# Random Variables Fundamental in Probability and $\Sigma$ -Complete Convergence

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# Abstract

The aim of this paper is to study some necessary and sufficient conditions for fundamental (Cauchy) in probability sequences of random variables. In this way, we will be able to deduce some relationships between certain types of convergence and these sequences of random variables characterized because in their definition the random variable limit does not appear. Finally, we introduce the concept of a  $\Sigma$ -completely convergent sequence and a sufficient condition for it.

Keywords: fundamental in probability, Cauchy in probability, convergence in probability,  $\Sigma$ -complete convergence

### 1. Introduction

In the framework of convergence of random variables, there are some concepts which do not involve the limit of such sequences, viz (Sokol & Rønn-Nielsen, 2013, p. 14):

- (i)  $\{X_t\}_{t=1}^{\infty}$  is Cauchy in probability if for every  $\epsilon > 0$ ,  $P[|X_r X_s| > \epsilon]$  tends to 0 as r and s tend to infinity.
- (ii)  $\{X_t\}_{t=1}^{\infty}$  is almost surely Cauchy if  $P\left[\{X_t\}_{t=1}^{\infty} \text{ is Cauchy}\right] = 1$ .
- (iii)  $\{X_t\}_{t=1}^{\infty}$  is Cauchy in  $L^p$   $(p \ge 1)$  if  $E[|X_r X_s|^p]$  tends to 0 as r and s tend to infinity.

It is well-known that Cauchy with respect to these three modes of convergence are convergent according to their respective modes. Nevertheless, in this paper we will focus on the Cauchy in probability sequences and we will study their convergence not in a classical way but introducing a novel mode of convergence.

In effect, in modern Finance the most usual mode of convergence is almost sure convergence but in the characterization of certain financial parameters it is necessary to turn to stronger modes of convergence and divergence (Montrucchio, 2004, pp. 645-663; Montrucchio & Privileggi, 2001, pp. 158-188; Cruz Rambaud, 2013, pp. 306-327). Thus, this paper introduces a mode of convergence more restricted than almost sure and in probability convergence, the so-called  $\Sigma$ -complete convergence. Consequently, although some results in this work are well-known, the main novelty is the provided methodology using certain  $\Sigma$ -completely convergent subsequences.

The organization of this paper is as follows. Section 2 presents the concept of a fundamental (Cauchy) sequence of random variables from three points of view, demonstrating that all definitions are equivalent (Lemma 1). Moreover, Corollary 3 shows that a sequence of random variables is fundamental in probability if and only if it is convergent in probability. Despite this is a well-known result, all our approach is around the  $\Sigma$ -complete convergence of a subsequence (necessary condition: Theorem 1) or the entire sequence (sufficient condition: Theorem 2). Section 3

formally introduces the concept of complete and  $\Sigma$ -complete convergence, and provides a sufficient condition for a subsequence being  $\Sigma$ -completely convergent to 0, involving the sequence of the partial sums of the series. Finally, Section 4 summarizes and concludes.

## 2. Fundamental (Cauchy) in Probability Sequences

**Definition 1** (Billingsley, 1995, p. 272) A sequence  $\{X_t\}_{t=1}^{\infty}$  of random variables is said to be *fundamental in probability* if for every  $\epsilon > 0$  there exists a  $t_{\epsilon}$  such that  $P[|X_r - X_s| > \epsilon] < \epsilon$ , for  $r, s \ge t_{\epsilon}$ .

Nevertheless, Resnick (1999, p. 171) gives the following

**Definition 2** A sequence  $\{X_t\}_{t=1}^{\infty}$  of random variables is said to be *Cauchy in probability* if for every  $\epsilon > 0$  and  $\delta > 0$  there exists a  $t_{\epsilon,\delta}$  such that  $P[|X_r - X_s| > \epsilon] < \delta$ , for  $r, s \ge t_{\epsilon,\delta}$ .

On the other hand, let X be the set of the random variables over a probability space  $(\Omega, \mathcal{F}, P)$ . If  $X, Y \in X$ , the distance between X and Y, d(X, Y) (Lukács, 1975, p. 62), is defined as the greatest lower bound of all x > 0 such that

$$\frac{P[|X - Y| > x]}{x} < 1$$

This distance is a semimetric (Aliprantis & Border, 2006, p. 23) because it is possible that d(X, Y) = 0 but  $X \neq Y$ . In effect, this is the case of two equivalent random variables. Theorem 3.1.2 in Lukács (1975, p. 64) shows that X is complete and so it gives rise to a Fréchet metric space (Rudin, 1991, p. 33).

