

The Burr XII Negative Binomial Distribution with Applications to Lifetime Data

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Abstract

A five-parameter model, called the Burr XII negative binomial distribution, is defined and studied. The new model contains as special cases some important lifetime distributions discussed in the literature, such as the log-logistic, Weibull, Pareto type II and Burr XII distributions, among several others. We derive the ordinary and incomplete moments, generating and quantile functions, mean deviations, reliability and two types of entropy. The order statistics and their moments are investigated. The method of maximum likelihood is proposed for estimating the model parameters. We obtain the observed information matrix. An application to real data demonstrates that the new distribution can provide a better fit than other classical lifetime models.

Keywords: Negative Binomial distribution, Burr XII distribution, Maximum likelihood, observed information matrix

1. Introduction

The negative binomial distribution has been widely used in mixing procedures of distributions. Several new models have been proposed and applied in survival analysis. Percontini *et al.* (2013) pioneered a family of univariate distributions generated by compounding the negative binomial distribution with any continuous model. For any baseline cumulative distribution function (cdf) $G(x)$, and $x \in \mathbb{R}$, they defined the *G-Negative Binomial* (G-NB) family of distributions with probability density function (pdf) $f(x)$ and cdf $F(x)$ given by

$$f(x) = \frac{s\beta}{[(1-\beta)^{-s} - 1]} g(x) \{1 - \beta[1 - G(x)]\}^{-s-1} \quad (1)$$

and

$$F(x) = \frac{(1-\beta)^{-s} - \{1 - \beta[1 - G(x)]\}^{-s}}{[(1-\beta)^{-s} - 1]}, \quad (2)$$

respectively. The corresponding hazard rate function (hrf) is

$$h(x) = \frac{s\beta g(x) \{1 - \beta[1 - G(x)]\}^{-s-1}}{\{1 - \beta[1 - G(x)]\}^{-s} - 1}.$$

The G-NB family has the same parameters of the G distribution plus two additional shape parameters $s > 0$ and $\beta \in (0, 1)$. If X is a random variable having pdf (1), we write $X \sim \text{G-NB}(s, \beta)$. This generalization is obtained by increasing the number of parameters compared to the G model, this increase being the price to pay for adding more flexibility to the generated distribution. A first positive point of the G-NB model is that it includes the G distribution as a sub-model when $s = 1$ and $\beta \rightarrow 0$. A second one is that it includes as special cases important lifetime models published in recent years.

Zimmer *et al.* (1998) introduced the three-parameter Burr XII distribution. This distribution, having logistic and Weibull as sub-models, is a very popular distribution for modeling lifetime data and phenomenon with monotone failure rates. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, it does not provide a reasonable parametric fit for modeling phenomena with non-monotone failure rates such as the bathtub-shaped and the unimodal failure rates that are common in reliability and biological studies. Such bathtub hazard curves have nearly flat middle portions and the corresponding densities have a positive anti-mode. Unimodal failure rates can be observed in the course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

In this paper, we use Percontini *et al.* (2013)'s generator to define a new model, so-called the *Burr XII negative binomial* (BXIINB) distribution, which generalizes the Burr XII (BXII) distribution.

The cdf and pdf of the BXII distribution are given by

$$G(x; c, k, a) = 1 - \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k} \quad \text{and} \quad g(x; c, k, a) = cka^{-c} x^{c-1} \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k-1}, \quad (3)$$

respectively, where $k > 0$ and $c > 0$ are shape parameters and $a > 0$ is a scale parameter. The n th moment about zero of the BXII distribution (for $n < kc$) is given by

$$\mu'_n = a^n k B(k - nc^{-1}, nc^{-1} + 1). \quad (4)$$

Some other mathematical properties of this distribution are discussed by Silva *et al.* (2008) and Paranaíba *et al.* (2011).

The BXII cdf has closed-form, thus it simplifies the computation of the percentiles and the likelihood function for censored data. This distribution has algebraic tails which are effective for modeling failures that occur with lesser frequency than with those models based on exponential tails. Hence, it represents a good distribution for modeling failure time data (Zimmer *et al.*, 1998). Shao (2004a) discussed maximum likelihood estimation for the BXII distribution. Shao *et al.* (2004b) studied models for extremes using the BXII distribution with application to flood frequency analysis. According to Soliman (2005), this distribution covers the curve shape characteristics for a large number of distributions. The versatility and flexibility of the BXII distribution turns it quite attractive as a tentative model for fitting data real whose underlying distribution is unknown. Wu *et al.* (2007) studied the estimation problems for its model parameters based on progressive type II censoring with random removals, where the number of units removed at each failure time has a discrete uniform distribution. Silva *et al.* (2008) proposed a location-scale regression model based on this distribution, refereed to as the log-Burr XII regression model, which is a feasible alternative to the log-logistic regression model. Silva *et al.* (2011) defined a residual for the log-Burr XII regression model whose empirical distribution is close to normality. More recently, Paranaíba *et al.* (2011, 2012) studied the beta Burr XII and Kumaraswamy Burr XII distributions.

We define a new five-parameter BXIINB distribution inserting equations (3) into (1) and (2). Then, the pdf and cdf of the generated distribution (for $x > 0$) are given by

$$f(x) = \frac{s\beta c k a^{-c}}{[(1-\beta)^{-s} - 1]} x^{c-1} \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k-1} \left\{1 - \beta \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k}\right\}^{-s-1} \quad (5)$$

and

$$F(x) = \frac{(1-\beta)^{-s} - \left\{1 - \beta \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k}\right\}^{-s}}{[(1-\beta)^{-s} - 1]}, \quad (6)$$

respectively, where $a > 0$ is a scale parameter and $s > 0$, $c > 0$, $k > 0$ and $\beta \in (0, 1)$ are shape parameters. Henceforth, the random variable X having density function (5) is denoted by $X \sim \text{BXIINB}(s, \beta, c, k, a)$.

Some possible shapes of the density function (5) for selected parameter values, including well-known distributions, are displayed in Figure 1.

The BXIINB hazard rate function (hrf) (for $x > 0$) is given by

$$h(x) = \frac{s\beta c k a^{-c} x^{c-1} \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k-1} \left\{1 - \beta \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k}\right\}^{-s-1}}{-1 + \left\{1 - \beta \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k}\right\}^{-s}}. \quad (7)$$

