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Marshall-Olkin Extended Burr Type XII Distribution

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Abstract

Marshall and Olkin (1997) proposed a new method to establish more flexible new families of distributions by adding a parameter to a distribution. In this article, Marshall-Olkin extended Burr type XII (MOEBXII) distribution is introduced. Properties of MOEBXII distribution are studied and analyzed. Based on complete sample, maximum likelihood and Bayesian estimators of the parameters are derived. Application to a real data set is carried out to illustrate the flexibility of the model.

Keywords: Marshall-Olkin extended Burr type XII (MOEBXII) distribution, maximum likelihood estimators, Bayesian estimators

1. Introduction

Marshall and Olkin (1997) introduced a new family of distributions by adding a parameter to obtain new families of distributions which are more flexible and represent a wide range of behavior than the original distributions. Burr type XII distribution plays major role in the analyses of lifetime and survival data. Shao et al. (2004) studied the models for extended three parameter of Burr type XII distribution and used this distribution to model extreme event with application to flood frequency. The flexibility of Burr type XII distribution has been studied by Rodriguez (1977). The Burr type XII has been widely used in various fields of sciences, such as in actuarial science, forestry, ecoloxicology, reliability and survival analysis.

Many researchers used the Marshall-Olkin method to propose new distributions and studied their properties and parameter estimation. These include Marshall-Olkin extended Weibull which can be obtained as a compound distribution with mixing exponential distribution (Ghitany et al., 2005), Marshall-Olkin extended Pareto (Ghitany, 2005), Marshall-Olkin extended gamma (Ristic et al., 2007), Marshall-Olkin extended Lomax using censored data (Ghitany et al., 2007), Marshall-Olkin extended exponential distribution (Srivastava et al., 2011), and Marshall-Olkin extend Uniform distribution (Jose & Krishna, 2011). Moreover, the reliability properties of the extended linear failure-rate distributions were studied by Ghitany and Kotz (2007). Jayakumar and Mathew (2008) proposed a method based on adding two parameters in to a family distribution and considered as generalization to the method suggested by Marshall and Olkin (1997). Gupta et al. (2010) estimated the reliability from Marshall-Olkin extended Lomax distribution. Gupta et al. (2010) studied the effect of the tilt parameter on the monotnicity of the failure rate and estimated the turning points of the failure rate of the extended Weibull distribution. Srivastava and Kumar (2011) estimated the two-parameter of the Marshall-Olkin extended Weibull using two method, maximum likelihood estimate and Bayes estimate and compared them. Gui (2013) introduced Marshall-Olkin power log-normal distribution and studied its statistical properties of the new distribution. Cordeiro and Lemonte (2013) studied some mathematical properties of Marshall-Olkin extended weibull distribution which was introduced by Marshall and Olkin (1997). Also, they determined the moments of the order statistics and discussed the estimation of the parameters using maximum likelihood method.

The main purpose of this article is to extend Burr Type XII to a three parameters distribution which can be more effective in lifetime applications.

This article will be organized as follows: In Section 2, the Marshall-Olkin Extended Burr Type XII (MOEBXII) distribution is introduced and some of its properties are studied theory and along with a graphical description.

In Section 3, two methods of estimation are discussed: The maximum likelihood and the Bayesian methods. Application to a real data set is performed in Section 4. Finally, concluding remarks are given in Section 5.

2. Marshall-Olkin Extended Burr Type XII Distribution and Some Properties

Marshall and Olkin (1997) have suggested new families of distributions which count on adding a parameter to a family of distributions and called them extended distributions. Using the survival function of any distribution, $\bar{F}(x)$, then the survival function of the new distribution $\bar{G}(x;\alpha)$ is as follows:

$$\bar{G}(x;a) = \frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}; -\infty < x < \infty, 0 < \alpha < \infty \tag{1}$$

where, $\alpha = 1 - \bar{\alpha}$ is an additional positive parameter and they have called it the tilt parameter. The cumulative distribution function (CDF) and the probability density function (PDF) for the new distribution are given by:

$$G(x;\alpha) = \frac{F(x)}{1-\bar{\alpha}\bar{F}(x)} = 1 - \frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)} \tag{2}$$

and

$$g(x;\alpha) = \frac{\alpha f(x)}{\left[1-\bar{\alpha}\bar{F}(x)\right]^2} \tag{3}$$

where, $F(x)$ is the CDF of any distribution.

