

The Explicit Solution and Precise Distribution of CKLS Model under Girsanov Transform

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Abstract

The relation between CKLS model and CIR model will be investigated in this paper. It will be shown that under a suitable transformation, any CKLS model of order $\frac{1}{2} < \gamma < 1$ or $\gamma > 1$ corresponds to a CIR model under a new probability space. Moreover, the explicit solution and the precise distribution of the CKLS model at any time t are obtained under the new probability space. The moment estimation of CKLS model will be given finally.

Keywords: interest rate; CKLS model; CIR model; Girsanov transform; martingale.

1. Introduction

Suppose that $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0})$ is a right continuous filtered probability space satisfying the usual conditions. In 1992, Chan, Karolyi, Longstaff and Sanders (Chan, Karolyi, Longstaff & Sanders, 1992) suggested modelling the behavior of the instantaneous interest rate by the following stochastic differential equation

$$dr_t = (a - br_t)dt + \sigma r_t^\gamma dB_t, \quad (1)$$

where the initial interest rate $r_0 > 0$, $a > 0$, $b \in \mathbb{R}$, $\sigma > 0$ and $\gamma \geq \frac{1}{2}$. B_t denotes an 1-dimensional standard \mathcal{F}_t -Brownian motion.

This so called CKLS model is a very important model in both theory and application, it contains many important models in finance. These models can be obtained from (1) by simply placing the appropriate restrictions on the four parameters a, b, σ and γ . For example, if $\gamma = 0$, it becomes to a Vasicek model; $\gamma = 1, a = 0$ a geometric Brownian motion; $\gamma = \frac{1}{2}$ a CIR model, etc..

There are many literatures concerned about the solutions as well as other properties of CKLS model. For example, Choi and Wirjanto (Choi & Wirjanto 2007) presented an analytic approximation formula for pricing zero-coupon bonds under CKLS models, Tangman, Thakoor, Dookhitram and Bhuruth (Tangman, Thakoor, Dookhitram & Bhuruth, 2011) considered fast approximations of bond option prices under CKLS models, while Tang and Chen (Tang & Chen, 2009) studied the parameter estimation and bias correction for diffusion processes which contain CKLS models as special cases, Khor, Pooi and Ng (Khor, Pooi & Ng, 2013) studied the bond (European) option pricing under the CKLS model.

It is well known that there is a unique non negative solution of the equation (1). By Mao (Mao, 2007) Chapter 9, we also know that the solution $r_t > 0$ for all $t \geq 0$ almost surely.

The motivation of this work is the following:

In Lamberton and Lapeyre (Lamberton & Lapeyre, 1996), Proposition 6.2.5, the authors presented the Laplace transform of r_t and $\int_0^t r_s ds$ in the case of CIR model (that is, $\gamma = \frac{1}{2}$), and finally they obtained the pricing formula of zero-coupon bond and the bond option. So a natural question raises: if we can transform CKLS model into a CIR model, then we can use the known results of CIR model to the CKLS model.

In the following, we assume that r_t is the unique positive solution of SDE (1).

Our main result is the following theorem.

Theorem 1 Let $\gamma > 1$, or $\gamma \in (\frac{1}{2}, 1)$ and $\frac{\gamma}{\sigma} \geq 1, b > 0$. Define $f(x) = \frac{C^2}{4(1-\gamma)^2} x^{2(1-\gamma)}$. Then there exist a new filtered probability space $(\Omega, \mathcal{F}, Q; (\mathcal{F}_t)_{t \geq 0})$ and a Q Brownian motion \bar{B}_t such that $Y_t = f(r_t)$ satisfies the following CIR model:

$$dY_t = \left(\frac{\sigma^2 C^2}{4} + 2b(1-\gamma)Y_t \right) dt + \sigma C \sqrt{Y_t} d\bar{B}_t. \quad (2)$$

Remark 1 When $\gamma \in (\frac{1}{2}, 1)$ and $\frac{\gamma}{\sigma} \geq 1, b > 0$, equation (2) is not the usual mean reverting square root process any more since the coefficient of Y_t is $2b(1-\gamma) > 0$.