Firstly, we will demonstrate that the similar definitions 1 and 2, and the property for the sequence of random variables to be fundamental (Cauchy) in the Fréchet metric space are in fact equivalent.

Lemma 1 The following three conditions are equivalent:

- (i)  $\{X_t\}_{t=1}^{\infty}$  is fundamental in probability.
- (*ii*)  $\{X_t\}_{t=1}^{\infty}$  is Cauchy in probability.

(iii)  $\{X_t\}_{t=1}^{\infty}$  is fundamental (Cauchy) in the Fréchet metric space X, that is, for every  $\epsilon > 0$  there exists a  $t_{\epsilon}$  such that  $d(X_r, X_s) \leq \epsilon$ , for  $r, s \geq t_{\epsilon}$ .

*Proof.* (i)  $\Rightarrow$  (ii). If  $\{X_t\}_{t=1}^{\infty}$  is fundamental in probability, for every  $\epsilon > 0$  and  $\delta > 0$ :

- There exists a  $t_{\epsilon}$  such that  $P[|X_r X_s| > \epsilon] < \epsilon$ , for  $r, s \ge t_{\epsilon}$ .
- There exists a  $t_{\delta}$  such that  $P[|X_r X_s| > \delta] < \delta$ , for  $r, s \ge t_{\delta}$ .

There are three possibilities:

(1)  $\epsilon = \delta$ , in whose case the conclusion is obvious:  $t_{\epsilon,\delta} := t_{\epsilon} = t_{\delta}$ .

- (2)  $\epsilon < \delta$ , in whose case we can take  $t_{\epsilon,\delta} := t_{\epsilon}$ . Therefore, for  $r, s \ge t_{\epsilon}$ ,  $P[|X_r X_s| > \epsilon] < \epsilon < \delta$ .
- (3)  $\epsilon > \delta$ , in whose case we can take  $t_{\epsilon,\delta} := t_{\delta}$ . Therefore, taking into account that

$$[|X_r - X_s| > \epsilon] \subseteq [|X_r - X_s| > \delta],$$

for  $r, s \ge t_{\delta}$ , one has

$$P[|X_r - X_s| > \epsilon] \le P[|X_r - X_s| > \delta] < \delta.$$

(ii)  $\Rightarrow$  (iii). Let  $\epsilon$  be any positive real number. By (ii), there exists a  $t_{\epsilon,\epsilon} := t_{\epsilon}$  such that

$$P[|X_r - X_s| > \epsilon] < \epsilon,$$

for  $r, s \ge t_{\epsilon}$ . Therefore,

$$d(X_r, X_s) = \inf\left\{x : \frac{P[|X_r - X_s| > x]}{x}\right\} \le \epsilon$$

and (iii) holds.

(iii)  $\Rightarrow$  (i). Let  $\epsilon$  be any positive real number. By (iii), there exists a  $t_{\epsilon}$  such that

$$d(X_r, X_s) \leq \epsilon,$$

for  $r, s \ge t_{\epsilon}$ . By the definition of the distance, there exists a  $\delta$  ( $0 < \delta < \epsilon$ ) such that

$$d(X_r, X_s) \le \frac{P[|X_r - X_s| > \delta]}{\delta} < 1.$$

Observe that

$$\frac{P[|X_r - X_s| > \epsilon]}{\epsilon} < \frac{P[|X_r - X_s| > \delta]}{\delta} < 1,$$

that is,

 $P[|X_r - X_s| > \epsilon] < \epsilon$ 

and so  $\{X_t\}_{t=1}^{\infty}$  is fundamental in probability.

In what follows, we will indistinctly use the three definitions.

**Theorem 1** Let  $\{X_t\}_{t=1}^{\infty}$  be a sequence of random variables. If  $\{X_t\}_{t=1}^{\infty}$  is fundamental in probability, for any summable sequence of positive numbers  $\{\epsilon_k\}_{k=1}^{\infty}$ , there exists a subsequence  $\{X_{t_k}\}_{k=1}^{\infty}$  such that  $\sum_{k=1}^{\infty} P[|X_{t_{k+1}} - X_{t_k}| > \epsilon_k] < \infty$ .