The relationship between the hazard rate of the original distribution $h(x)$ and the hazard rate of the Marshall-Olkin extended distribution $r(x;\alpha)$ is given as:

$$r(x;\alpha) = \frac{h(x)}{1-\bar{\alpha}\bar{F}(x)} \tag{4}$$

The CDF for the MOEBXII distribution is obtained from (2) as:

$$G(x;\alpha,c,k) = \frac{1-(1+x^c)^{-k}}{1-(1-\alpha)(1+x^c)^{-k}}; \tag{5}$$

where x, α, c and k > 0.

The PDF associated with (3) is given by:

$$g(x;\alpha,c,k) = \frac{\alpha ckx^{c-1}(1+x^c)^{-(k+1)}}{[1-(1-\alpha)(1+x^c)^{-k}]^2}; \tag{6}$$

where $x > 0$, α,c and $k > 0$.

If X is a random variable with density (6), then the MOEBXII distribution will be denoted as $X \sim$ MOEBXII (α, c, k).

The following distributions are sub-models of the MOEBXII distribution:

(i) When$\alpha = 1$, the MOEBXII becomes BurrXII distribution with two parameters c and k.

(ii) When $c = 1$, the MOEBXII becomes the Marshall - Olkin extended Lomax distribution (MOEL) with PDF given as:

$$g(x;\alpha,k) = \frac{\alpha k(1+x)^{-(k+1)}}{[1-(1-\alpha)(1+x)^{-k}]^2} \tag{7}$$

The CDF and PDF of the MOEBXII (α, c, k), with different values of the parameters, are plotted in Figure 1 and Figure 2, which display the CDF and PDF when the new parameter α has different values (0.8, 3 and 5) with different values of c and k.

The survival function $\bar{G}(\cdot)$ and hazard rate $r(\cdot)$ of the distribution are respectively given by:

$$\bar{G}(x;\alpha,c,k) = \frac{\alpha(1+x^c)^{-k}}{[1-(1-\alpha)(1+x^c)^{-k}]} \tag{8}$$

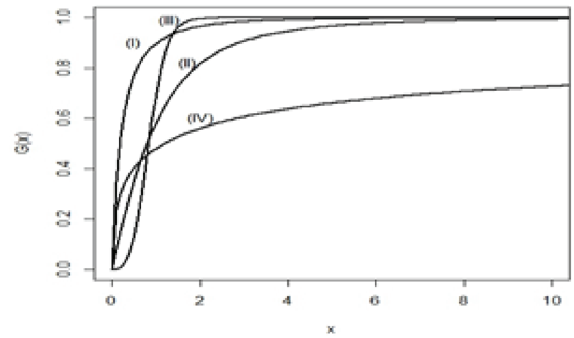

Figure 1. Plot of the CDF of the MOEBXII distribution. (I) $\alpha = 0.8$, $k = 3$, $c = 0.9$, (II) $\alpha = 5$, $k = 0.9$, $c = 3$, (III) $\alpha = 3$, $k = 3$, $c = 3$, (IV) $\alpha = 0.8$, $k = 0.8$, $c = 0.5$

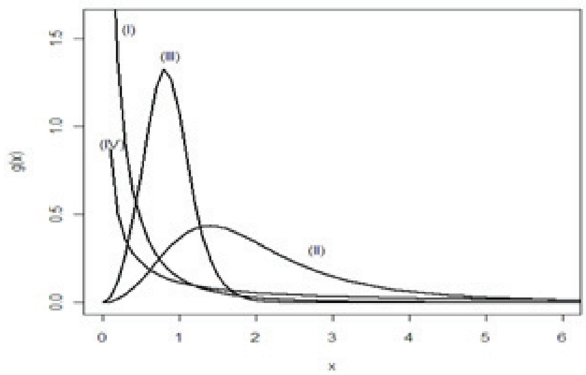

Figure 2. Plot of the PDF of the MOEBXII distribution. (I) $\alpha = 0.8$, $k = 3$, $c = 0.9$, (II) $\alpha = 5$, $k = 0.9$, $c = 3$, (III) $\alpha = 3$, $k = 3$, $c = 3$, (IV) $\alpha = 0.8$, $k = 0.8$, $c = 0.5$

$$r(x; \alpha, c, k) = \frac{kcx^{c-1}}{\left[1 - (1-\alpha)(1+x^c)^{-k}\right](1+x^c)} \tag{9}$$

From (9), it can be verified that:

(i) For $c, k < 1$ or $c < 1, k > 1$ with different values of α, the $r(x; \alpha, c, k)$ is monotone decreasing (L-shaped).