In general, we can not solve equation (1) explicitly except for some special cases (e.g., $\gamma = 1$), but by a suitable measure transform, we can get the explicit solution of our CKLS model under the new probability space.

Theorem 2 Assume that $\gamma > 1$, or $\gamma \in (\frac{1}{2}, 1)$ and $\frac{\gamma}{\sigma} \geq 1, b > 0$. Under the probability measure Q (which is the same as in Theorem 1), the explicit solution of equation (1) is

$$r_t = \left| r_0^{1-\gamma} e^{-(\gamma-1)bt} + \sigma|\gamma-1| \int_0^t e^{-b(\gamma-1)(t-s)} d\bar{B}_s \right|^{\frac{1}{1-\gamma}}.$$

We can also get the distribution of r_t .

Theorem 3 Assume that $\gamma > 1$, or $\gamma \in (\frac{1}{2}, 1)$ and $\frac{\gamma}{\sigma} \geq 1, b > 0$. Under the probability measure Q , the precise density function of the solution r_t of equation (1) is

$$g_t(x) = g_{\delta, \zeta}(f(x)/L) |f'(x)|/L, \quad \forall x > 0,$$

where f is the same as in Theorem 1, $g_{\delta, \zeta}$ is the density of the non-central chi-square law with δ degrees of freedom and parameter ζ ,

$$\delta = C^2, \quad L = \frac{\sigma^2 C^2}{8b(\gamma-1)} (1 - e^{2b(1-\gamma)t}), \quad \zeta = \frac{8Y_0 b(\gamma-1)}{\sigma^2 C^2 (e^{2b(\gamma-1)t} - 1)}.$$

Finally, let us consider the moment estimation of r_t . We have

Theorem 4 Assume that $1 < \gamma \leq \frac{3}{2}$, or $\frac{1}{2} \leq \gamma < 1$ and $(2\gamma+1)\sigma^2 \leq 2a$. Then

$$\mathbb{E} \int_0^t r_s^{-2\gamma} ds \vee \mathbb{E} \int_0^t r_s^{2(\gamma-1)} ds < \infty, \quad \forall t > 0,$$

where $a \vee b := \max(a, b)$.

The rest of the paper is organized as follows. Firstly, we study the relation between CKLS model and CIR model in Section 2, we derive the transform under which an arbitrary CKLS model can be transformed to a CIR model formally. In Section 3 we will prove that the condition of Girsanov transform is satisfied (that is, R_t defined in Section 2 is a true martingale under Q). After this, we prove Theorem 2 and Theorem 3 in Section 4. Finally we give the moment estimation of CKLS model, which is our Theorem 4.

2. The Relation between CKLS Model and CIR Model

Suppose $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a differentiable function such that

$$x^\gamma f'(x) = C \sqrt{f(x)}, \quad C > 0.$$

Then we have

$$f(x) = \frac{C^2}{4(1-\gamma)^2} (x^{1-\gamma} + C')^2,$$

where C' is an arbitrary constant. Take $C' = 0$ for simplicity. Then

$$f(x) = \frac{C^2}{4(1-\gamma)^2} x^{2(1-\gamma)}.$$

We also have

$$f'(x) = \frac{C^2}{2(1-\gamma)} x^{1-2\gamma}, \quad f''(x) = \frac{C^2(1-2\gamma)}{2(1-\gamma)} x^{-2\gamma}.$$

It's clear that $f(x)$ is strictly monotone on $(0, \infty)$, so we have

$$f^{-1}(x) = \left| \frac{2(\gamma-1)}{C} \right|^{\frac{1}{1-\gamma}} x^{\frac{1}{2(1-\gamma)}}.$$

