE), and standard error. There are twelve observed variables in the model. Nine indicators,$x_1, \cdots, x_6$, for the three latent exogenous variables, ξ_1, ξ_2 and ξ_3. Three observed indicators , $y_1, \cdots, y_3$, for the latent endogenous variable η. The observed variable covariance matrix contains ($\frac{12(12+1)}{2} = 78$) unique elements. The model contains 44 parameters to be estimated : eight of the the twelve factor loading , twelve error variances, eight factor variances , three covariance between main effects, three covariance betweeen main effects and the product of the latent variables three way interaction term ,and three covariance between two way interactions term. All variables were simulated to come the following population parameters

$$\eta = 0.3\xi_1 + 0.4\xi_2 + 0.5\xi_3 + 0.1\xi_1\xi_2 + 0.2\xi_1\xi_3 + 0.2\xi_2\xi_3 + \gamma_7\xi_1\xi_2\xi_3 + \zeta \tag{5}$$

Where ξ_1, ξ_2 and ξ_3 are standard normal variables. The values of γ_1 to γ_6 paths were chosen based on values used by Klein and Muthn (2007). The values of γ_7 varied depending on the magnitude of the interaction effect size.

The errors for the 12 indicators in the measurement model were generated with the variances of the errors chosen so that the reliability of each indicator is 0.64. These population values are chosen so that the variances of the factor indicators are one which makes the parameter values more easily interpretable. Reliability is calculated as the ratio of the variance of the factor indicator explained by the factor to the total variance of the factor indicator using the following formula,

$$\frac{\lambda^2 * \psi}{\lambda^2 * \psi + \theta}$$

where λ is the factor loading, ψ is the factor variance, and θ is the residual variance of the factor indicators. We have used indicator reliability because of it has been shown to affect power to detect interaction effect in a latent variable interaction model (Harring et al., 2012). We chose the indicator reliabilities to be equal across the 12 indicator variables. The latent factor, ξ_1, ξ_2, ξ_3, were generated under the distributional study conditions with mean 0 and variance 1.

The error term ζ was generated from a normal distribution with mean 0 and variance 0.4 which is the same value used by (Klein & Muthn, 2007).

Sample size (n=50 n=100, n=250,n=500) were used in the current study. Past simulation studies investigating interactions between two latent variables have used similar sample sizes (Klein & Muthn, 2007; Marsh et al., 2004). The loading of 0.8 was selected to represent adequate loading size and is comparable to what has been used in previous studies (Klein & Muthn, 2007; Little et al., 2006; Marsh et al., 2004).

In the first simulation study,the correlation between the two first-order latent variables ξ_1, ξ_2 and ξ_3 were set equal to the values used by (Klein & Muthn, 2007):$\phi_{11} = \phi_{22} = \phi_{33} = 1, \phi_{12} = 0.3, \phi_{13} = 0.1, \phi_{32} = 0.2$. When first-order latent variables are strongly related, the standard errors associated with the gamma estimates will become very large (Cohen et al., 2003). Thus, for the current study a larger value for $\phi_{12}, \phi_{13}, \phi_{32}$ were selected to investigate the robustness of the standard errors when the covariance of the latent exogenous factors were high.

The effect size represents the additional variance that the three way interaction effect term explains in η above and beyond that which can be explained by the first-order effects and the other three two way interaction term (Marsh et al., 2004) as shown below.

$$R^2_{\gamma 7} = \gamma_7^2 [\frac{\phi_{11}(\phi_{22}\phi_{33} + 2\phi_{23}\phi_{23}) + 2\phi_{12}(\phi_{12}\phi_{33} + 2\phi_{13}\phi_{23}) + 2\phi_{13}(2\phi_{12}\phi_{32} + \phi_{13}\phi_{22})}{\sigma^2_\eta}]$$

(Jaccard & Wan, 1995) did a review of the social science literature and found that interaction effect sizes typically accounted for 0.05 and 0.1 of the variance in the dependent variable in the case of two-way latent interaction effects. In the case of three-way interaction effects, the current study chose similar effect sizes for interaction effects in which the proportion of variance in η accounted for by the interaction effect was set equal to .0 (to investigate Type I error rates), .05, and .10 (to investigate power)