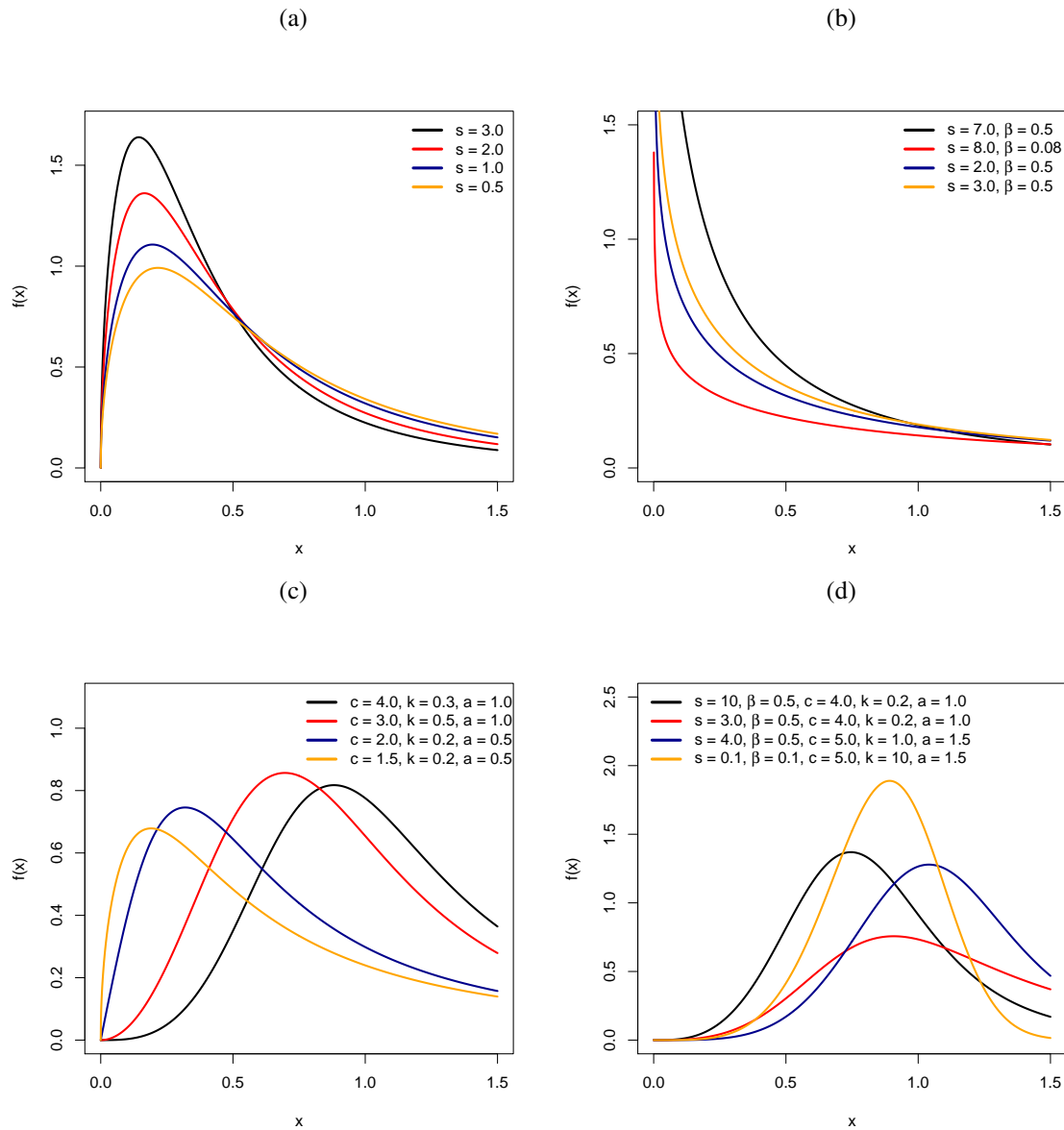


Figure 1. Plots of the BXIINB density function for some parameter values. (a) For $\beta = 0.5$, $c = 1.5$, $k = 1.2$ and $a = 1$. (b) For $c = 0.8$, $k = 0.5$ and $a = 2$. (c) For $s = 2$ and $\beta = 0.5$.

Special BXIINB hazard rate shapes are displayed in Figure 2 for some parameter values. This hrf can be increasing, decreasing, constant and unimodal.

Some distributions arise as special cases of the BXIINB model:

- When $s = 1$ and $\beta \rightarrow 0$, the BXII distribution is a limiting case of (5);
- For $s = 1$, $\lambda \rightarrow 0$, $a = m^{-1}$ and $k = 1$, it becomes the *log-logistic* (LL) distribution, whose survival function is $S(t; m, c) = [1 + (tm)^c]^{-1}$;
- For $s = 1$, $\lambda \rightarrow 0$ and $c = 1$, it reduces to the *Pareto type II* (PII) distribution;
- If $k \rightarrow \infty$ in addition to $s = 1$, $\lambda \rightarrow 0$, it becomes the *Weibull* distribution;
- When $s = 1$ in (6), we obtain the *Burr XII geometric* (BXIIG) distribution;
- For $s \rightarrow \infty$ and $\beta = \lambda/s$, equation (6) becomes the *Burr XII Poisson* (BXIIP) distribution.

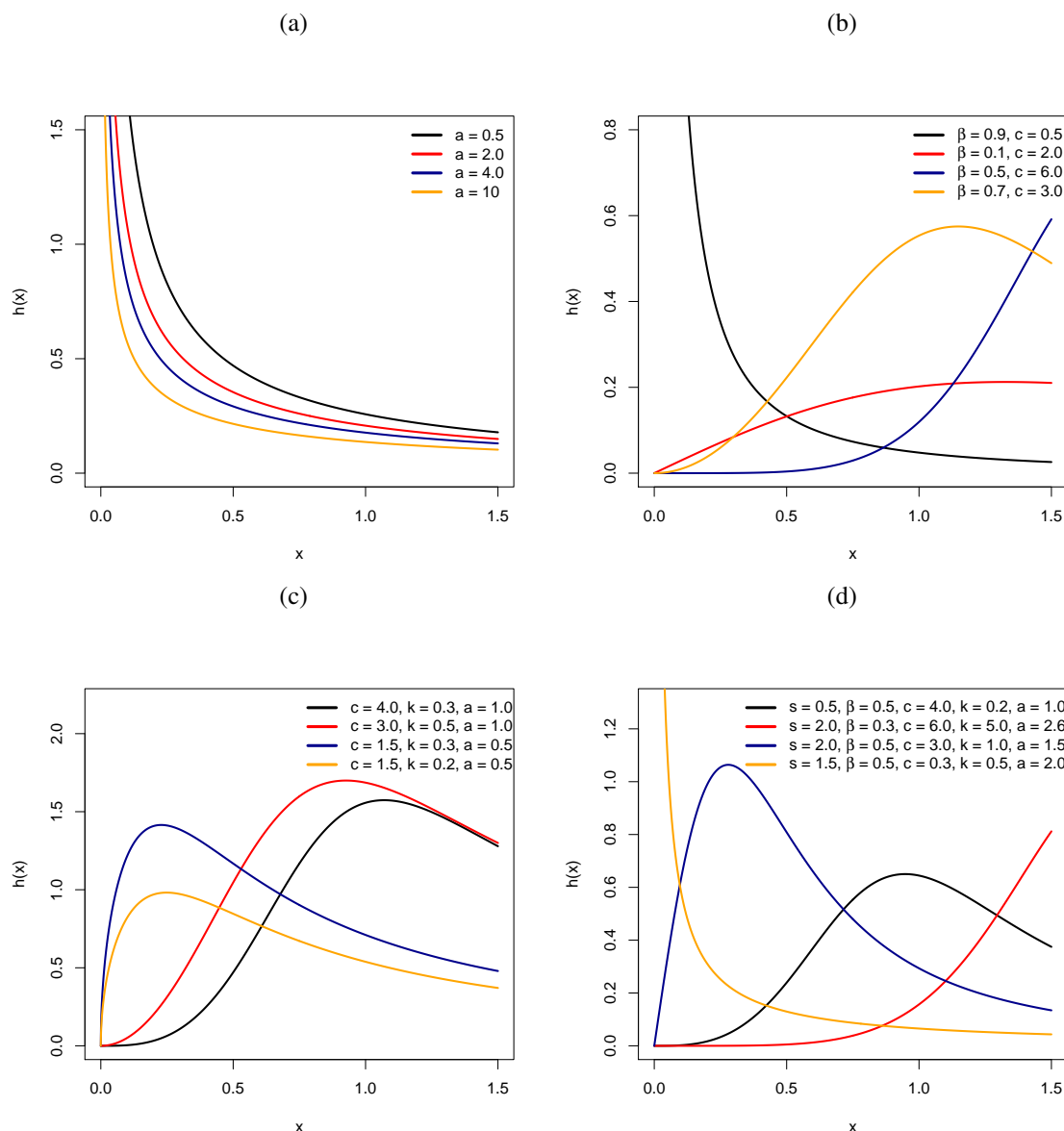


Figure 2. Plots of the BXIINB hrf for some parameter values. (a) For $s = 2$, $\beta = 0.5$, $c = 0.5$ and $k = 0.5$. (b) For $s = 2$, $k = 0.5$ and $a = 2$. (c) For $s = 3$ and $\beta = 0.5$.