By Itô's formula, we have

$$\begin{aligned} df(r_t) &= [f'(r_t)(a - br_t) + \frac{\sigma^2}{2} f''(r_t) r_t^{2\gamma}] dt + \sigma f'(r_t) r_t^\gamma dB_t \\ &= \left[\frac{aC^2}{2(1-\gamma)} r_t^{1-2\gamma} + \frac{bC^2}{2(1-\gamma)} r_t^{2-2\gamma} + \frac{\sigma^2}{2} \frac{C^2(1-2\gamma)}{2(1-\gamma)} \right] dt + \sigma C \sqrt{f(r_t)} dB_t. \end{aligned}$$

Denote $Y_t = f(r_t)$. Then $r_t = f^{-1}(Y_t) = \left| \frac{2(\gamma-1)}{C} \right|^{\frac{1}{1-\gamma}} Y_t^{\frac{1}{2(1-\gamma)}}$. Therefore,

$$dY_t = \left[\frac{\sigma^2 C^2 (1-2\gamma)}{4(1-\gamma)} + \frac{bC^2}{2(1-\gamma)} \left(\frac{2(\gamma-1)}{C} \sqrt{Y_t} \right)^2 + \frac{aC^2}{2(1-\gamma)} r_t^{1-2\gamma} \right] dt + \sigma C \sqrt{Y_t} dB_t. \quad (3)$$

Define

$$q(r_t) := \left(\frac{a}{\sigma} r_t^{-\gamma} - \frac{\gamma\sigma}{2} r_t^{\gamma-1} \right) \text{sgn}(\gamma-1), \quad (4)$$

$$\bar{B}_t := B_t - \int_0^t q(r_s) ds$$

and

$$R_t := \exp \left\{ \int_0^t q(r_s) dB_s - \frac{1}{2} \int_0^t q(r_s)^2 ds \right\}.$$

It's clear that R_t is an \mathcal{F}_t local martingale with respect to probability measure P . If $\{R_t\}$ is a real martingale with respect to P , then by Girsanov transform, there exists a probability measure Q on Ω such that $\frac{dQ}{dP}|_{\mathcal{F}_t} = R_t$ and \bar{B}_t is an \mathcal{F}_t Brownian motion with respect to Q . Then Y_t satisfies the following CIR model

$$dY_t = \left(\frac{\sigma^2 C^2}{4} + 2b(1-\gamma)Y_t \right) dt + \sigma C \sqrt{Y_t} d\bar{B}_t.$$

So the key point is to prove that $\{R_t\}$ is a true martingale, which will be fulfilled in the next Section.

3. R_t is a True Martingale with Respect to Probability P

To prove that R_t is a true martingale, let us consider the following auxiliary equation

$$d\tilde{r}_t = (a - b\tilde{r}_t + q(\tilde{r}_t)\sigma\tilde{r}_t^\gamma)dt + \sigma\tilde{r}_t^\gamma dB'_t, \quad \tilde{r}_0 = r_0, \quad (5)$$

where B'_t is Brownian motion, q are defined as in Section 2. Since we have known that, with probability one, the solution of equation (1) will never leave the state space $(0, \infty)$, by Mijatović and Urusov (Mijatović & Urusov, 2012) Corollary 2.2, we only need to prove that the solution of equation (5) will also never leave the state space $(0, \infty)$.

3.1 In Case $\gamma > 1$

If $\gamma > 1$, then equation (5) could be transformed to

$$d\tilde{r}_t = \left(2a - b\tilde{r}_t - \frac{\gamma\sigma^2}{2} \tilde{r}_t^{2\gamma-1} \right) dt + \sigma\tilde{r}_t^\gamma dB'_t. \quad (6)$$

For this equation, since $\mu(x) = 2a - bx - \frac{\gamma\sigma^2}{2}x^{2\gamma-1}$ satisfies

$$(x - y)(\mu(x) - \mu(y)) \leq 0 \leq K(x - y)^2, \forall x, y$$

and x^γ is local Lipschitz continuous, by Gyöngy and Krylov (Gyöngy & Krylov, 1996) Corollary 2.7, there exists a unique solution of equation (5). We now prove that the solution $\tilde{r}_t > 0, \forall t \geq 0$ almost surely. Define

$$\tau_k := \inf\{t > 0, \tilde{r}_t \notin (\frac{1}{k}, k)\}, k \geq k_0$$

where k_0 is large enough such that $r_0 \in (\frac{1}{k_0}, k_0)$.