(ii) For $c, k > 1$ or $c > 1, k < 1$ with different values of α, the $r(x; \alpha, c, k)$ increases to maximum and then decreases rapidly, and this is more clear when $\alpha \leq 1$ (see Figure 3).

According to Marshall and Olkin (1997), the relationship between the hazard rate function and survival function of extended distributions with the original distributions depending on range of the new parameter $\alpha > 1$ or $0 < \alpha < 1$.

For $0 < \alpha < 1$, it can be shown that

$$kcx^{c-1}(1+x^c)^{-1} \leq r(x; \alpha, c, k) \leq \left(kcx^{c-1}(1+x^c)^{-1}\right)/\alpha, x > 0 \tag{10}$$

$$(1+x^c)^{-\frac{k}{\alpha}} \leq \bar{G}(x; \alpha, c, k) \leq (1+x^c)^{-k}, x > 0 \tag{11}$$

While for $\alpha > 1$, it follows that

$$\left(kcx^{c-1}(1+x^c)^{-1}\right)/\alpha \leq r(x; \alpha, c, k) \leq kcx^{c-1}(1+x^c)^{-1}, x > 0 \tag{12}$$

$$(1+x^c)^{-k} \leq \bar{G}(x; \alpha, c, k) \leq (1+x^c)^{-\frac{k}{\alpha}}, x > 0 \tag{13}$$

Some authors obtained the relationship between $r(\cdot)$ and $\bar{G}(\cdot)$ of extended distributions with the original distributions depending on range of the new parameter such as Marhall-Olkin (1997) and Corderio, Lemonte, and Ortega (2013).

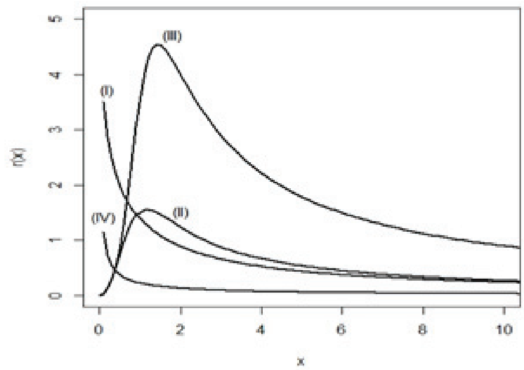

Figure 3. Plot of the hazard rate of the MOEBXII distribution. (I) $\alpha = 0.8$, $k = 3$, $c = 0.9$, (II) $\alpha = 0.8$, $k = 0.9$, $c = 3$, (III) $\alpha = 3$, $k = 3$, $c = 3$, (IV) $\alpha = 0.8$, $k = 0.8$, $c = 0.5$

The qth quantile of the MOEBXII is given by

$$x_q = G^{-1}(q) = \left[\left(1 + \frac{\alpha q}{1-q}\right)^{1/2} - 1\right]^{1/2} ; 0 \leq q < 1 \tag{14}$$

where $G^{-1}(\cdot)$ is the inverse distribution function. In particular, the median of the MOEBXII distribution is given by

$$median(x) = \left[(1+\alpha)^{1/k} - 1\right]^{1/c} \tag{15}$$

The mode for this distribution can be found by solving the following equation $d \log g(x; \alpha, c, k)/dx = 0$

$$\frac{d \log g(x; \alpha, c, k)}{dx} = (c-1)\frac{1}{x} - c(k+1)\frac{x^{c-1}}{1+x^c} - 2(1-\alpha)kc\frac{(1+x^c)^{-(k+1)}x^{c-1}}{\left[1-(1-\alpha)(1+x)^{-k}\right]} = 0 \tag{16}$$

Table 1 displays the mode values when $c > 1$ with selected values of k and α. This shows that the PDF of the distribution is unimodal under these specific values.

Table 1. The results of mode for MOEBXII with different selected values of parameters

Parameters	Mode
$k = 0.9, \alpha = 0.8, c = 1.5$	0.3086
$k = 1.2, \alpha = 1.2, c = 2$	0.5960

3. Estimation of the Parameters of MOEBXII Distribution

In this section, parameters estimation of the MOEBXII model is conducted using the maximum likelihood and Bayes methods.