The BXIINB distribution is well-motivated for industrial applications and biological studies. As a first example, consider that the number, say N , of carcinogenic cells for an individual left active after an initial treatment follows a random variable Z with zero truncated negative binomial distribution and let X_i be the time spent for the i th carcinogenic cell to produce a detectable cancer mass, for $i \geq 1$. If $\{X_1, \dots, X_N\}$ is a sequence of independent and identically distributed (iid) random variables independent of N having the BXII distribution, then the time to relapse of cancer of a susceptible individual can be modeled by the BXIINB distribution. Another example considers that the failure of a device occurs due to the presence of an unknown number N of initial defects of the same kind, which can be identifiable only after causing failure and are repaired perfectly. Define by X_i the time to the failure of the device due to the i th defect, for $i \geq 1$. If we assume that the X_i 's are iid random variables independent of N , which follow a BXII distribution, then the time to the first failure is appropriately modeled by the BXIINB distribution. In fact, for reliability studies, the random variable $X = \min\{X_i\}_{i=1}^N$ can be used in serial systems with identical components, which appear in many industrial applications and biological organisms.

In this paper, we study some mathematical properties of the BXIINB model and illustrate its potentiality. The

paper is organized as follows. In Section 2, we demonstrate that the pdf of X can be expressed as a mixture of BXII densities. The quantile function is obtained in Section 3. Explicit expressions for the ordinary and incomplete moments, generating function and mean deviations are derived in Sections 4, 5 and 6, respectively. The Rényi and Shannon entropies and the reliability are determined in Sections 7 and 8, respectively. In Section 9, we investigate the order statistics. Maximum likelihood estimation of the model parameters is performed and the observed information matrix is derived in Section 10. In Section 11, we illustrate the potentiality of the BXIINB distribution by means of a real data set. Finally, some conclusions are addressed in Section 12.

2. Useful Expansion

For any real a and $|z| < 1$, we have the binomial expansion

$$(1-z)^{-a} = \sum_{k=0}^{\infty} (a)_k \frac{z^k}{k!}, \quad (8)$$

where $(a)_0 = 1$ and $(a)_k = a(a+1)(a+2)\dots(a+k-1) = \Gamma(a+k)/\Gamma(a)$ is the ascending factorial. Using the power series (8), we can write (5) as

$$f(x) = \sum_{i=0}^{\infty} \frac{c k s \beta^{i+1} (s+1)_i}{a^c [(1-\beta)^{-s} - 1] i!} x^{c-1} \left[1 + \left(\frac{x}{a} \right)^c \right]^{-k(1+i)-1},$$

which gives an infinite linear combination

$$f(x) = \sum_{i=0}^{\infty} \omega_i(s, \beta) g(x; c, (i+1)k, a), \quad (9)$$

where

$$\omega_i(s, \beta) = \frac{s \beta^{i+1} (s+1)_i}{[(1-\beta)^{-s} - 1] (i+1)!} \quad (10)$$

and $g(x; c, (i+1)k, a)$ denotes the BXII density function with parameters $c, (i+1)k$ and a . We can verify that $\sum_{i=0}^{\infty} \omega_i(s, \beta) = 1$. So, equation (9) reveals that the BXIINB density can be expressed as a linear combination of BXII densities. By integrating (9), we obtain

$$F(x) = \sum_{i=0}^{\infty} \omega_i(s, \beta) G(x; c, (i+1)k, a), \quad (11)$$

where $G(x; c, (i+1)k, a)$ denotes the BXII cumulative distribution with parameters $c, (i+1)k$ and a .

3. Quantile Function

Percontini *et al.* (2013) demonstrated that the G-NB quantile function can be expressed (for $0 < u < 1$) as

$$Q(u) = Q_G \left\{ 1 - \frac{1}{\beta} \left[1 - (1-\beta)(1-u[1-(1-\beta)^s])^{-\frac{1}{s}} \right] \right\},$$

where $Q_G(u)$ is the baseline quantile function. The BXII quantile function is $Q_{BXII}(u) = a \left[(1-u)^{-1/k} - 1 \right]^{1/c}$. Then, the BXIINB quantile function reduces to

$$Q(u) = a \left\{ \left[\frac{1}{\beta} \left\{ 1 - \bar{\beta} \left[1 - u(1-\bar{\beta}^s) \right]^{-\frac{1}{s}} \right\} \right]^{-1/k} - 1 \right\}^{1/c}, \quad (12)$$

where $\bar{\beta} = 1 - \beta$.

Quantiles of interest can be obtained from (12) by substituting appropriate values for u . In particular, the median of X comes when $u = 0.5$.

The motivation for using quantile-based measures is because of the non-existence of classical kurtosis for many generalized distributions. The Bowley's skewness is based on quantiles (Kenney and Keeping, 1962):

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}$$

and the Moors kurtosis (Moors, 1988) is based on octiles:

$$M = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}.$$

Plots of the skewness and kurtosis for the BXIINB distribution, for some choices of β , c , k and a as functions of s , and for some choices of s , c , k and a as functions of β are displayed in Figure 3. These plots indicate that there is a great flexibility of the skewness and kurtosis curves for this distribution.