The squared multiple correlation R^2 is

$$\begin{aligned} R^2 = \gamma_1^2\phi_{11} + \gamma_2^2\phi_{22} + \gamma_3^2\phi_{33} + \gamma_4^2(\phi_{11}\phi_{22} + \phi_{12}^2) + \gamma_5^2(\phi_{11}\phi_{33} + \phi_{13}^2) \\ + \gamma_6^2(\phi_{22}\phi_{33} + \phi_{23}^2) + \gamma_7^2[\phi_{11}(\phi_{22}\phi_{33} + 2\phi_{32}) \\ + 2\phi_{12}(\phi_{12}\phi_{33} + 2\phi_{13}\phi_{23}) + 2\phi_{13}(2\phi_{12}\phi_{32} \\ + \phi_{13}\phi_{22})] + 2[\gamma_1\gamma_2\phi_{12} + \gamma_1\gamma_3\phi_{13} + \gamma_2\gamma_3\phi_{23} \\ + \gamma_1\gamma_7(\phi_{11}\phi_{23} + \phi_{12}\phi_{13} + \phi_{13}\phi_{12}) \\ + \gamma_2\gamma_7(\phi_{21}\phi_{23} + \phi_{22}\phi_{13} + \phi_{23}\phi_{12}) \\ + \gamma_3\gamma_7(\phi_{31}\phi_{23} + \phi_{32}\phi_{13} + \phi_{33}\phi_{12})]/\sigma^2_\eta \end{aligned}$$

For the interaction effect size 0,0.05,0.1 and the population variance covariance matrix defined above, squared multiple correlation is ,65.95%, 71.61%, 74.65%,72.86%,79.93%,and 83.01 respectively.

The design of study is 3(effect size)x 4 (sample size)x 2(indicator reliability)x 2(latent covariance) completely crossed factorial design resulting in 48 possible combinations (Table 1). Once the data were generated, they were analyzed with Mplus 7.4.

For each of the 48 possible condition combinations, 500 data sets were generated with Mplus version 7.4. This decision was based on the number of replications used in previous studies for latent interaction, and factors that are known to influence the number of necessary replications for Monte Carlo simulations. For instance, (Powell & Schafer, 2001) conducted a meta analysis of 219 simulation studies in structural equation modeling and reported that the number of replications used in these studies ranged from 20 to 1,000, with the median number of replications being 200. Similarly, (Bandalos, 2006) suggested that 500 replications were large for SEM Monte Carlo simulation studies. She argued that this number of replications would provide stable standard error estimates even when data were generated to come from a non-normal distribution. To check the stability of the model estimation, we have used different seeds to implement the same Monte Carlo simulations, and the model results basically remain unchanged. Thus we conclude that Monte Carlo simulation results are stable.

3. Results

3.1 Bias, Standard Error and RMSE for Main Effects' Regression Coefficients

While the bias of the γ_7 parameter was the primary interest, bias was also examined for the main effects. Bias of the main effects,γ_1, γ_2, and γ_3, were examined across different conditions (see table 2). With small sample size(i.e, n=50)and moderate reliability (reliability=0.64), this bias was very high.The resulting overestimation decreased as reliability of the indicators and sample size increased, but kept increasing as the interaction effect size for the three-way interaction term (($R^2_\gamma 7$)) and co-variance between latent exogenous variables increased. That is, bias decreased as $\phi_{12}, \phi_{13}, \phi_{23}$ decreased. In reference to the criterion of .05, the estimation method in this study (LMS) produced unbiased estimates for sample size 500 and also for n=250 with high reliability (0.84). Therefore. with moderate reliability and small sample size(i.e, n=50), the bias estimates for γ_1, γ_2, and γ_3 resulting from LMS approach cannot be trusted.

