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Measuring Average Rate of Return of Pensions: A Discrete, Stochastic and Continuous Price Index Approaches

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Abstract

In this paper the problem of the proper construction of the average rate of return (ARR) of pension (or investment) funds is considered, using a chain price index approach. Some known formulas of the ARR can be expressed by chain indices. The paper proposes and discusses a continuous-time formula. The prices and the number of the participating units are assumed to be continuous-time stochastic processes. Using the Ito theorem (Ito, 1951) it is proved that the relative change in net assets of funds equals a product of relative changes in unit prices and number of fund clients. Simulation study compares the discrete time formulas and the continuous formula in some illustrative case.

Keywords: investment funds, pension funds, average rate of return of funds, price index theory, continuous time stochastic model, Wiener process, chain indices, minimal rate of return

1. Introduction

Efficiency of pension or investment funds is measured in many different ways (Białek, 2008, 2009). These measures should be properly defined. Efficiency in this context is meant to indicate changes of fund assets connected with any investment. The information about the average return of a group of funds is very important both for fund clients and fund managers. Firstly, it allows to compare the financial outcome of the given fund to the rest of funds. It helps customers in making a decision about money allocation. Secondly, the average return of investment funds from different sectors (manufacturing, agricultural, service etc.) provides important information about the financial situation within these sectors. And thirdly, for pension funds, we can find legal regulations defining the *minimal rate of return* of funds based on the average rate of return. For example, in the Polish legal regulations (Polish Pension Reform Package, 1997) the minimal rate of return is defined as a half of the average return of a group of funds or the average return minus four percentage points (depending on which of these values is higher). In case of deficit the corresponding fund has to cover it. It is always a very dangerous situation for the fund (Note 1). Under the Polish law the average rate of return (ARR) of a group of pension funds is defined as follows:

$$\bar{r}_0(T_1, T_2) = \sum_{i=1}^{n} \frac{1}{2} r_i(T_1, T_2) \cdot \left(\frac{A_i(T_1)}{\sum_{i=1}^{n} A_i(T_1)} + \frac{A_i(T_2)}{\sum_{i=1}^{n} A_i(T_2)}\right) \tag{1}$$

where $r_i(T_1, T_2)$ denotes the rate of return of the i^{th} fund during a given time period $[T_1, T_2]$ and $A_i(t)$ denotes the value of i^{th} fund assets at time t. Since 2004 the results of funds for the last 36 months are verified twice a year. There are may arguments for searching new definitions of the ARR of a group of funds (Gajek & Kałuszka, 2000), our propositions for a discrete time can be found in (Białek, 2009). The paper is organized as follows: Section 2 gives the economic postulates for the ARR and presents some discrete formulas. It is shown that the discrete measures can be expressed by chain indices. Section 3 presents and discusses an analogical formula for the continuous time, where the prices and the number of the participating units are assumed to be continuous-time stochastic processes. Simulation study results are given in Section 4, where it is shown that presented discrete formulas and the continuous formula approximate each other.

2. Economic Postulates and Discrete Formulas of the ARR

At first sight the problem of constructing the ARR of funds seems to be straightforward. But if we look at postulates of Gajek and Kałuszka propose seven economic postulates (Białek, 2005) and they prove that the Polish measure violates four of these postulates. Moreover, Gajek and Kałuszka (2001) propose their own definition of the average rate of return of funds:

$$\bar{r}_{GK}(T_1, T_2) = \prod_{t=T_1}^{T_2-1}(1 + \sum_{i=1}^{n} A_i^*(t) r_i(t, t+1)) - 1, \tag{2}$$

where

$$A_i^*(t) = \frac{A_i(t)}{\sum_{i=1}^{n} A_i(t)} = \frac{p_i(t) q_i(t)}{\sum_{i=1}^{n} p_i(t) q_i(t)}, \tag{3}$$