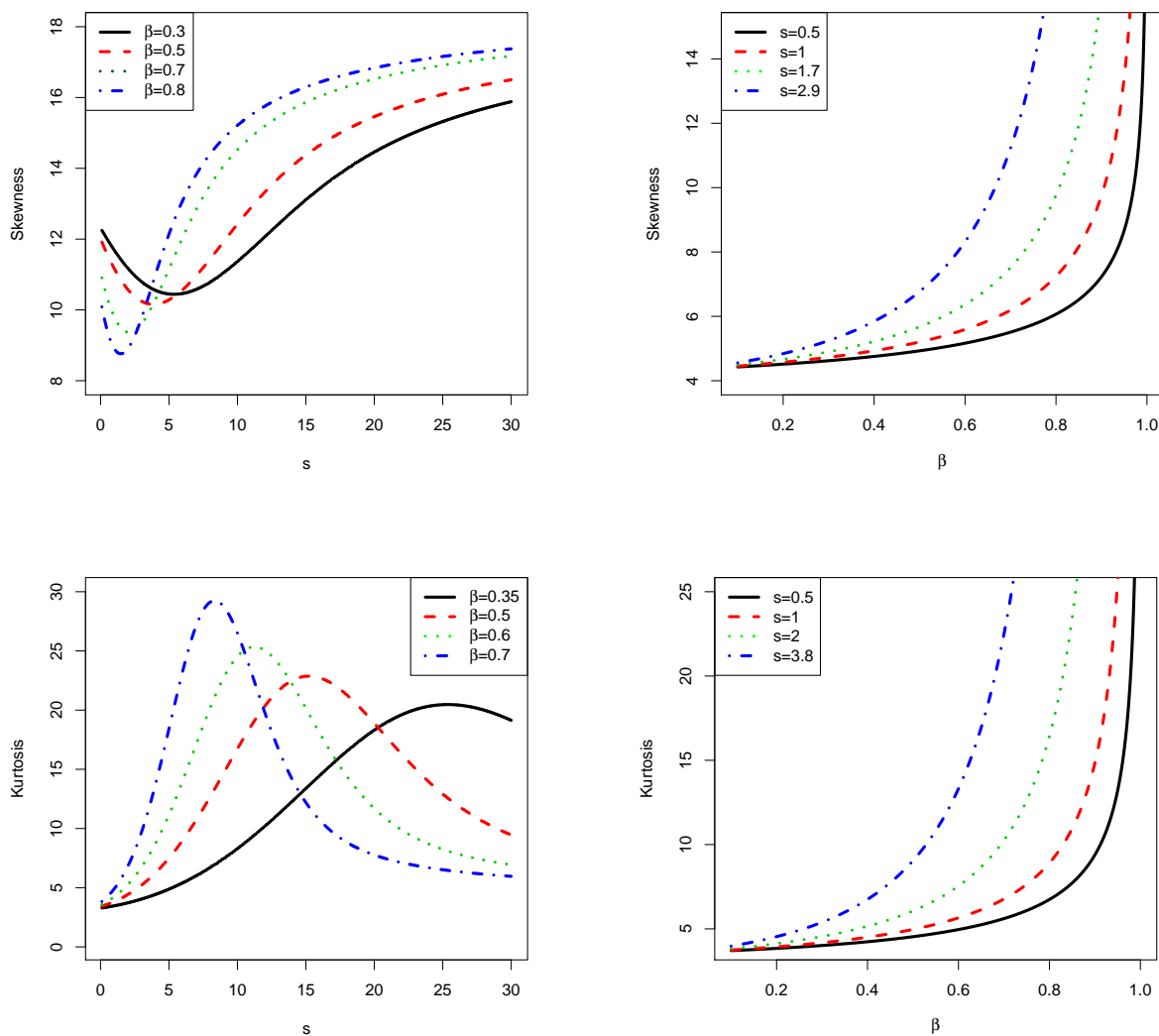


Figure 3. Plots of the BXIINB skewness and kurtosis as functions of s for selected values of β and as functions of β for selected values of s .

4. Moments

The n th ordinary moment of X can be obtained from equation (9) as

$$E(X^n) = \sum_{i=0}^{\infty} \omega_i(s, \beta) E(Y_{i+1}^n),$$

where Y_{i+1} denotes the BXII random variable with density function $g(x; c, (i+1)k, a)$. Thus, using (4) (for $n < kc$), we obtain

$$E(X^n) = \sum_{i=0}^{\infty} \omega_i(s, \beta) a^n (i+1)k B\left[(i+1)k - \frac{n}{c}, 1 + \frac{n}{c}\right]. \quad (13)$$

The incomplete moments play an important role for measuring inequality, for example, income quantiles and Lorenz and Bonferroni curves. These curves depend on the first incomplete moment of the distribution. The n th incomplete moment of X follows from equation (9) as

$$\begin{aligned} m_n(y) &= E(X^n | X < y) = \sum_{i=0}^{\infty} \omega_i(s, \beta) \int_0^y x^n g(x; c, (i+1)k, a) dx \\ &= kc \sum_{i=0}^{\infty} (i+1) \omega_i(s, \beta) \int_0^y x^n \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k(i+1)-1} a^{-c} x^{c-1} dx. \end{aligned}$$

Setting $u = a^{-c} x^c$, the last equation becomes

$$\begin{aligned} m_n(y) &= k \sum_{i=0}^{\infty} (i+1) \omega_i(s, \beta) \int_0^{a^{-c} y^c} a^n u^{\frac{n}{c}} (1+u)^{-k(i+1)-1} du \\ &= k \sum_{i=0}^{\infty} \frac{(i+1)c \omega_i(s, \beta)}{a^c (c+n)} y^{c+n} {}_2F_1\left[1 + (i+1)k, 1 + \frac{n}{c}, 2 + \frac{n}{c}; -a^{-c} y^c\right], \end{aligned} \quad (14)$$

where ${}_2F_1$ is the hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}$$

and $(a)_k = a(a+1) \dots (a+k-1)$ denotes the ascending factorial.

5. Generating Function

A first representation for the moment generating function (mgf) $M(t)$ of X can be obtained from equation (9) as an infinite weighted sum

$$M(t) = \sum_{i=0}^{\infty} \omega_i(s, \beta) M_{i+1}(t), \quad (15)$$

where $M_{i+1}(t)$ is the mgf of the BXII($c, (i+1)k, a$) distribution and $\omega_i(s, \beta)$ is defined by (10). We provide a simple representation for the mgf $M_{BXII}(t)$ of the BXII(c, k, a) distribution. We can write for $t < 0$

$$M_{BXII}(t) = ck \int_0^{\infty} e^{ayt} y^{c-1} (1+y^c)^{-(k+1)} dy.$$

Now, we require the Meijer G -function defined by

$$G_{p,q}^{m,n} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j + t) \prod_{j=1}^n \Gamma(1 - a_j - t)}{\prod_{j=n+1}^p \Gamma(a_j + t) \prod_{j=m+1}^q \Gamma(1 - b_j - t)} x^{-t} dt,$$

where $i = \sqrt{-1}$ is the complex unit and L denotes an integration path (Gradshteyn and Ryzhik, 2000, Section 9.3). The Meijer G -function contains as particular cases many integrals with elementary and special functions (Prudnikov *et al.*, 1986).

We now assume that $c = m/k$, where m and k are positive integers. This condition is not restrictive since every positive real number can be approximated by a rational number. Using the integral (21) given in Appendix A, we conclude (for $t < 0$) that

$$M_{BXII}(t) = mI\left(-at, \frac{m}{k} - 1, \frac{m}{k}, -k - 1\right).$$

Hence, from equation (15), the mgf of the BXIINB(s, β, c, k, a) distribution (for $t < 0$) follows as

$$M(t) = m \sum_{i=0}^{\infty} w_i(s, \beta) I\left(-at, \frac{m}{(i+1)k} - 1, \frac{m}{(i+1)k}, -(i+1)k - 1\right). \quad (16)$$

Equation (16) is the main result of this section.

6. Mean Deviation

The scatter in a population can be measured by the totality of deviations from the mean and median. It is possible to derive the mean deviations about the mean and about the median from the relations $\delta_1(X) = E(|X - \mu'_1|) = 2\mu'_1 F(\mu'_1) - 2m_1(\mu'_1)$ and $\delta_2(X) = E(|X - M|) = \mu'_1 - 2m_1(M)$, respectively, where $\mu'_1 = E(X)$ comes from (13) with $n = 1$, the median M comes from (12) as

$$Q(u) = a \left\{ \left[\frac{1}{\beta} \left\{ 1 - \bar{\beta} \left[1 - 1/2(1 - \bar{\beta}^s) \right]^{-\frac{1}{s}} \right\} \right]^{-1/k} - 1 \right\}^{1/c},$$

$F(\mu'_1)$ is easily obtained from (6) and $m_1(\cdot)$ is determined from (14) with $n = 1$. The first incomplete moment is useful to obtain the Bonferroni (B) and Lorenz (L) curves since they have applications in several areas. For a positive random variable X , they are defined by $B(\pi) = m_1(q)/(\pi\mu'_1)$ and $L(\pi) = m_1(q)/\mu'_1$, where $q = F^{-1}(\pi) = Q(\pi)$ is calculated from (12). The measures $B(\pi)$ and $L(\pi)$ are determined from (14) by $m_1(Q(\pi))$.