3.1 Maximum Likelihood Estimation of MOEBXII Distribution

Let $\underline{X} = (X_1, X_2, ..., X_n)$ be a random sample of size n from MOEBXII (α, c, k), then the log-likelihood function $l(\alpha, c, k)$ can be written as:

$$\begin{aligned} l(a, c, k) &= n(\log a + \log c + \log k) + (c-1)\textstyle\sum_{i=1}^{n} \log(x_i) \\ & -(k+1)\textstyle\sum_{i=1}^{n} \log\left(j_1\left(x_i; c\right)\right) - 2\textstyle\sum_{i=1}^{n} \log\left(w_1\left(x_i; a, c, k\right)\right) \end{aligned} \tag{17}$$

where for $i = 1, 2, ..., n$,

$$\varphi_1(x_i; c) = (1 + x_i^c), \omega_1(x_i; \alpha, c, k) = 1 - (1-\alpha)(1 + x_i^c)^{-k} \tag{18}$$

Equation (17) can be maximized with respect to α, c and k. That is, solve the following three non-linear equations using iterative procedure:

$$\frac{\partial l(\hat{\alpha}, \hat{c}, \hat{k})}{\partial \hat{\alpha}} = \frac{n}{\hat{\alpha}} - 2\sum_{i=1}^{n} \frac{(\varphi_1(x_i; \hat{c}))^{-\hat{k}}}{\omega_1(x_i; \hat{\alpha}, \hat{c}, \hat{k})} = 0 \tag{19}$$

$$\frac{\partial l(\hat{\alpha},\hat{c},\hat{k})}{\partial\hat{c}} = \frac{n}{\hat{c}} + \sum_{i=1}^{n}\log(x_i) - (\hat{k}+1)\sum_{i=1}^{n}\frac{x_i^{\hat{c}}\log(x_i)}{\varphi_1(x_i;\hat{c})} - 2\hat{k}(1-\hat{\alpha})\sum_{i=1}^{n}\frac{x_i^{\hat{c}}\,(\varphi_1(x_i;\hat{c}))^{-(\hat{k}+1)}\log(x_i)}{\omega_1(x_i;\hat{\alpha},\hat{c},\hat{k})} = 0 \tag{20}$$

$$\frac{\partial l(\hat{\alpha},\hat{c},\hat{k})}{\partial\hat{k}} = \frac{n}{\hat{k}} - \sum_{i=1}^{n}\log(\varphi_1(x_i;\hat{c})) - 2(1-\hat{\alpha})\sum_{i=1}^{n}\frac{(\varphi_1(x_i;\hat{c}))^{-\hat{k}}\log(\varphi_1(x_i;\hat{c}))}{\omega_1(x_i;\hat{\alpha},\hat{c},\hat{k})} = 0 \tag{21}$$

It is clear that, the system of nonlinear Equations (19-21) does not have an analytic solution in α, c and k. Therefore, a numerical method is needed to obtain the solution. This problem is solved by using *nlminb* finction in R program.

The invariance property of MLE's can then be applied to obtain the ML estimators for $\bar{G}(\cdot)$ and $r(\cdot)$, which are given in (8) and (9), respectively, for some x_0.

The approximate confidence interval of the parameters $\underline{\theta}_1$ can be obtained based on the asymptotic distribution of the ML estimates of $\underline{\theta}_1$. Using the large sample and under appropriate regularity conditions, the ML estimates for the parameters $\hat{\underline{\theta}}_{1ML}$ have approximately multivariate normal distribution with mean $\underline{\theta}_1 = (\alpha, c, k)$ and asymptotic variance-covariance matrix which is equivalent to the inverse of Fisher information matrix. Then the $100(1-\gamma/2)\%$ approximate confidence interval of the parameters $\underline{\theta}_1 = (\alpha, c, k)$ are:

$$\hat{\alpha}_{ML} \pm z_{\gamma/2}\sqrt{var(\hat{\alpha}_{ML})}, \quad \hat{c}_{ML} \pm z_{\gamma/2}\sqrt{var(\hat{c}_{ML})} \quad \text{and} \quad \hat{k}_{ML} \pm z_{\gamma/2}\sqrt{var(\hat{k}_{ML})} \tag{22}$$

3.2 Bayesian Estimation of MOEBXII Distribution

Assume that the vector of unknown parameters $\underline{\theta}_1 = (\alpha, c, k)$ have independent prior distribution and that $\alpha \sim uniform(a_1, b_1)$, $c \sim uniform(a_2, b_2)$ and $k \sim uniform(a_3, b_3)$.