The column labeled SE-Bias in table 2 stands for standard error bias for the estimates of γ_1, γ_2, and γ_3. It was found that, this bias is very large (in absolute value) with small sample size(n=50) and moderate reliability indicators, indicating that the LMS approach underestimated standard errors. With the same sample size(n=50), the standard error bias for the estimate of first-order effects decreased as $((R^2_\gamma 7))$ and reliability increased. For all sample size under study,this bias increased as the covariance of latent exogenous increased which is consistent with result of (Cohen et al., 2003). In reference to the criterion of 0.1, the LMS produced unbiased estimates for main effects with sample size 100 and greater but underestimated standard errors because most values were negatives. However, the standard error estimates were fairly accurate when the sample size was 500.

3.2 Bias, Standard Error and RMSE for Three-way Interaction Term Regression Coefficients

Table 3 shows the latent moderated structural equations (LMS) approach parameter estimates of γ_7 in the all conditions understudy. Perhaps not too surprisingly,bias was greatest for sample size 50 coupled with moderate indicator reliability, but reduced almost by 10% when reliability of the indicators was good (e.g., reliability = 0.84). In the same conditions the standard error and root mean square error reduced by 8% and 10 % respectively for the increment of indicator reliability from 0.64 to 0.84. As anticipated, bias across conditions decreased as sample size increased, but there was a pattern indicative of diminishing returns for sample sizes larger than 500. The increment of interaction effects size resulted increased bias, standard errors and RMSE at small sample size(i.e,n=50), but showed inconstant pattern for the sample size greater than 50. Similarly, for the increase of the covariance between latent factors, the bias, standard error and RMSE reduced for small sample. However, this properties showed inconstant pattern for the others samples size in study.

3.3 Type I Error Rates and Empirical Power

As previously stated, the proportion of variance in η accounted for by the three-way interaction effect was set equal to .00 (to investigate Type I error rates), .05, and .10 (to investigate power).The empirical Type I error rates of the nominal size = .05 two-sided tests (under the null hypothesis, H0 : $\gamma_7 = 0$) when using the LMS procedure are given in Table 4. The Type I error rate was computed as the proportion of converged solutions that had a statistically significant three-way interaction effect (at the .05 level) in the simulated data when H0 was true. In addition, empirical power (probability of rejecting a false null hypothesis, H0 : $\gamma_7 = 0$) was represented by the proportion of converged solutions that have a statistically significant interaction effect in the simulated data when H0 was false (Marsh et al., 2004)and tabulated in table 5 under the 5% and 10% effect size conditions.

Type I error rates.

When the sample size was 50,100 and indicator reliability 0.64, type I error rates closest to the desired level, but increased as the covariance between latent factors and indicator reliability increased. Moreover, when sample size 100 and reliability was 0.84 the approach in this study (LMS)had very high Type I error rates, rejecting 10% of true models. In this condition(indicator reliability 0.84), the approach under study rejected the null hypothesis (with all the samples) more frequently than the nominal level would predict, except when coupled with moderate reliability. In general, in this study the type I error increased as latent factor covariance and indicators reliability increased(see table 4).

Empirical power

Empirical power is represented by the proportion of converged solutions that have a significant interaction effect in the simulated data when the population interaction effect is not equal to zero. Empirical power rates for effect size $R^2_\gamma 7 = 0.05$ and $R^2_\gamma 7 = 0.1$ were computed using an level of .05, and are shown in table 5. As anticipated, empirical power increased as the size of the effect increased from 5% to 10% across methods and conditions. That is, when medium to large three-way interaction effects exist in the population, the methods were able to detect them with a great deal of certainty for moderate sample sizes under high reliability. This was the case even when the sample size was extremely small (n = 50) and the indicators were moderate(reliability = .64). Predictably, power increased as reliability and sample size increased. Power for the LMS approach under study increased as $R^2_\gamma 7$ increased, sample size increased, and $\phi_{12}, \phi_{13}, \phi_{23}$ increased.

4. Discussion

Although many simulation studies have been conducted to study latent interaction effects in nonlinear SEM, majority of these studies has focused on two-way latent interactions and quadratic effects. In current study an examination of three-way continuous latent interaction effects was conducted via monte carlo simulation using latent moderated structural method. The simulated data were varied as a function of the size of the three-way interaction term effect, sample size, indicator reliability and the size of the relation between first-order latent variables.