We only need to prove

$$\tau_\infty := \lim_{k \rightarrow \infty} \tau_k = \infty.$$

If not, we can choose $T > 0, 0 < \varepsilon < 1$ such that $P(\tau_\infty \leq T) \geq \varepsilon$. Then there exists $k_1 \geq k_0$ such that

$$P(\tau_k \leq T) \geq \varepsilon \quad \forall k \geq k_1.$$

Define

$$V(x) = \sqrt{x} - 1 - \frac{1}{2} \log x, \quad x > 0.$$

Then if $\tilde{r}_t > 0$, Itô's formula yields that

$$\begin{aligned} dV(\tilde{r}_t) &= \frac{1}{2}(\tilde{r}_t^{-\frac{1}{2}} - \tilde{r}_t^{-1})(2a - b\tilde{r}_t - \frac{\gamma\sigma^2}{2}\tilde{r}_t^{2\gamma-1})dt \\ &\quad + \frac{\sigma^2}{4}(-\frac{1}{2}\tilde{r}_t^{-\frac{3}{2}} + \tilde{r}_t^{-2})\tilde{r}_t^{2\gamma}dt + \frac{\sigma}{2}(\tilde{r}_t^{-\frac{1}{2}} - \tilde{r}_t^{-1})\tilde{r}_t^\gamma dB'_t \\ &= F(\tilde{r}_t)dt + \frac{\sigma}{2}(\tilde{r}_t^{-\frac{1}{2}} - \tilde{r}_t^{-1})\tilde{r}_t^\gamma dB'_t, \end{aligned}$$

where

$$F(x) = \frac{1}{2}(x^{-\frac{1}{2}} - x^{-1})(2a - bx - \frac{\gamma\sigma^2}{2}x^{2\gamma-1}) + \frac{\sigma^2}{4}(-\frac{1}{2}x^{-\frac{3}{2}} + x^{-2})x^{2\gamma}.$$

Since the coefficients of the highest and lowest order of $F(x)$ are both negative (the coefficients are $-\frac{\sigma^2(2\gamma+1)}{8}, -a$, respectively), then $F(x)$ is bounded, say by K , on $x \in (0, \infty)$. Thus,

$$dV(\tilde{r}_t) \leq Kdt + \frac{\sigma}{2}(\tilde{r}_t^{-\frac{1}{2}} - \tilde{r}_t^{-1})\tilde{r}_t^\gamma dB'_t$$

as long as $\tilde{r}_t \in (0, \infty)$. So we have

$$\mathbb{E}(V(\tilde{r}_{T \wedge \tau_k})) \leq V(r_0) + KT.$$

Set $\Omega_k = \{\tau_k \leq T\}$, then $P(\Omega_k) \geq \varepsilon$ for $k \geq k_1$. Since

$$\tilde{r}(\tau_k, \omega) = k \text{ or } \frac{1}{k},$$

then

$$V(\tilde{r}(\tau_k, \omega)) \geq (\sqrt{k} - 1 - \frac{1}{2} \log k) \wedge (\frac{1}{2} \log k + \sqrt{\frac{1}{k}} - 1).$$

Therefore,

$$\begin{aligned} V(r_0) + KT &\geq \mathbb{E}(V(\tilde{r}_{T \wedge \tau_k})) \geq \mathbb{E}(1_{\Omega_k} V(\tilde{r}(\tau_k, \omega))) \\ &\geq \varepsilon[(\sqrt{k} - 1 - \frac{1}{2} \log k) \wedge (\frac{1}{2} \log k + \sqrt{\frac{1}{k}} - 1)]. \end{aligned}$$

Letting $k \rightarrow \infty$ leads to a contradiction

$$\infty > V(r_0) + KT \geq \infty.$$

So $\tau_\infty = \infty$. That is, with probability 1, $r_t > 0$ for all $t \geq 0$. Therefore, R_t is a true martingale with respect to P .