*Proof.* Let  $\{\epsilon_t\}_{t=1}^{\infty}$  be an arbitrary summable sequence of positive real numbers. As  $\{X_t\}_{t=1}^{\infty}$  is fundamental in probability, for  $\epsilon_1$  there exists a  $t_{\epsilon_1}$  such that  $P[|X_r - X_s| > \epsilon_1] < \epsilon_1$ , for  $r, s \ge t_{\epsilon_1}$ . Take  $t_1 := t_{\epsilon_1}$ . Therefore,  $P[|X_r - X_{t_1}| > \epsilon_1] < \epsilon_1$ , for  $r \ge t_1$ . Now for  $\epsilon_2$  there exists a  $t_{\epsilon_2}$  such that  $P[|X_r - X_s| > \epsilon_2] < \epsilon_2$ , for  $r, s \ge t_{\epsilon_2}$ . Take  $t_2 := \max\{t_{\epsilon_2}, t_1 + 1\}$ . Thus,  $P[|X_r - X_{t_{\epsilon_2}}| > \epsilon_2] < \epsilon_2$ , for  $r \ge t_{\epsilon_2}$  and  $P[|X_{t_2} - X_{t_1}| > \epsilon_1] < \epsilon_1$ . Next for  $\epsilon_3$  there exists a  $t_{\epsilon_3}$  such that  $P[|X_r - X_s| > \epsilon_3] < \epsilon_3$ , for  $r, s \ge t_{\epsilon_3}$ . Take  $t_3 := \max\{t_{\epsilon_3}, t_2 + 1\}$ . Thus,  $P[|X_r - X_{t_{\epsilon_3}}| > \epsilon_3] < \epsilon_3$ , for  $r \ge t_{\epsilon_3}$  and  $P[|X_{t_3} - X_{t_2}| > \epsilon_2] < \epsilon_2$ . We can continue this process infinitely and obviously  $\sum_{k=1}^{\infty} P[|X_{t_{k+1}} - X_{t_k}| > \epsilon_k] < \infty$  holds.

**Corollary 1** Let  $\{X_t\}_{t=1}^{\infty}$  be a sequence of random variables. If  $\{X_t\}_{t=1}^{\infty}$  is fundamental in probability, there exists a random variable X and a subsequence  $\{X_{t_k}\}_{k=1}^{\infty}$  such that  $X_{t_k} \to X$  almost surely.

*Proof.* It is an immediate consequence of the former Theorem 1 and Theorem 4.2.2 in Lukács (1975, pp. 81-82).  $\Box$ 

**Corollary 2** Let  $\{X_t\}_{t=1}^{\infty}$  be a sequence of random variables. If  $\{X_t\}_{t=1}^{\infty}$  is fundamental in probability, then it converges in probability to a random variable X.

*Proof.* In effect, if  $\{X_t\}_{t=1}^{\infty}$  is fundamental in probability, each subsequence  $\{X_{t_k}\}_{k=1}^{\infty}$  is also fundamental in probability. By Corollary 1, each subsequence contains another subsequence which converges almost surely to a random variable X. In these circumstances, Theorem 2.4.4 in Lukács (1975, p. 49) applies.

**Corollary 3** Let  $\{X_t\}_{t=1}^{\infty}$  be a sequence of random variables.  $\{X_t\}_{t=1}^{\infty}$  is fundamental in probability if and only if it converges in probability to a random variable X.

*Proof.* In effect, assume that  $\{X_t\}_{t=1}^{\infty}$  converges in probability to a random variable X and consider an arbitrary  $\epsilon > 0$  and  $\delta > 0$ . By hypothesis, for  $\frac{\epsilon}{2}$  and  $\frac{\delta}{2}$  there exists a  $t_{\epsilon,\delta}$  such that  $P[|X_t - X| > \frac{\epsilon}{2}] < \frac{\delta}{2}$ , for  $t \ge t_{\epsilon,\delta}$ . Thus, taking into account that:

$$[|X_r - X_s| > \epsilon] \subseteq \left[|X_r - X| > \frac{\epsilon}{2}\right] \cup \left[|X_s - X| > \frac{\epsilon}{2}\right],$$

for  $r, s \ge t_{\epsilon,\delta}$ , one has

 $P[|X_r - X_s| > \epsilon] \le P\left[|X_r - X_s| > \frac{\epsilon}{2}\right] + P\left[|X_r - X_s| > \frac{\epsilon}{2}\right] < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$ 

The rest of the proof is Corollary 2.

*Remark* An easier and faster proof of corollaries 2 and 3 could be considered taking into account that the space X is complete with respect to the Fréchet metric introduced in this Section.