The findings in the Monte Carlo simulation study indicated that, when indicator reliability was moderate and three-way interaction effect present in the generating population-generating model(i.e, $R^2_{\gamma 7} \neq 0.00$), the LMS method led to biased estimate of interaction effect. As with past simulation studies in two-way interaction, indicator variable reliability tended to have the greatest impact on the ability of the LMS to accurately and precisely estimate the three-way interaction effect with size of the relation between the first-order latent variable exerting less influence. Moreover,Parameter estimates for the LMS approach became less biased as the size of the interaction effect and the correlation between the first-order latent variables decreased. We observed that this result for three-way interaction was similar to previous findings of two -way interaction in which the LMS approach was found to result in unbiased estimates of the interaction effect across all sizes of the interaction effect (Klein & Moosbrugger, 2000; Klein & Muthn, 2007).

This finding suggest that the method appeared to control Type I error fairly by reducing the size of the relation between first-order latent variables. Hence moderate indicator reliability, and small sample sizes appear to have the greatest negative impact on the estimation accuracy, precision, and deflation of standard errors of the three-way interaction parameter. Because the method investigated here performed poorly under these circumstances, if data exhibit these characteristics in practice, statistical conclusions should be made cautiously.

5. Conclusion and Recommendations

It is the conclusion of the authors that the latent moderated structural approach can be used to study three-way continuous latent interaction in nonlinear structural equation modeling using Mplus software. The approach had no model convergence problems across the conditions in the study and did not produced unrealistic estimates. However, because of the complexity of the model, it took along time to get monte carlo simulation output.

In the conditions considered in the current study, the method led to the least biased estimates of the interaction effect, and accurate standard error estimates, particularly when the sample size was 250 or greater and the indicator reliability was high. Additionally, the latent moderated structural approach accurately estimated first-order effects provided that the sample size was 250 or greater. For the small size(i.e, n=50), the bias for interaction effects and exogenous regression coefficients was high. But for the same sample size, the method had less bias (approximately less than 2%) in estimating the exogenous covariances. This bias increased as the interaction effect size ($R^2_{\gamma 7}$) increased and decrease when sample size increased.

Type I error rates were close to the desired alpha level, particularly when the sample size was 250 or greater. Comparing to other conditions in the study, when indicator reliability were low and the sample size was 50, the method had low power to detect true three-way interaction effects and a sample size of at least 250 was necessary to have acceptable power(greater than 0.8).

Based on these findings,high indicator reliability and a sample size of 250 or more is recommended for use with the latent moderated structural method, although it performs fairly well with sample sizes of 100. It also recommended that,under small sample (i.e,n=50),the method provided sufficient power to detect the three-way interaction effects when high indicator reliability and the covariance of the exogenous latent variable was increased.

Acknowledgment

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List of Tables

Table 1. Summary of Manipulated Features

Factor	1	2	3	4			
Sample size	50	100	250	500			
Indicator reliability	0.64	0.84					
Effect size(($R^2_\gamma 7$))	0.00	0.05	0.10				
$\phi_{12}, \phi_{13}, \phi_{23}$	0.3	0.1	0.2		0.6	0.4	0.5
Distribution of ξ_1, ξ_2, ξ_3	Normal						
Factor loading	0.8						
Estimation method	LMS						

Table 2. Parameter Estimates for the first-order main effects γ_1 to γ_3 when $R^2_{\gamma 7} = 0.05, 0.1$ and with different covarince of exogenous latent variables