and $p_i(t)$ denotes the value of the participation unit of the i^{th} fund at time t and $q_i(t)$ denotes the number of units of the i^{th} fund at time t. In this Section, a group of funds is considered as an aggregate containing n commodities (funds) with prices $p_i(t)$ and quantities $q_i(t)$, where $t \in [T_1, T_2]$. Let us denote by $P^L(t, t+1)$ the Laspeyres price index, where

$$P^L(t, t+1) = \frac{\sum_{i=1}^{n} q_i(t) p_i(t+1)}{\sum_{i=1}^{n} q_i(t) p_i(t)}, \tag{4}$$

and the logarithmic Laspeyres price index as follows (Lippe, 2007):

$$P^{LL}(t, t+1) = \prod_{i=1}^{n} (\frac{p_i(t+1)}{p_i(t)})^{A_i^*(t)}, \tag{5}$$

Equation (2) could be incorporated with Equation (5), using the Laspeyres chain index (Białek, 2012) as follows:

$$\prod_{t=T_1}^{T_2-1} P^L(t, t+1) - 1 = \prod_{t=T_1}^{T_2-1}(1 + \sum_{i=1}^{n} \frac{q_i(t) p_i(t)}{\sum_{i=1}^{n} q_i(t) p_i(t)} \cdot \frac{p_i(t+1) - p_i(t)}{p_i(t)}) - 1 = \bar{r}_{GK}(T_1, T_2). \tag{6}$$

Białek proposes another definition of the ARR, where $\bar{r}_B$ can be written using Equation (5) as follows (Białek, 2012):

$$\bar{r}_B(T_1, T_2) = \prod_{t=T_1}^{T_2-1} P^{LL}(t, t+1) - 1 = \prod_{t=T_1}^{T_2-1} \prod_{i=1}^{n} (\frac{p_i(t+1)}{p_i(t)})^{A_i^*(t)} - 1 = \prod_{t=T_1}^{T_2-1} \exp(\sum_{i=1}^{n} A_i^*(t) \ln \frac{p_i(t+1)}{p_i(t)}) - 1. \tag{7}$$

The two measures $\bar{r}_{GK}$ and $\bar{r}_B$ satisfy all the postulates from Gajek and Kałuszka, 2000). It can be shown that $\bar{r}_B(T_1, T_2) \leq \bar{r}_{GK}(T_1, T_2)$. Moreover, if $p_i(t+1) \approx p_i(t)$ for each $i$and $t \in [T_1, T_2]$, then $\bar{r}_B(T_1, T_2) \approx \bar{r}_{GK}(T_1, T_2)$. Gajek and Kałuszka claim that the Polish measure defined in (1) overestimates the real value of the average rate of return of funds. Gajek and Kałuszka consider not only the discrete stochastic model but they also propose continuous (deterministic and stochastic) measures (Gajek & Kałuszka, 2002). In the next Section, an original, stochastic and continuous measure of the average return is presented. It seems to be a natural next step in using the chain index theory for constructing the average rate of return of funds.

3. Continuous Time Stochastic Model

Let $\{p_i(t)\colon t \geq 0\}$ denote the stochastic process of the price of unit of i^{th} fund ($i = 1, 2, ..., n$) defined on a probability space $(\Omega, \Im, P)$ and let $\{q_i(t)\colon t \geq 0\}$ denote the stochastic process of the number of units of i^{th} fund defined on the same probability space. Let $F = \{\Im_t\colon t = 0, 1, 2, ...\}$ be a filtration, i.e. each $\Im_t$ is an algebra of Ω with $\Im_0 \subseteq \Im_s \subseteq \Im_t \subseteq \Im$ for any$s < t$. Without loss of generality, $\Im_0 = \{\emptyset, \Omega\}$ is assumed. The filtration F describes how the information about the market is revealed to the observer. Processes $p_i(t)$ and $q_i(t)$ are assumed to be progressively measurable with respect to the family $\{\Im_t\colon t \geq 0\}$. In practice, the price and quantity processes have positive values. Thus in finance, the processes of share prices are often described by the geometric Brownian (Wiener) motion (Note 2) (also known as exponential Brownian motion) as follows (Koo, 1998):