3.2 In Case $\frac{1}{2} \leq \gamma < 1$

In this case, equation (5) becomes to

$$d\tilde{r}_t = \left(\frac{\gamma\sigma}{2} \tilde{r}_t^{2\gamma-1} - b\tilde{r}_t \right) dt + \sigma \tilde{r}_t^\gamma dB'_t. \quad (7)$$

Note that the method we used in Subsection 3.1 to prove the positivity of the solution can not be used in the present case. Since

$$\sigma x^\gamma > 0, \quad \forall x > 0 \quad \text{and} \quad \frac{1}{\sigma^2 x^{2\gamma}}, \frac{\frac{\gamma\sigma}{2} x^{2\gamma-1} - bx}{\sigma^2 x^{2\gamma}} \in L^1_{loc}(0, \infty),$$

where $L^1_{loc}(0, \infty)$ denotes the class of locally integrable functions, i.e. the functions $(0, \infty) \rightarrow \mathbb{R}$ that are integrable on compact subsets of $(0, \infty)$. By (Engelbert & Schmidt, 1985), (Engelbert & Schmidt, 1991) or (Karatzas & Shreve, 1991) Chapter 5, Theorem 5.15, there exists a unique in law weak solution that possibly exits its state space $(0, \infty)$.

We now prove that the solution $\tilde{r}_t > 0, \forall t \geq 0$ almost surely by using Feller's test for explosions.

Define the scale function

$$p(x) = \int_1^x \exp \left\{ -2 \int_1^y \frac{\frac{\gamma\sigma}{2} z^{2\gamma-1} - bz}{\sigma^2 z^{2\gamma}} dz \right\} dy. \quad (8)$$

Compute

$$p(x) = \exp \left\{ -\frac{b}{\sigma^2(1-\gamma)} \right\} \int_1^x y^{-\frac{\gamma}{\sigma}} \exp \left\{ \frac{b}{\sigma^2(1-\gamma)} y^{2(1-\gamma)} \right\} dy. \quad (9)$$

In the case when $\frac{\gamma}{\sigma} \geq 1$ and $b > 0$, it is ease to see that

$$\lim_{x \downarrow 0} p(x) = -\infty \quad \text{and} \quad \lim_{x \uparrow \infty} p(x) = \infty.$$

By (Karatzas & Shreve, 1991) Proposition 5.22,

$$P(\tau_\infty = \infty) = 1,$$

where τ_∞ is defined as Subsection 3.1. That is, the solution of equation will never leave the state space $(0, \infty)$. Therefore, R_t is a true martingale. The proof of Theorem 1 is then complete.

4. Explicit Solution and Precise Distribution of r_t under Probability Q

Proof of Theorem 2 By Theorem 1, we know that $Y_t = f(r_t)$ satisfies the CIR model. By Itô's formula

$$\begin{aligned} d\sqrt{Y_t} &= \frac{1}{2\sqrt{Y_t}} \left[\left(\frac{\sigma^2 C^2}{4} + 2b(1-\gamma)Y_t \right) dt + \sigma C \sqrt{Y_t} d\bar{B}_t \right] - \frac{1}{8Y_t^{\frac{3}{2}}} \sigma^2 C^2 Y_t dt \\ &= b(1-\gamma) \sqrt{Y_t} dt + \frac{\sigma C}{2} d\bar{B}_t. \end{aligned}$$

So $\sqrt{Y_t}$ is an Ornstein-Uhlenbeck process, whose solution is

$$\sqrt{Y_t} = \sqrt{Y_0} e^{b(1-\gamma)t} + \frac{\sigma C}{2} \int_0^t e^{b(1-\gamma)(t-s)} d\bar{B}_s.$$

Here \sqrt{x} should be understood as a real number a such that $a^2 = x$, otherwise we will get a contradiction since the left hand side of the equation is non negative while the right hand side will be negative with positive probability.