7. Entropies

The entropy of a random variable X with density function $f(x)$ is a measure of variation of the uncertainty. Two popular entropy measures are due to Shannon and Rényi (Shannon, 1951; Rényi, 1961). A large value of the entropy indicates the greater uncertainty in the random variable. The Rényi entropy is defined by (for $\delta > 0$ and $\delta \neq 1$)

$$I_R(\delta) = \frac{1}{(1-\delta)} \log \left(\int_0^{\infty} f^{\delta}(x) dx \right).$$

Based on the pdf (5), the Rényi entropy of X is given by

$$I_R(\delta) = \frac{J}{(1-\delta)} \left[\int_0^{\infty} x^{\delta(c-1)} \left[1 + \left(\frac{x}{a} \right)^c \right]^{-\delta(k+1)} \left\{ 1 - \beta \left[1 + \left(\frac{x}{a} \right)^c \right]^{-k} \right\}^{-\delta(s+1)} dx \right],$$

where $J = \left[\frac{s\beta c k a^{-c}}{(1-\beta)^{-s}-1} \right]^{\delta}$. Using (8), we obtain

$$I_R(\delta) = \frac{J}{(1-\delta)} \left\{ \sum_{i=0}^{\infty} \frac{[\delta(k+1)]_i \beta^i}{i!} \int_0^{\infty} x^{\delta(c-1)} \left[1 + \left(\frac{x}{a} \right)^c \right]^{-\delta(k+1)-ki} dx \right\}.$$

So, if $\delta > (k+1-c^{-1})^{-1}$, the BXIINB Rényi entropy becomes

$$I_R(\delta) = \frac{J}{(1-\delta)} \left\{ a^{c+\delta} c^{-1} \sum_{i=0}^{\infty} \frac{[\delta(k+1)]_i \beta^i}{i!} B \left[\frac{\delta}{c} + 1, \delta(k+1-c^{-1}) + ki - 1 \right] \right\}.$$

The Shannon entropy of X is given by

$$\begin{aligned} E\{-\log[f(X)]\} &= -\log(s\beta c k a^{-c}) + \log[(1-\beta)^{-s}-1] \\ &\quad - (c-1)E[\log(X)] + (k+1)E\left\{\log\left[1 + \left(\frac{X}{a}\right)^c\right]\right\} \\ &\quad + (s+1)E\left\{\log\left\{1 - \beta\left[1 + \left(\frac{X}{a}\right)^c\right]^{-k}\right\}\right\}. \end{aligned}$$

Using (9), we have

$$\begin{aligned} E[\log(X)] &= \sum_{i=0}^{\infty} \omega_i(s, \beta) \int_0^{\infty} \log(x) g(x; c, (i+1)k, a) dx \\ &= \sum_{i=0}^{\infty} \omega_i(s, \beta) k(i+1) \int_0^{\infty} ca^{-c} x^{c-1} \log(x) \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k(i+1)-1} dx. \end{aligned}$$

Setting $u = a^c x^c$, we obtain

$$\begin{aligned} E[\log(X)] &= \sum_{i=0}^{\infty} \omega_i(s, \beta) k(i+1) \int_0^{\infty} \log(a u^{\frac{1}{c}}) (1+u)^{-k(i+1)-1} du \\ &= \frac{1}{c} \sum_{i=0}^{\infty} \omega_i(s, \beta) [\gamma + c \log(a) + \psi((i+1)k)], \end{aligned} \quad (17)$$

where $\gamma = 0.5772\dots$ is the Euler's constant and $\psi(\cdot)$ is the digamma function.

Now, setting $v = 1 + a^c x^c$, we have

$$\begin{aligned} E\left\{\log\left[1 + \left(\frac{X}{a}\right)^c\right]\right\} &= \sum_{i=0}^{\infty} \omega_i(s, \beta) k(i+1) \int_1^{\infty} \log(v) v^{-k(i+1)-1} dv \\ &= \sum_{i=0}^{\infty} \frac{\omega_i(s, \beta)}{k(i+1)}. \end{aligned}$$

Analogously, setting $v = 1 + a^c x^c$, we obtain

$$\begin{aligned} E\left(\log\left\{1 - \beta\left[1 + \left(\frac{X}{a}\right)^c\right]^{-k}\right\}\right) &= \sum_{i=0}^{\infty} \omega_i(s, \beta) k(i+1) \int_1^{\infty} \log(1 - \beta v^{-k}) v^{-k(i+1)-1} dv \\ &= \sum_{i=0}^{\infty} \omega_i(s, \beta) \beta \Phi(\beta, 1, i+2) + \log(1 - \beta), \end{aligned}$$

where $\Phi(\cdot)$ is the Hurwitz-Lerch transcendent function given by $\Phi(z, s, \alpha) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s}$. Then, we can express the Shannon entropy of X as

$$\begin{aligned} E\{-\log[f(X)]\} &= -\log(s\beta c k a^{-c}) + \log[(1 - \beta)^{-s} - 1] - (c-1) \sum_{i=0}^{\infty} \omega_i(s, \beta) \\ &\quad \times \left[\frac{\gamma + c \log(a) + \psi((i+1)k)}{c} \right] + (k+1) \sum_{i=0}^{\infty} \frac{\omega_i(s, \beta)}{k(i+1)} \\ &\quad + (s+1) \sum_{i=0}^{\infty} \omega_i(s, \beta) \beta \Phi(\beta, 1, i+2) + \log(1 - \beta). \end{aligned}$$

8. Reliability

In the context of reliability, the stress-strength model describes the life of a component which has a random strength X_1 that is subjected to a random stress X_2 . The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X_1 > X_2$. Hence, $R = Pr(X_2 < X_1)$ is a measure of component reliability. It has many applications especially in engineering concepts. Here, we obtain the reliability R when X_1 and X_2 have independent BXIINB(s_1, β_1, c, k, a) and BXIINB(s_2, β_2, c, k, a) distributions. The reliability is defined by

$$R = \int_0^{\infty} f_1(x) F_2(x) dx.$$

The pdf of X_1 and cdf of X_2 are obtained from equations (9) and (11) as

$$f_1(x) = \sum_{m=0}^{\infty} \omega_m(s_1, \beta_1) g(x; c, (m+1)k, a) \quad \text{and} \quad F_2(x) = \sum_{n=0}^{\infty} \omega_n(s_2, \beta_2) G(x; c, (n+1)k, a),$$

where

$$\omega_m(s_1, \beta_1) = \frac{s_1 \beta_1^{m+1} (s_1 + 1)_m}{[(1 - \beta_1)^{-s_1} - 1] (m + 1)!},$$

and $\omega_n(s_2, \beta_2)$ comes analogously. Hence,

$$\begin{aligned} R &= \sum_{m,n=0}^{\infty} \omega_m(s_1, \beta_1) \omega_n(s_2, \beta_2) k(m+1) \int_0^{\infty} c a^{-c} x^{c-1} \left[1 + \left(\frac{x}{a} \right)^c \right]^{-k(m+1)-1} \\ &\quad \times \left\{ 1 - \left[1 + \left(\frac{x}{a} \right)^c \right]^{-k(n+1)} \right\} dx. \end{aligned}$$

Setting $u = a^{-c} x^c$, we have

$$R = \sum_{m,n=0}^{\infty} \omega_m(s_1, \beta_1) \omega_n(s_2, \beta_2) k(m+1) \int_0^{\infty} (1+u)^{-k(m+1)-1} \left[1 - (1+u)^{-k(n+1)} \right] du.$$

Hence, the BXIINB reliability is given by

$$\begin{aligned} R &= \sum_{m,n=0}^{\infty} \omega_m(s_1, \beta_1) \omega_n(s_2, \beta_2) k(m+1) \left\{ -\frac{1}{k(m+n+2)} - \Gamma[k(n+1)+1] \right. \\ &\quad \times {}_2\tilde{F}_1[1, (m+n+2)k+1; (n+1)k+2; 1] \left. \right\}, \end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function and ${}_2\tilde{F}_1(\cdot)$ is the regularized hypergeometric function defined by

$${}_2\tilde{F}_1(a, b; c; x) = \frac{{}_2F_1(a, b; c; x)}{\Gamma(c)}.$$

9. Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. The density function $f_{i:n}(x)$ of the i th order statistic, for $i = 1, \dots, n$, from iid random variables X_1, \dots, X_n having the G-NB distribution is given by (Percontini *et al.*, 2013)

$$f_{i:n}(x) = \sum_{r,v=0}^{\infty} \sum_{j=0}^{n-i} m_{r,v,j} (r+v+j+i) g(x) G(x)^{r+v+j+i}, \quad (18)$$

where

$$\begin{aligned} m_{r,v,j} &= \frac{(-1)^j i (r+1) \omega_r \gamma_{j+i-1,v}}{(r+v+j+i)} \binom{n}{i} \binom{n-i}{j}, \\ \omega_r &= \frac{(-1)^r s}{(r+1)[(1-\beta)^{-s} - 1]} \sum_{l=r}^{\infty} \frac{(s+1)_l \beta^{l+1}}{l!} \binom{l}{r}, \end{aligned}$$

$$\gamma_{j+i-1,0} = \omega_0^{j+i-1} \text{ and } \gamma_{j+i-1,v} = (v \omega_0)^{-1} \sum_{m=1}^v [m(j+i) - v] \omega_m \gamma_{j+i-1,v-m}.$$