$$dp_i(t) = \alpha_i p_i(t) dt + \beta_i p_i(t) dW_i(t), i \in \{1, 2, ..., n\}, \tag{8}$$

where the percentage drift α_i and the percentage volatility β_i are constants, $W_i(t)$ denotes the standard Wiener process. For an arbitrary initial real value $p_i(0)$ the stochastic differential Equation (8) has the analytic solution (under Ito's interpretation: (Ito, 1951)),

$$p_i(t) = p_i(0) \exp((\alpha_i - \frac{\beta_i^2}{2})t + \beta_i W_i(t)), i \in \{1, 2, ..., n\}. \tag{9}$$

Thus the price processes described in (9) have always positive values and additionally $p_i(t)$ is log-normally distributed (Oksendal, 2002).

Let us assume that not only prices of units are described by the geometric Wiener process but also processes of number of units of funds are described as follows:

$$dq_i(t) = \gamma_i q_i(t)dt + \theta_i q_i(t)dW_i(t), i \in \{1, 2, ..., n\}, \tag{10}$$

where the percentage drift γ_i and the percentage volatility θ_i are constants. Hence,

$$q_i(t) = q_i(0)\exp((\gamma_i - \frac{\theta_i^2}{2})t + \theta_i W_i(t)), i \in \{1, 2, ..., n\}. \tag{11}$$

Under above assumptions the following definition of the ARR of a group of funds on a time interval $[T_1, T_2]$ is proposed:

$$\begin{aligned} R_P(T_1, T_2) = \quad & \exp[\textstyle\int_{T_1}^{T_2}(\sum_{i=1}^n A_i^*(t)\alpha_i + \frac{1}{2}\sum_{i=1}^n A_i^*(t)\beta_i\theta_i - \frac{1}{2}\sum_{i=1}^n (A_i^*(t))^2\beta_i^2 - \\ & -\textstyle\frac{1}{2}\sum_{i=1}^n (A_i^*(t))^2\beta_i\theta_i)dt + \sum_{i=1}^n \int_{T_1}^{T_2} A_i^*(t)\beta_i dW_i(t)] - 1, \end{aligned} \tag{12}$$

where, $A_i^*(t)$, α_i , β_i, θ_i, $W_i(t)$as defined above, and the integral on the right side of formula (12) is the Ito integral (Karatzas & Shreve, 1991). In Equation (12), reducing the random factor connected with the Wiener process to equal zero, i.e., $\beta_i(t) = 0$ and assuming $\alpha_i = dp_i(t)/dt$, we obtain,

$$R_P(T_1, T_2) = \exp(\int_{T_1}^{T_2}\sum_{i=1}^n A_i^*(t)\alpha_i dt) - 1 = \exp(\int_{T_1}^{T_2}\sum_{i=1}^n A_i^*(t)dp_i(t)) - 1 = P^{Div}(T_1, T_2) - 1, \tag{13}$$

where P^{Div} denotes the continuous Divisia price index (Banerjee, 1979; Hulten, 1973).

When $\beta_i(t) = 0$ Equation (9) becomes:

$$\frac{p_i(T_2)}{p_i(T_1)} = \exp[\alpha_i(T_2 - T_1)]. \tag{14}$$

Using Equation (14), Equation (13) is reduced to:

$$R_p(T_1, T_2) = \prod_{i=1}^n (\frac{p_i(T_2)}{p_i(T_1)})^{w_i} - 1 = P^{CD} - 1, \tag{15}$$

where,

$$w_i = \frac{\int_{T_1}^{T_2} A_i^*(t)dt}{T_2 - T_1},$$

and

$$\sum_{i=1}^n w_i = \frac{1}{T_2 - T_1}\sum_{i=1}^n \int_{T_1}^{T_2} A_i^*(t)dt = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2}\sum_{i=1}^n A_i^*(t)dt = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} dt = 1. \tag{16}$$