Thus

$$r_t = \left| \frac{2(\gamma-1)}{C} \right|^{\frac{1}{1-\gamma}} Y_t^{\frac{1}{2(1-\gamma)}} = \left| r_0^{1-\gamma} e^{-(\gamma-1)bt} + \sigma|\gamma-1| \int_0^t e^{-b(\gamma-1)(t-s)} d\bar{B}_s \right|^{\frac{1}{1-\gamma}}.$$

We complete the proof.

Proof of Theorem 3 Define $L = \frac{\sigma^2 C^2}{8b(\gamma-1)}(1 - e^{2b(1-\gamma)t})$. By (Lamberton & Lapeyre, 1996) Proposition 6.2.5 with $\mu = 0$, we know that Y_t/L satisfies the non-central chi-square law with δ degrees of freedom and parameter ζ , where

$$\delta = C^2, \zeta = \frac{8Y_0b(\gamma-1)}{\sigma^2 C^2(e^{2b(\gamma-1)t} - 1)}.$$

Thus the density of Y_t/L is given by

$$g_{\delta, \zeta}(x) = \frac{e^{-\zeta/2}}{2\zeta^{\delta/4-1/2}} e^{-x/2} x^{\delta/4-1/2} \left(\frac{\sqrt{x\zeta}}{2}\right)^{\delta/2-1} \sum_{n=0}^{\infty} \frac{(\sqrt{x\zeta}/2)^{2n}}{n! \Gamma(\delta/2 + n)}, \quad x > 0,$$

where $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$.

Therefore

$$g_t(x) = g_{\delta, \zeta} \left(\frac{f(x)}{L} \right) \left| \frac{f'(x)}{L} \right|, \quad x > 0.$$

We then complete the proof of Theorem 3.

Remark 2 Let us denote $P(t, T)$ the price of zero-coupon bond paying 1 dollar at a maturity date T . If we take the risk premium as $q(r_t)$, defined by (4), then Q is the so-called risk neutral probability. Thus,

$$\begin{aligned} P(t, u) &= \mathbb{E}^Q(e^{-\int_t^u r_s ds} | \mathcal{F}_t) = \mathbb{E}^Q(e^{-\int_0^{u-t} r_s ds}) \\ &= \mathbb{E}^P(e^{-\int_0^{u-t} \left| r_0^{1-\gamma} e^{-(\gamma-1)bs + \sigma[\gamma-1] \int_0^s e^{-b(\gamma-1)(s-l)} dB_l \right|^{\frac{1}{1-\gamma}} ds}). \end{aligned}$$

5. Moment Estimations of r_t

Proof of Theorem 4 By Itô's formula, it follows that

$$\begin{aligned} dr_t^{-2\gamma} &= (-2\gamma r_t^{-2\gamma-1}(a - br_t) + \frac{2\gamma(2\gamma+1)\sigma^2}{2} r_t^{-2})dt - 2\gamma\sigma r_t^{-\gamma-1} dB_t \\ &= -2a\gamma(r_t^{-2\gamma})^{1+\frac{1}{2\gamma}} dt + 2b\gamma r_t^{-2\gamma} dt + \gamma(2\gamma+1)\sigma^2(r_t^{-2\gamma})^{\frac{1}{\gamma}} dt + M_t. \end{aligned} \quad (10)$$

5.1 Case I: $\frac{1}{2} \leq \gamma < 1$

If $\frac{1}{2} \leq \gamma < 1$, since

$$(x^{-2\gamma})^{\frac{1}{\gamma}} \leq (x^{-2\gamma})^{1+\frac{1}{2\gamma}} + 1, \quad \forall x > 0,$$

we have

$$\mathbb{E}(r_t^{-2\gamma}) \leq r_0^{-2\gamma} + \gamma(2\gamma+1)\sigma^2 t + 2b\gamma \int_0^t \mathbb{E}(r_s^{-2\gamma}) ds$$

in case $(2\gamma+1)\sigma^2 \leq 2a$.