	$\phi_{12} = 0.3$ $\phi_{13} = 0.1$ $\phi_{23} = 0.2$						$\phi_{12} = 0.6$ $\phi_{13} = 0.4$ $\phi_{23} = 0.5$			
			$R^2_{\gamma 7} = 0.05$		$R^2_{\gamma 7} = 0.1$		$R^2_{\gamma 7} = 0.05$		$R^2_{\gamma 7} = 0.1$	
Rel.	N	Par.	Bias	SE-Bias	Bias	SE-Bias	Bias	SE-Bias	Bias	SE-Bias
0.64	50	γ_1	33.32	-0.929	55.146	-0.90	95.33	-0.968	142.297	-0.969
		γ_2	44.115	-0.944	48.110	-0.932	3.482	-0.958	-11.977	-0.962
		γ_3	40.259	-0.948	56.899	-0.940	58.602	-0.943	78.215	-0.939
	100	γ_1	0.182	-0.049	0.183	-0.047	0.207	-0.051	0.212	-0.045
		γ_2	0.141	-0.029	0.143	-0.028	0.148	-0.007	0.155	-0.008
		γ_3	0.172	-0.009	0.176	-0.026	0.198	-0.026	0.2	-0.056
	250	γ_1	0.069	-0.035	0.069	-0.027	0.081	-0.029	0.083	-0.026
		γ_2	0.051	-0.055	0.049	-0.058	0.054	-0.059	0.054	-0.063
		γ_3	0.062	-0.015	0.062	-0.017	0.007	-0.029	0.075	-0.025
	500	γ_1	0.039	-0.024	0.039	-0.025	0.045	-0.033	0.046	-0.037
		γ_2	0.023	-0.017	0.024	-0.016	0.022	-0.023	0.023	-0.024
		γ_3	0.031	-0.008	0.030	-0.009	0.036	-0.019	0.036	-0.018
0.84	50	γ_1	0.193	-0.126	0.197	-0.126	0.200	-0.138	0.209	-0.142
		γ_2	0.171	-0.109	0.174	-0.106	0.163	-0.109	0.169	-0.108
		γ_3	0.204	-0.074	0.209	-0.015	0.209	-0.089	0.217	-0.097
	100	γ_1	0.087	-0.052	0.088	-0.049	0.092	-0.057	0.094	-0.058
		γ_2	0.076	-0.097	0.077	-0.096	0.073	-0.066	0.075	-0.067
		γ_3	0.084	-0.006	0.085	-0.005	0.089	-0.024	0.092	-0.026
	250	γ_1	0.038	-0.045	0.038	-0.043	0.042	-0.036	0.043	-0.036
		γ_2	0.031	-0.053	0.031	-0.057	0.031	-0.055	0.032	-0.056
		γ_3	0.034	-0.008	0.034	-0.010	0.038	-0.015	0.039	-0.015
	500	γ_1	0.023	-0.035	0.024	-0.035	0.025	-0.029	0.026	-0.030
		γ_2	0.014	-0.027	0.014	-0.027	0.013	-0.034	0.014	-0.032
		γ_3	0.018	0.011	0.018	0.011	0.021	0.000	0.021	0.000

Table 3. Parameter Estimates for nonlinear effects γ_7 across the study conditions

	$\phi_{12} = 0.3$ $\phi_{13} = 0.1$ $\phi_{23} = 0.2$						$\phi_{12} = 0.6$ $\phi_{13} = 0.4$ $\phi_{23} = 0.5$		
Rel.	$R^2_{\gamma 7}$	N	Parameter	Bias	SE-Bias	RMSE	Bias	SE-Bias	RMSE
0.64	0.0	50	γ_7	2.0698	-0.924	84.206	-0.471	-0.853	53.911
		100	γ_7	-0.0154	-0.089	0.150	-0.014	-0.120	0.117
		250	γ_7	-0.007	-0.029	0.069	-0.007	-0.079	0.055
		500	γ_7	-0.0003	-0.026	0.047	-0.003	-0.092	0.037
	0.05	50	γ_7	58.87	-0.96	138.74	35.704	-0.901	62.404
		100	γ_7	-0.025	-0.158	0.173	-0.004	-0.208	0.152
		250	γ_7	-0.033	-0.068	0.079	-0.029	-0.096	0.069
		500	γ_7	0.002	-0.019	0.053	-0.006	-0.060	0.045
	0.1	50	γ_7	73.838	-0.965	249.624	46.579	-0.925	107.664
		100	γ_7	0.004	-0.186	0.202	0.036	-0.323	0.216
		250	γ_7	-0.02	-0.079	0.092	-0.013	-0.102	0.081
		500	γ_7	0.003	-0.017	0.059	0.000	-0.032	0.052
0.84	0.0	50	γ_7	0.003	-0.171	0.182	0.002	-0.189	0.151
		100	γ_7	-0.004	-0.187	0.104	-0.003	-0.158	0.081
		250	γ_7	-0.003	-0.080	0.053	-0.004	0.088	0.041
		500	γ_7	0.000	-0.065	0.036	-0.0016	-0.077	0.028
	0.05	50	γ_7	0.088	-0.174	0.198	0.102	-0.179	0.171
		100	γ_7	-0.005	-0.176	0.114	0.000	-0.126	0.093
		250	γ_7	-0.012	-0.089	0.060	-0.016	-0.080	0.051
		500	γ_7	0.004	-0.042	0.040	-0.002	-0.041	0.035
	0.1	50	γ_7	0.085	-0.175	0.214	0.102	-0.176	0.193
		100	γ_7	0.002	-0.169	0.125	0.007	-0.114	0.106
		250	γ_7	-0.006	-0.091	0.068	-0.008	-0.075	0.060
		500	γ_7	0.005	-0.029	0.045	0.002	-0.020	0.040