So, applying (3) in equation (18), the density function $f_{i:n}(x)$ of the i th order statistic, (for $i = 1, \dots, n$) from iid random variables X_1, \dots, X_n having the BXIINB distribution can be reduced to

$$\begin{aligned} f_{i:n}(x) &= \sum_{r,v=0}^{\infty} \sum_{j=0}^{n-i} m_{r,v,j} (r+v+j+i) \sum_{t=0}^{r+v+j+i-1} (-1)^t c k a^{-c} x^{c-1} \binom{r+v+j+i-1}{t} \\ &\quad \times \left[1 + \left(\frac{x}{a} \right)^c \right]^{-k(t+1)-1} \\ &= \sum_{r,v=0}^{\infty} \sum_{j=0}^{n-i} \sum_{t=0}^{r+v+j+i-1} \phi_t(r, v, j) g(x; c, (t+1)k, a), \end{aligned} \quad (19)$$

where

$$\phi_t(r, v, j) = \frac{(-1)^t}{t+1} (r+v+j+i) m_{r,v,j} \binom{r+v+j+i-1}{t}.$$

Equation (19) reveals that the pdf of the BXIINB order statistics is a triple linear combination of BXII density functions. So, several mathematical quantities of these order statistics, such as the ordinary, incomplete and factorial moments, mgf and mean deviations, can be obtained from those BXII quantities. Clearly, the cdf of $X_{i:n}$ can be expressed as

$$F_{i:n}(x) = \sum_{r,v=0}^{\infty} \sum_{j=0}^{n-i} \sum_{t=0}^{r+v+j+i-1} \phi_t(r, v, j) G(x; c, (t+1)k, a).$$

For example, from equation (19), the moments of the BXIINB order statistics can be expressed directly in terms of the BXII moments (for $p < kc$) as

$$\begin{aligned} E(X_{i:n}^p) &= \sum_{r,v=0}^{\infty} \sum_{j=0}^{n-i} \sum_{t=0}^{r+v+j+i-1} \phi_t(r, v, j) \int_0^{\infty} x^p g(x; c, (t+1)k, a) \\ &= \sum_{r,v=0}^{\infty} \sum_{j=0}^{n-i} \sum_{t=0}^{r+v+j+i-1} \phi_t(r, v, j) (t+1)k a^p B\left[(t+1)k - \frac{p}{c}, 1 + \frac{p}{c}\right]. \end{aligned}$$

10. Maximum Likelihood Estimation

We determine the maximum likelihood estimates (MLEs) of the parameters of the BXIINB distribution from complete samples only. Let x_1, \dots, x_n be a random sample of size n from the BXIINB(s, β, c, k, a) distribution. The log-likelihood function for the vector of parameters $\theta = (s, \beta, c, k, a)^T$ can be expressed as

$$\begin{aligned} l(\theta) &= n \log(s\beta c k a^{-c}) - n \log[(1-\beta)^{-s} - 1] + (c-1) \sum_{i=1}^n \log(x_i) \\ &\quad - (k+1) \sum_{i=1}^n \log\left[1 + \left(\frac{x_i}{a}\right)^c\right] - (s+1) \sum_{i=1}^n \log\left\{1 - \beta\left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}. \end{aligned} \quad (20)$$

The components of the score vector $U(\theta)$ are

$$\begin{aligned} U_s(\theta) &= \frac{n}{s} + \frac{n(1-\beta)^{-s} \log(1-\beta)}{(1-\beta)^{-s} - 1} - \sum_{i=1}^n \log\left\{1 - \beta\left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}, \\ U_\beta(\theta) &= \frac{n}{\beta} - \frac{n s (1-\beta)^{-s-1}}{(1-\beta)^{-s} - 1} + (s+1) \sum_{i=1}^n \left\{ \frac{\left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}}{1 - \beta\left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}} \right\}, \\ U_c(\theta) &= \frac{n}{c} - n \log(a) + \sum_{i=1}^n \log(x_i) - (k+1) \sum_{i=1}^n \left[\frac{\left(\frac{x_i}{a}\right)^c \log\left(\frac{x_i}{a}\right)}{1 + \left(\frac{x_i}{a}\right)^c} \right] \\ &\quad - (s+1) \sum_{i=1}^n \left\{ \frac{\beta k \left(\frac{x_i}{a}\right)^c \log\left(\frac{x_i}{a}\right) \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1}}{1 - \beta\left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}} \right\}, \\ U_k(\theta) &= \frac{n}{k} - \sum_{i=1}^n \log\left[1 + \left(\frac{x_i}{a}\right)^c\right] - (s+1) \sum_{i=1}^n \left\{ \frac{\beta\left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k} \log\left[1 + \left(\frac{x_i}{a}\right)^c\right]}{1 - \beta\left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}} \right\}, \end{aligned}$$

$$U_a(\theta) = -\frac{cn}{a} + (k+1) \sum_{i=1}^n \left\{ \frac{c x_i \left(\frac{x_i}{a}\right)^{c-1}}{a^2 \left[1 + \left(\frac{x_i}{a}\right)^c\right]} \right\} + (s+1) \sum_{i=1}^n \left\{ \frac{\beta c k x_i \left(\frac{x_i}{a}\right)^{c-1}}{a^2 \left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}} \right. \\ \left. \times \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1} \right\}.$$

Let $\widehat{\theta}$ be the MLE of θ . The log-likelihood can be maximized either directly by using the Proc NLMixed of SAS or the sub-routine MaxBFGS of the Ox program (see, Doornik, 2007) or by solving the nonlinear likelihood equations $U_s(\theta) = U_k(\theta) = U_c(\theta) = U_p(\theta) = 0$ simultaneously. For interval estimation and hypothesis tests on the model parameters, we require the 5×5 unit observed information matrix $J(\theta) = -\{U_{pq}\}$, where p, q vary over the elements in $\theta = (s, \beta, c, k, a)$. The elements U_{pq} are given in Appendix B. Based on the approximate multivariate normal $N_5(0, J(\widehat{\theta})^{-1})$ distribution, we can construct confidence regions for the model parameters. We can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain likelihood ratio (LR) statistics for testing some sub-models of the BXIINB distribution. Hypothesis tests of the type $H_0 : \psi = \psi_0$ versus $H_1 : \psi \neq \psi_0$, where ψ is a vector formed with some components of θ and ψ_0 is a specified vector, can be based on LR statistics. For example, the test of $H_0 : s = 1$ and $\beta \rightarrow 0$ versus $H_1 : H_0 \text{ is not true}$ is equivalent to compare the BXIINB and BXII distributions and then the LR statistic reduces to

$$w = 2\{\ell(\widehat{s}, \widehat{\beta}, \widehat{c}, \widehat{k}, \widehat{a}) - \ell(1, 0, \widetilde{c}, \widetilde{k}, \widetilde{a})\},$$

where $\widehat{s}, \widehat{\beta}, \widehat{c}, \widehat{k}$ and \widehat{a} are the MLEs under H_1 and $\widetilde{c}, \widetilde{k}$ and \widetilde{a} are the estimates under H_0 .