Gronwall's lemma yields that

$$\mathbb{E}(r_t^{-2\gamma}) \leq \Psi(t) + 2b\gamma \int_0^t \Psi(s) e^{2b\gamma(t-s)} ds,$$

where $\Psi(t) = r_0^{-2\gamma} + \gamma(2\gamma+1)\sigma^2 t$. It's clear that the right hand side of the inequality is locally integrable with respect to t . Therefore,

$$\mathbb{E} \int_0^t r_s^{-2\gamma} ds < \infty$$

holds for any $t > 0$.

On the other hand, by Itô's formula,

$$dr_t^{2(\gamma-1)} = \left(2(\gamma-1)r_t^{2\gamma-3}(a - br_t) + \frac{2(\gamma-1)(2\gamma-3)\sigma^2}{2} r_t^{4\gamma-4} \right) dt + 2(\gamma-1)\sigma r_t^{3\gamma-3} dB_t.$$

Notice that $2\gamma - 3 \leq 4\gamma - 4 < 0$, and if $(2\gamma + 1)\sigma^2 \leq 2a$, then

$$\frac{2(\gamma - 1)(2\gamma - 3)\sigma^2}{2} \leq 2a(1 - \gamma).$$

Thus $\mathbb{E}(r_t^{2(\gamma-1)}) \leq r_0^{2(\gamma-1)} + (\gamma - 1)(2\gamma - 3)\sigma^2 t + 2b(1 - \gamma) \int_0^t \mathbb{E}(r_s^{2(\gamma-1)}) ds$.

By using Gronwall's lemma again,

$$\mathbb{E}(r_t^{2(\gamma-1)}) \leq \tilde{\Psi}(t) + 2b(1 - \gamma) \int_0^t \tilde{\Psi}(s) e^{2b(1-\gamma)(t-s)} ds,$$

where $\tilde{\Psi}(t) = r_0^{2(\gamma-1)} + (\gamma - 1)(2\gamma - 3)\sigma^2 t$.

So we have proved Theorem 4 in case that $\frac{1}{2} \leq \gamma < 1$, $(2\gamma + 1)\sigma^2 \leq 2a$.

5.2 Case II: $1 < \gamma < \frac{3}{2}$

Note that $0 < \frac{1}{\gamma} < 1$ in this case. Since

$$x^{\frac{1}{\gamma}} \leq x + 1, \quad \forall x > 0,$$

by (10) we have

$$\mathbb{E}(r_t^{-2\gamma}) \leq r_0^{-2\gamma} + \gamma(2\gamma + 1)\sigma^2 t + \gamma(2b + (2\gamma + 1)\sigma^2) \int_0^t \mathbb{E}(r_s^{-\gamma}) ds.$$

Gronwall's lemma yields that

$$\mathbb{E}(r_t^{-2\gamma}) \leq \Psi(t) + \gamma(2b + (2\gamma + 1)\sigma^2) \int_0^t \Psi(s) e^{\gamma(2b+(2\gamma+1)\sigma^2)(t-s)} ds,$$

where $\Psi(t) = r_0^{-2\gamma} + \gamma(2\gamma + 1)\sigma^2 t$. Then

$$\mathbb{E} \int_0^t r_s^{-2\gamma} ds < \infty$$

holds for any $t > 0$.

On the other hand, by (1),

$$\mathbb{E}(r_t) = r_0 + at - b \int_0^t \mathbb{E}(r_s) ds.$$

Thus $\mathbb{E}(r_t) = \frac{a}{b} + (r_0 - \frac{a}{b})e^{-bt}$. Since $1 < \gamma \leq \frac{3}{2}$,

$$\mathbb{E}(r_t^{2(\gamma-1)}) \leq 1 + \mathbb{E}(r_t) = 1 + \frac{a}{b} + (r_0 - \frac{a}{b})e^{-bt}.$$

We complete the proof of Theorem 4.

From this, we know that $\mathbb{E} \int_0^t q(r_s)^2 ds < \infty$ for any $t > 0$ if $\frac{1}{2} \leq \gamma < 1$ and $(2\gamma + 1)\sigma^2 \leq 2a$, or $1 < \gamma < \frac{3}{2}$. Therefore, we have proved that $\int_0^t q(r_s) dB_s$ is a true martingale.

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