Table 4. Type I error rates for $R^2_{\gamma 7} = 0$ and with different covarince of exogenous latent variables and reliability of the latent indicators

	$\phi_{12} = 0.3$ $\phi_{13} = 0.1$ $\phi_{23} = 0.2$		$\phi_{12} = 0.6$ $\phi_{13} = 0.4$ $\phi_{23} = 0.5$
Reliability	N	Type I error	type I error
0.64	50	0.029	0.034
	100	0.064	0.068
	250	0.070	0.070
	500	0.066	0.082
0.84	50	0.080	0.092
	100	0.100	0.108
	250	0.082	0.076
	500	0.080	0.080

Table 5. Type I error rates for $R^2_{\gamma 7} = 0$ and with different covarince of exogenous latent variables and reliability of the latent indicators

		$\phi_{12} = 0.3$ $\phi_{13} = 0.1$ $\phi_{23} = 0.2$		$\phi_{12} = 0.6$ $\phi_{13} = 0.4$ $\phi_{23} = 0.5$	
		Power		Power	
Reliability	N	$R^2_{\gamma 7} = 0.05$	$R^2_{\gamma 7} = 0.1$	$R^2_{\gamma 7} = 0.05$	$R^2_{\gamma 7} = 0.1$
0.64	50	0.099	0.162	0.151	0.238
	100	0.344	0.526	0.468	0.666
	250	0.746	0.942	0.884	0.982
	500	0.972	1.000	0.996	1.000
0.84	50	0.302	0.507	0.414	0.612
	100	0.602	0.832	0.704	0.908
	250	0.940	0.994	0.982	1.000
	500	1.000	1.000	1.000	1.000

References

Bandalos, D. L. (2006). The use of Monte Carlo studies in structural equation modeling research. In G. R. Hancock & R. O. Mueller (Eds.), *Structural equation modeling: A second course (pp. 385-426)*. Greenwich, CT: Information Age Publishing

Bohnstedt, G. W., & Goldberger, A. S. (1969). On the exact covariance of products of random variables. *Journal of the American Statistical Association, 64*, 325-328.

Cohen, J., Cohen, P., West, S. G., & Aiken, L. S. (2003). *Applied multiple regression/correlation analysis for the behavioral sciences (3rd ed.)*. Mahwah, NJ: Lawrence Erlbaum Associates.

Gerry Gray (1999). Covariances in Multiplicative Estimates. *Transactions of the American Fisheries Society,128*(3), 475-482. http://dx.doi.org/10.1577/1548-8659(1999)128<0475:CIME>2.0.CO;2

Harring, J. R., Weiss, B. A., Hsu, J. C. (2012). A comparison of methods for estimating quadratic effects in nonlinear structural equation models. *Psychological Methods, 17*(2), 193-214. http://dx.doi.org/10.1037/a0027539

Hayduk, L. A. (1987). Structural equation modeling with LISREL. *Essentials and advances*. Johns Hopkins University; Baltimore, MD.