11. Application

In this section, we illustrate the usefulness of the BXIINB distribution applied to a real data set. These data on failure times are reported in the book “Weibull Models” by Murthy *et al.* (2004, p. 297):

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

We also fit the density functions of the *Exponentiated Burr XII Poisson* (Exp-BXIIP), *Beta Burr XII* (BBXII), *Kumaraswamy Burr XII* (KwBXII) and *McDonald Burr XII* (McBXII) distributions given by

$$f_{\text{Exp-BXIIP}}(x; c, k, a, \alpha, \lambda) = \frac{c k a^{-c} \alpha \lambda}{1 - e^{-\lambda}} x^{c-1} \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k-1} \left\{1 - \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k}\right\}^{\alpha-1} \\ \times \exp\left\{-\lambda \left[1 - \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k}\right]^{\alpha}\right\},$$

$$f_{\text{BBXII}}(x; c, k, a, d, b) = \frac{c k a^{-c}}{B(d, b)} x^{c-1} \left[1 + \left(\frac{x}{a}\right)^c\right]^{-kb-1} \left\{1 - \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k}\right\}^{d-1},$$

$$f_{\text{KwBXII}}(x; c, k, a, d, b) = d b c k a^{-c} x^{c-1} \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k-1} \left\{1 - \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k}\right\}^{\alpha-1} \\ \times \left\{1 - \left\{1 - \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k}\right\}^d\right\}^{b-1}$$

and

$$f_{\text{McBXII}}(x; c, k, a, d, b, \alpha) = \frac{c k a^{-c} \alpha}{B(d, b)} x^{c-1} \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k-1} \left\{1 - \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k}\right\}^{d\alpha-1} \\ \times \left\{1 - \left\{1 - \left[1 + \left(\frac{x}{a}\right)^c\right]^{-k}\right\}^{\alpha}\right\}^{b-1},$$

respectively, where all parameters are positive.

Further, we apply the Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics described in details in Chen and Balakrishnan (1995) to verify which distribution fits better to these data. Table 1 lists the MLEs, their standard errors in parentheses and the statistics W^* and A^* . These statistics indicate that the BXIINB distribution is the best model to describe the data.

Table 1. MLEs and the measures W^* and A^*

Distribution	Estimates					Statistics	
BXIINB	\hat{s}	$\hat{\beta}$	\hat{c}	\hat{k}	\hat{a}	W^*	A^*
	19.1368 (18.9036)	0.9607 (0.0868)	2.4613 (0.2213)	0.2565 (1.0065)	19.7483 (23.7111)	0.0602	0.6047
Exp-BXIIP	\hat{c}	\hat{k}	\hat{a}	$\hat{\alpha}$	$\hat{\lambda}$	W^*	A^*
	6.7808 (0.0079)	9.5540 (15.5828)	5.6803 (1.5101)	0.2673 (0.0548)	0.6577 (1.0049)	0.0961	0.6838
BBXII	\hat{c}	\hat{k}	\hat{a}	\hat{d}	\hat{b}	W^*	A^*
	7.3601 (0.0046)	5.5584 (3.1054)	4.3671 (0.0046)	0.2180 (0.0278)	0.4961 (0.2190)	0.2317	1.9095
KwBXII	\hat{c}	\hat{k}	\hat{a}	\hat{d}	\hat{b}	W^*	A^*
	5.9787 (0.0464)	6.1332 (6.2411)	6.2593 (0.0454)	0.3216 (0.0529)	1.8375 (1.1272)	0.0777	0.6050
McBXII	\hat{c}	\hat{k}	\hat{a}	\hat{d}	\hat{b}	$\hat{\alpha}$	A^*
	6.3865 (0.0140)	11.0209 (0.0671)	6.4909 (0.0140)	0.5411 (0.7742)	1.6177 (0.3525)	0.5016 (0.6585)	0.8799

More information is provided by a visual comparison of the fitted densities to the histogram of the data. The plots of the fitted BXIINB, Exp-BXIIP, BBXII, KwBXII and McBXII density functions are displayed in Figure 4. They show that the new distribution provides a good fit to these data and that it is also a very competitive model to the other lifetime distributions.

12. Concluding Remarks

We define and study a new five-parameter lifetime distribution, referred to as the Burr XII negative binomial (BXI-INB) distribution, which extends several distributions widely used in the lifetime literature. The proposed model is much more flexible than the Burr XII, Weibull and log-logistic models. It is useful for modeling lifetime data with a unimodal-shaped hazard rate function. We provide a mathematical treatment of the distribution including expansions for the BXIINB density function and plots of its skewness and kurtosis. We derive explicit expressions for the ordinary and incomplete moments, generating and quantile functions, order statistics and their moments, Rényi and Shanon entropies, mean deviations and reliability. The estimation of the model parameters is approached by the method of maximum likelihood and the observed information matrix is derived. One application of the BXIINB distribution shows that it could provide a better fit than other statistical models used in lifetime data analysis.

Appendix A

We have the following result which holds for m and k positive integers, $\mu > -1$ and $p > 0$ (Prudnikov *et al.*, 1992, p. 21)

$$\begin{aligned}
 I\left(p, \mu, \frac{m}{k}, \nu\right) &= \int_0^\infty \exp(-px) x^\mu (1 + x^{\frac{m}{k}})^\nu dx \\
 &= \frac{k^{-\nu} m^{\mu + \frac{1}{2}}}{(2\pi)^{\frac{(m-1)}{2}} \Gamma(-\nu) p^{\mu+1}} \times \\
 &\quad G_{k+m, k}^{k, k+m} \left(\frac{m^m}{p^m} \middle| \begin{matrix} \Delta(m, -\mu), \Delta(k, \nu + 1) \\ \Delta(k, 0) \end{matrix} \right), \quad (21)
 \end{aligned}$$

where $\Delta(k, a) = \frac{a}{k}, \frac{a+1}{k}, \dots, \frac{a+k}{k}$.

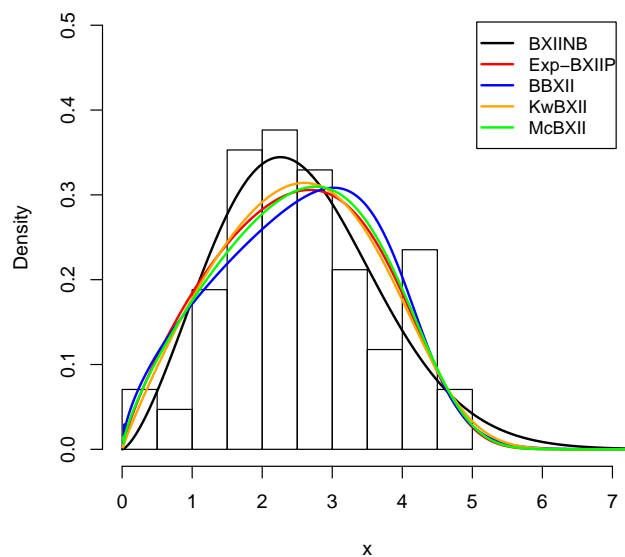


Figure 4. Some fitted densities to the histogram of the current data.