**Theorem 2** Let  $\{X_t\}_{t=1}^{\infty}$  be a sequence of random variables. A sufficient condition for  $\{X_t\}_{t=1}^{\infty}$  being fundamental in probability is that there exists a sequence of positive real numbers  $\{\epsilon_t\}_{t=1}^{\infty}$  such that

$$\sum_{t=1}^{\infty} \epsilon_t < \infty \text{ and } \sum_{t=1}^{\infty} P[|X_{t+1} - X_t| > \epsilon_t] < \infty.$$

*Proof.* Let be  $\epsilon > 0$ . As  $\lim_{t\to\infty} \sum_{k=t}^{\infty} \epsilon_k = 0$ , there exists a  $t_0$  such that, for every  $t \ge t_0$ ,  $\sum_{k=t}^{\infty} \epsilon_k < \epsilon$ . Let r and s be two positive integer numbers such that  $r, s \ge t_0$ . There is not loss of generality in considering r < s. Obviously,  $\sum_{k=s}^{r-1} \epsilon_k < \sum_{k=t_0}^{\infty} \epsilon_k$ . Therefore,

$$P[|X_r - X_s| > \epsilon] < P\left[|X_r - X_s| > \sum_{k=s}^{r-1} \epsilon_k\right] < \sum_{k=s}^{r-1} P[|X_{k+1} - X_k| > \epsilon_k].$$

As  $\lim_{t\to\infty} \sum_{k=t}^{\infty} P[|X_{k+1} - X_k| > \epsilon_k] = 0$ , there exists a  $t_1$  such that, for every  $t \ge t_1$ ,  $\sum_{k=t}^{\infty} P[|X_{k+1} - X_k| > \epsilon_k] < \epsilon$ . Therefore, for every r and s such that  $r, s \ge \max\{t_0, t_1\} := t_{\epsilon}$ ,

$$\sum_{k=s}^{r-1} P[|X_{k+1} - X_k| > \epsilon_k] < \sum_{k=t_{\epsilon}}^{\infty} P[|X_{k+1} - X_k| > \epsilon_k] < \epsilon$$

and so  $\{X_t\}_{t=1}^{\infty}$  is fundamental in probability.

### **3.** Σ-Completely Convergent Sequences

**Definition 3** A sequence of random variables  $\{X_t\}_{t=1}^{\infty}$  is said to be *completely convergent* (Note 1) to X if it satisfies any of the three following equivalent conditions:

- (i) For every  $\epsilon > 0$ ,  $\sum_{t=1}^{\infty} P[|X_t X| > \epsilon] < \infty$ .
- (ii) For every  $\epsilon > 0$ ,  $\lim_{t\to\infty} \sum_{k=t}^{\infty} P[|X_k X| > \epsilon] = 0$ .
- (iii) There exists a sequence of positive real numbers  $\{\epsilon_t\}_{t=1}^{\infty}$  such that

$$\lim_{t\to\infty}\epsilon_t=0 \text{ and } \sum_{t=1}^{\infty}P\left[|X_t-X|>\epsilon_t\right]<\infty.$$

These three conditions are equivalent by virtue of Proposition 2 in Cruz Rambaud (2011, pp. 215-221). Another definition very close to the former one is the following (see Cruz Rambaud & Rodríguez López-Cañizares, 2012, pp. 35-42).

**Definition 4** A sequence of random variables  $\{X_t\}_{t=1}^{\infty}$  is said to be  $\Sigma$ -completely convergent to X (denoted by  $X_t \xrightarrow{\Sigma-c} X$ ) if there exists a sequence of positive real numbers  $\{\epsilon_t\}_{t=1}^{\infty}$  such that

$$\sum_{t=1}^{\infty} \epsilon_t < \infty \text{ and } \sum_{t=1}^{\infty} P\left[|X_t - X| > \epsilon_t\right] < \infty.$$

This definition is not equivalent to that of the convergence in probability because Cruz Rambaud and Rodríguez López-Cañizares (2012, pp. 35-42) provide an example in which they demonstrate that complete does not imply  $\Sigma$ -complete convergence. Finally, taking into account that in general convergence in probability does not imply complete convergence, we can deduce that definition 4 is more restricted than convergence in probability. Let be  $S_t = \sum_{k=1}^t X_k$ .

**Theorem 3** Let  $\{X_t\}_{t=1}^{\infty}$  be a sequence of positive random variables. If  $\{S_t\}_{t=1}^{\infty}$  is almost surely convergent to S then there exists a subsequence of  $\{X_t\}_{t=1}^{\infty}$  which is  $\Sigma$ -completely convergent to 0.