Jaccard, J., & Wan, C. K. (1995). Measurement error in the analysis of interaction effects between continuous predictors using multiple regression: Multiple indicator and structural equation approaches. *Psychological Bulletin, 117*, 348-357. http://dx.doi.org/10.1037/0033-2909.117.2.348

Kendall, M. G., & Stuart, A. (1958). *The advanced Theory of Statistics(vol.1)*. London:Griffin

Kenny, D. A., & Judd, C. M. (1984). Estimating the nonlinear and interactive effects of latent variables. *Psychological Bulletin, 96*, 201-210. http://dx.doi.org/10.1037/0033-2909.96.1.201

Klein, A. G., & Moosbrugger, H. (2000). Maximum likelihood estimation of latent interaction effects with the LMS method. *Psychometrika, 65*, 457-474. http://dx.doi.org/10.1007/BF02296338

Klein, A. G., & Muthn, B. (2007). Quasi-maximum likelihood estimation of structural equation models with multiple interaction and quadratic effects. *Multivariate Behavioral Research, 42*, 647-673. http://dx.doi.org/10.1080/00273170701710205

Little, T. D., Bovaird, J. A., & Widaman, K. F. (2006). On the merits of orthogonalizing powered and product terms: Implications for modeling interactions among latent variables. *Structural Equation Modeling: A Multidisciplinary Journal, 13*, 497-519. http://dx.doi.org/10.1207/s15328007sem1304_1

Marsh, H. W., Wen, Z., & Hau, K. T. (2004). Structural equation models of latent interactions: Evaluation of alternative estimation strategies and indicator construction. *Psychological Methods, 9*, 275-300. http://dx.doi.org/10.1037/1082-989X.9.3.275

Powell, D. A., & Schafer, W. D. (2001). The robustness of the likelihood ratio chi?square test for structural equation models: A meta-analysis. *Journal of Educational and Behavioral Statistics, 26*, 105-132. http://dx.doi.org/10.3102/10769986026001105

Weiss, B. A. (2010). A comparison of methods for testing interaction effects in structural equation modeling. University of Maryland; College Park. (Unpublished doctoral dissertation)

Appendix

Simulation code using Mplus version 7.4

```
TITLE: Monte Carlo simulation for three-way continuous latent interaction
    MONTECARLO: NAMES=x1-x9 y1-y3;
    NOBSERVATIONS = 250; ! for sample size 250
    NREPS = 500;
    SEED = 12345;
    ANALYSIS: ESTIMATOR = MLR;
    TYPE = RANDOM;
    ALGORITHM = INTEGRATION;
    MODEL POPULATION:
    [x1 − x9@0y1 − y3@0];
    xi1 BY x1-x3@0.8;
    xi2 BY x4-x6@0.8;
    xi3 BY x7-x9@0.8;
    eta BY y1-y3@0.8;
    xi1@1;
    xi2@1;
    xi3@1;
    eta@0.4;! we set the var(zeta)=0.4
    D — xi1 XWITH xi2;
    E — xi1 XWITH xi3;
    F — xi2 XWITH xi3;
    G — xi1 XWITH F;
    eta ON xi1@0.3 xi2@0.4 xi3@0.5 D@0.1 E@0.2 F@0.2 G@0.3;

x1-x9@0.36; y1-y3@0.36;
    xi1 WITH xi2@0.3 xi3@0.1;! for the first variance covariance condition
    xi2 WITH xi3@0.2;
    MODEL:
    [x1 − x9 ∗ 0y1 − y3 ∗ 0];
    xi1 BY x1-x3*0.8;
    xi2 BY x4-x6*0.8;
    xi3 BY x7-x9*0.8;
    eta BY y1-y3*0.8;
    xi1@1;
    xi2@1;
    xi3@1;
    eta@0.4;
    D — xi1 XWITH xi2;
    E — xi1 XWITH xi3;
    F — xi2 XWITH xi3;
    G — xi1 XWITH F;
    eta ON xi1*0.3 xi2*0.4 xi3*0.5 D*0.1 E*0.2 F*0.2 G*0.3;

x1-x9*0.36; y1-y3*0.36;
    xi1 WITH xi2*0.3 xi3*0.1;
    xi2 WITH xi3*0.2;
    OUTPUT: TECH9;
```