Appendix B

$$U_{ss} = -\frac{n \left\{ [(1-\beta)^s - 1]^2 - s^2(1-\beta)^s \log^2(1-\beta) \right\}}{s^2 [(1-\beta)^s - 1]^2},$$

$$U_{s\beta} = \frac{n [1 - (1-\beta)^s + s(1-\beta)^s \log(1-\beta)]}{(\beta-1) [(1-\beta)^s - 1]^2} + \sum_{i=1}^n \frac{\left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}}{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}},$$

$$U_{sc} = -\sum_{i=1}^n \frac{\beta k \left(\frac{x_i}{a}\right)^c \log\left(\frac{x_i}{a}\right) \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1}}{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}},$$

$$U_{sk} = -\sum_{i=1}^n \frac{\beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k} \log\left[1 + \left(\frac{x_i}{a}\right)^c\right]}{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}},$$

$$U_{sa} = \sum_{i=1}^n \frac{\beta c k x_i \left(\frac{x_i}{a}\right)^{c-1} \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1}}{a^2 \left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}},$$

$$U_{\beta\beta} = n \left\{ \frac{s [s(1-\beta)^s + (1-\beta)^s - 1]}{(\beta-1)^2 [(1-\beta)^s - 1]^2} - \frac{1}{\beta^2} \right\} + (s+1) \sum_{i=1}^n \frac{\left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-2k}}{\left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}^2},$$

$$U_{\beta c} = -(s+1) \sum_{i=1}^n \left(\frac{k \left(\frac{x_i}{a}\right)^c \log \left(\frac{x_i}{a}\right) \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1}}{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}} + \frac{\beta k \left(\frac{x_i}{a}\right)^c \log \left(\frac{x_i}{a}\right) \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-2k-1}}{\left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}^2} \right),$$

$$U_{\beta k} = -(s+1) \sum_{i=1}^n \left(\frac{\beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-2k} \log \left[1 + \left(\frac{x_i}{a}\right)^c\right]}{\left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}^2} + \frac{\left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k} \log \left[1 + \left(\frac{x_i}{a}\right)^c\right]}{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}} \right),$$

$$U_{\beta a} = (s+1) \sum_{i=1}^n \left(\frac{c k x_i \left(\frac{x_i}{a}\right)^{c-1} \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1}}{a^2 \left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}} + \frac{\beta c k x_i \left(\frac{x_i}{a}\right)^{c-1} \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-2k-1}}{a^2 \left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}^2} \right),$$

$$\begin{aligned} U_{cc} = & \frac{n}{c^2} + (s+1) \left[\sum_{i=1}^n \left(\frac{\beta^2 k^2 \left(\frac{x_i}{a}\right)^{2c} \log^2 \left(\frac{x_i}{a}\right) \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-2k-2}}{\left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}^2} - \frac{\beta k \left(\frac{x_i}{a}\right)^c \log^2 \left(\frac{x_i}{a}\right)}{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}} \right. \right. \\ & \times \left. \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1} - \frac{\beta k (-k-1) \left(\frac{x_i}{a}\right)^{2c} \log^2 \left(\frac{x_i}{a}\right) \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-2}}{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}} \right) \right] \\ & + (k+1) \left[\sum_{i=1}^n \left(\frac{\left(\frac{x_i}{a}\right)^c \log^2 \left(\frac{x_i}{a}\right)}{1 + \left(\frac{x_i}{a}\right)^c} - \frac{\left(\frac{x_i}{a}\right)^{2c} \log^2 \left(\frac{x_i}{a}\right)}{\left[1 + \left(\frac{x_i}{a}\right)^c\right]^2} \right) \right], \end{aligned}$$

$$\begin{aligned} U_{ck} = & - \sum_i \frac{\left(\frac{x_i}{a}\right)^c \log \left(\frac{x_i}{a}\right)}{1 + \left(\frac{x_i}{a}\right)^c} + (s+1) \sum_{i=1}^n \left(\frac{\beta^2 k \left(\frac{x_i}{a}\right)^c \log \left(\frac{x_i}{a}\right) \log \left[1 + \left(\frac{x_i}{a}\right)^c\right]}{\left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}^2} \right. \\ & \times \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-2k-1} - \frac{\beta \left(\frac{x_i}{a}\right)^c \log \left(\frac{x_i}{a}\right) \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1}}{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}} + \frac{\beta k \left(\frac{x_i}{a}\right)^c \log \left(\frac{x_i}{a}\right)}{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}} \\ & \times \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1} \log \left[1 + \left(\frac{x_i}{a}\right)^c\right] \right), \end{aligned}$$

$$\begin{aligned} U_{ca} = & -(s+1) \sum_{i=1}^n \left(\frac{\beta^2 c k^2 x_i \left(\frac{x_i}{a}\right)^{2c-1} \log \left(\frac{x_i}{a}\right) \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-2k-2}}{a^2 \left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}^2} - \frac{\beta c k x_i \left(\frac{x_i}{a}\right)^{c-1}}{\left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}} \right. \\ & \times \frac{1}{a^2} \log \left(\frac{x_i}{a}\right) \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1} - \frac{\beta c (-k-1) k x_i \left(\frac{x_i}{a}\right)^{2c-1} \log \left(\frac{x_i}{a}\right) \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-2}}{a^2 \left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}} \\ & \left. - \frac{\beta k \left(\frac{x_i}{a}\right)^c \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1}}{a \left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}} \right) - (k+1) \sum_i \left\{ \frac{c x_i \left(\frac{x_i}{a}\right)^{c-1} \log \left(\frac{x_i}{a}\right)}{a^2 \left[1 + \left(\frac{x_i}{a}\right)^c\right]} + \frac{\left(\frac{x_i}{a}\right)^c}{a \left[1 + \left(\frac{x_i}{a}\right)^c\right]} \right. \\ & \left. - \frac{c x_i \left(\frac{x_i}{a}\right)^{2c-1} \log \left(\frac{x_i}{a}\right)}{a^2 \left[1 + \left(\frac{x_i}{a}\right)^c\right]^2} \right\} + \frac{n}{a}, \end{aligned}$$

$$U_{kk} = -\frac{n}{k^2} + (s+1) \sum_{i=1}^n \left(\frac{\beta^2 \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-2k} \log^2 \left[1 + \left(\frac{x_i}{a}\right)^c\right]}{\left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}^2} + \frac{\beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}}{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}} \right. \\ \left. \times \log^2 \left[1 + \left(\frac{x_i}{a}\right)^c\right] \right),$$

$$U_{ka} = \sum_{i=1}^n \frac{c x_i \left(\frac{x_i}{a}\right)^{c-1}}{a^2 \left[1 + \left(\frac{x_i}{a}\right)^c\right]} - (s+1) \sum_{i=1}^n \left(\frac{\beta^2 c k x_i \left(\frac{x_i}{a}\right)^{c-1} \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-2k-1} \log \left[1 + \left(\frac{x_i}{a}\right)^c\right]}{a^2 \left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}^2} \right. \\ \left. - \frac{\beta c x_i \left(\frac{x_i}{a}\right)^{c-1} \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1}}{a^2 \left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}} + \frac{\beta c k x_i \left(\frac{x_i}{a}\right)^{c-1} \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1} \log \left[1 + \left(\frac{x_i}{a}\right)^c\right]}{a^2 \left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}} \right),$$

$$U_{aa} = \frac{cn}{a^2} - (s+1) \sum_{i=1}^n \left(-\frac{\beta^2 c^2 k^2 x_i^2 \left(\frac{x_i}{a}\right)^{2c-2} \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-2k-2}}{a^4 \left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}^2} + \frac{\beta c^2 (-k-1)k}{\left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}} \right. \\ \left. \times \frac{x_i^2 \left(\frac{x_i}{a}\right)^{2c-2} \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-2}}{a^4} + \frac{\beta c k (c-1) x_i^2 \left(\frac{x_i}{a}\right)^{c-2} \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1}}{a^4 \left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}} \right. \\ \left. + \frac{2\beta c k x_i \left(\frac{x_i}{a}\right)^{c-1} \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k-1}}{a^3 \left\{1 - \beta \left[1 + \left(\frac{x_i}{a}\right)^c\right]^{-k}\right\}} \right) + (k+1) \sum_{i=1}^n \left(\frac{c^2 x_i^2 \left(\frac{x_i}{a}\right)^{2c-2}}{a^4 \left[1 + \left(\frac{x_i}{a}\right)^c\right]^2} - \frac{c(c-1)}{\left[1 + \left(\frac{x_i}{a}\right)^c\right]} \right. \\ \left. \times \frac{x_i^2 \left(\frac{x_i}{a}\right)^{c-2}}{a^4} - \frac{2c x_i \left(\frac{x_i}{a}\right)^{c-1}}{a^3 \left[1 + \left(\frac{x_i}{a}\right)^c\right]} \right).$$

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