*Proof.* Assume that  $\{S_t\}_{t=1}^{\infty}$  is almost surely convergent to *S*. By Egoroff's Theorem (Billingsley, 1995, p. 187), for every  $\epsilon > 0$ , there exists an  $A_{\epsilon}$ , such that  $P(A_{\epsilon}) < \epsilon$  and  $\{S_t\}_{t=1}^{\infty}$  is uniformly convergent on  $A_{\epsilon}^c$ . If  $\{S_t\}_{t=1}^{\infty}$  is uniformly convergent on  $A_{\epsilon}$ ,  $\{S_t\}_{t=1}^{\infty}$  would be uniformly convergent on  $\Omega$  and so the conclusion holds. Therefore, we can suppose that  $\{S_t\}_{t=1}^{\infty}$  is not uniformly convergent on  $A_{\epsilon}$ . Consequently, there exists a  $\delta_{\epsilon} > 0$  such that, for every *r*, we can find a  $t(r, \epsilon) \ge r$  and an  $\omega(r, \epsilon) \in A_{\epsilon}$ , verifying

$$|S_{t(r,\epsilon)}(\omega(r,\epsilon)) - S(\omega(r,\epsilon))| > \delta_{\epsilon},$$

or equivalently,

$$\sum_{k=t(r,\epsilon)+1}^{\infty} X_k(\omega(r,\epsilon)) > \delta_\epsilon$$

Now, take a sequence  $\{\epsilon_s\}_{s=1}^{\infty}$  such that  $\sum_{s=1}^{\infty} \epsilon_s < \infty$ . Without loss of generality, we can choose  $\delta_{\epsilon_s} < \epsilon_s$ , for every *s*. Since in general

$$P[|S_t - S_{t-1}| > 2\delta_{\epsilon}] \le P[|S_t - S| > \delta_{\epsilon}] + P[|S_{t-1} - S| > \delta_{\epsilon}] < \epsilon + \epsilon = 2\epsilon$$

and

$$P[|S_t - S_{t-1}| > 2\delta_{\epsilon}] = P[|X_t| > 2\delta_{\epsilon}],$$

it remains:

$$P[|X_t| > 2\delta_{\epsilon}] < 2\epsilon.$$

Thus, we can deduce that:

$$\sum_{s=1}^{\infty} P[|X_{t(t_{s-1},\epsilon_s)}| > 2\epsilon_s] \le \sum_{s=1}^{\infty} P[|X_{t(t_{s-1},\epsilon_s)}| > 2\delta_{\epsilon_s}] < 2\sum_{s=1}^{\infty} \epsilon_s < \infty$$

and so it can be deduced that  $\{X_{t(t_{s-1},\epsilon_s)}\}_{s=1}^{\infty}$  is a subsequence of  $\{X_t\}_{t=1}^{\infty}$  which is  $\Sigma$ -completely convergent to 0. Figure 1 summarizes the modes of convergence studied in this paper.



Figure 1. Implications among modes of convergence

where each arrow means one convergence concept implies the other, and:

- "a.s." means "converges almost surely",
- "f.p." means "fundamental in probability",
- "P" means "converges in probability", and

• "Theorem 2" means that this result provides a sufficient condition for a sequence being fundamental in probability.

## 4. Conclusion

In this paper we have provided three equivalent definitions of the so-called fundamental (or Cauchy) in probability sequences of random variables, involving Fréchet metric spaces (Lemma 1). The main contribution of the first section in this paper is that a sequence of random variables is fundamental in probability if and only if it is convergent in probability (Corollary 3), involving  $\Sigma$ -completely convergent subsequences of  $\{X_t\}_{t=1}^{\infty}$ . Despite this result is well-known, our aim has been to demonstrate it from a classic point of view (around the concept of  $\Sigma$ -complete convergence) and starting from the defined Fréchet metric space X. The second concept presented in this paper is the corresponding to a  $\Sigma$ -completely convergent sequence of random variables which is a restriction of a completely convergent sequence. Theorem 3 provides a sufficient condition for a subsequence being  $\Sigma$ -completely convergent to 0.

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### Notes

Note 1. Some scholars use the term *complete convergence* to describe weak convergence (Billingsley, 1995, p. 328; Resnick, 1999, p. 250). Dugué (1957, pp. 127-138) uses the term *almost complete convergence*.

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