Then the joint posterior probability density function of α, c and k given $\underline{x}$ can written as:

$$\pi^*(\alpha, c, k \,|\, \underline{x}) = w\alpha^n c^n k^n \frac{\prod_{i=1}^{n}(x_i)^{c-1}\prod_{i=1}^{n}(\varphi_1(x_i;c))^{-(k+1)}}{\prod_{i=1}^{n}(\omega_1(x_i;\alpha,c,k))^2}\pi(\alpha,c,k) \tag{23}$$

where, $\pi(\alpha, c, k)$ is a joint prior distribution of the parameters, $\varphi_1(x_i; c)$, $\omega_1(x_i; \alpha, c, k)$ is defined in (18), and $\underline{w}$ is a normalizing constant.

Metro-Hastings function in R program is used to obtained the mean posterior estimates of the parameters $\underline{\theta}_1$ by Markov chain Monte Carlo techniques (MCMC).

4. Application on Electrical Insulating Data

In this section, an application to illustrate the of MOEBXII distribution using the data set of electrical insulating described in Lawless (2003) in which the length of time until "breakdown" is recorded. The data at 34 kilovolts with sample size $n = 19$ is used in this application.

4.1 Model Validation

To check the validity of the fitted model, we compute the Kolmogorov-Smirnov (K-S) statistics between the fitted distribution function when the parameters are obtained by ML method and the empirical distribution function which is displayed in Figure 4. The result of K-S test is $D = 0.2632$ with the corresponding *p-value* $= 0.1532$. Therefore, Figure 4 and K-S test indicate that MOEBXII model provides appropriate fit for this data set.

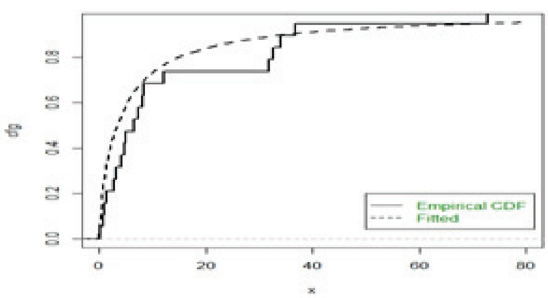

Figure 4. The plot for the fitted and empirical CDF of the MOEBXII distribution

Also, The K-S goodness of fit test is computed when the parameters are estimated by Bayesian method. The K-S test value is $D = 0.3158$ and the corresponding *p-value* = 0.3057. This indicates that the estimated model is appropriate for analyzing this data set.

4.2 Model Fitting

We fitted the same data set to demonstrate the flexibility of the MOEBXII distribution by comparing it with the original distribution BurrXII using maximum likelihood and Bayesian methods. Akaike information criterion (AIC) and Bayesian information criterion (BIC) are computed to compare between two distributions. The log likelihood, AIC, BIC of the estimate of the parameters using MLEs, and Baysian methods are presented in Table 2.

The results indicate that the MOEBXII distribution has the lowest AIC and BIC values compared the BurrXII distribution. Therefore, it can be chosen as it is more appropriate model.

Table 2. Maximum likelihood and Bayesian estimates with standard error and AIC, BIC of the BurrXII and MOEBXII distribution

Model	Maximum likelihood						Bayesian					
	$\hat{k}$	$\hat{c}$	$\widehat{\alpha}$	loglik	AIC	BIC	$\hat{k}$	$\hat{c}$	$\widehat{\alpha}$	loglik	AIC	BIC
Burr XII	1.25 (0.04)	0.60 (0.02)	-	-80.69	165.39	167.28 (0.07)	1.08 (0.01)	0.71	-	-78.60	161.20	163.09
MOEBX II	1.002 (0.09)	0.90 (0.05)	4 (0.05)	-69.77	145.55	148.38	1.17 (0.05)	0.81 (0.03)	3.77 (0.01)	-70.47	146.95	149.78

5. Concluding Remarks

In this paper, Marshall-Olkin extended Burr type XII distribution is introduced. Some of the statistical properties of the MOEBXII distribution are studied. Parameters estimation is obtained using MLE and Bayesian methods. A real data set is analyzed and has indicated that the MOEBXII model provides flexibility and better fit of the data.

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