and P^{CD} is the well known Cobb-Douglas price index (Lippe, 2007). From (6), (7), (13) and (15) we conclude that the stochastic proposition of the ARR is well-constructed. It has been proved (see Appendix) that in the stochastic case one of the most important postulates of Gajek and Kałuszka holds, namely:

$$R_A(T_1, T_2) + 1 = (R_P(T_1, T_2) + 1)(R_Q(T_1, T_2) + 1), \tag{17}$$

where $R_A(T_1, T_2)$ and $R_Q(T_1, T_2)$ denote respectively the relative change in net assets and the number of clients of funds.

4. Simulation Study

A group of $n = 4$ funds were considered, the time horizon of observations $T = 1$ and the following parameters of prices of units and numbers of units processes were assumed (Note 3):

(a) Prices of units:

$$\alpha_1 = 0,3, \beta_1 = 0,25, \alpha_2 = -0,12, \beta_2 = 0,05,$$

$$\alpha_3 = 0,55, \beta_3 = 0,3, \alpha_4 = -0,45, \beta_4 = 0,07,$$

(b) Numbers of units:

$$\gamma_1 = 0,25, \theta_1 = 0,05, \gamma_2 = -0,45, \theta_2 = 0,1,$$

$$\gamma_3 = 0,7, \theta_3 = 0,25, \gamma_4 = 0,3, \theta_4 = 0,03.$$

Without loss of generality, it is assumed that $p_i(0) = q_i(0) = 1$ for each $i \in \{1, 2, ..., 4\}$. Some realizations of prices and numbers of units proccesses are presented in Figure 1 and Figure 2.

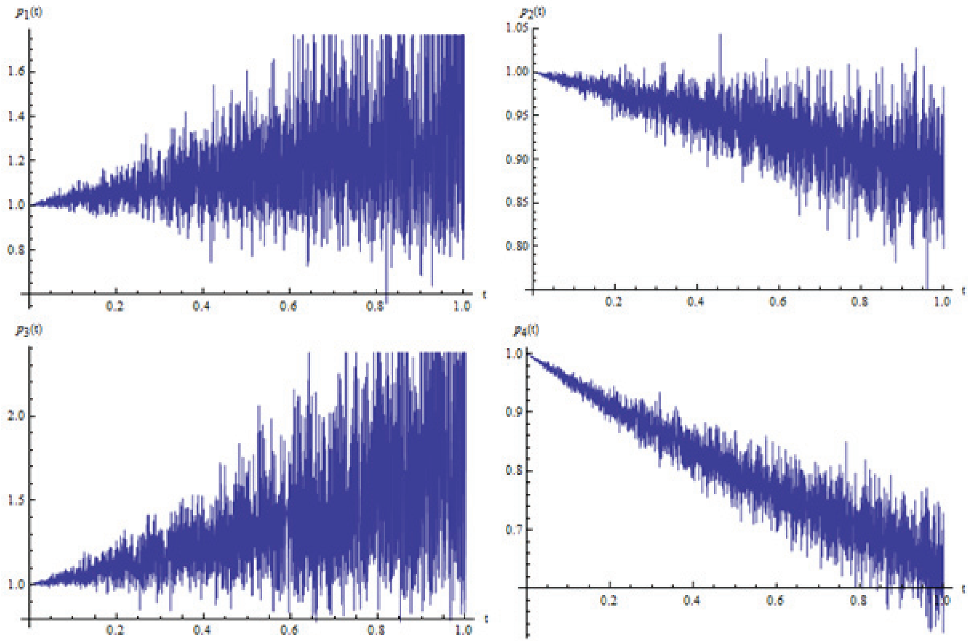

Figure 1. Some realizations of prices of units processes $p_1(t)$, $p_2(t)$, $p_3(t)$ and $p_4(t)$

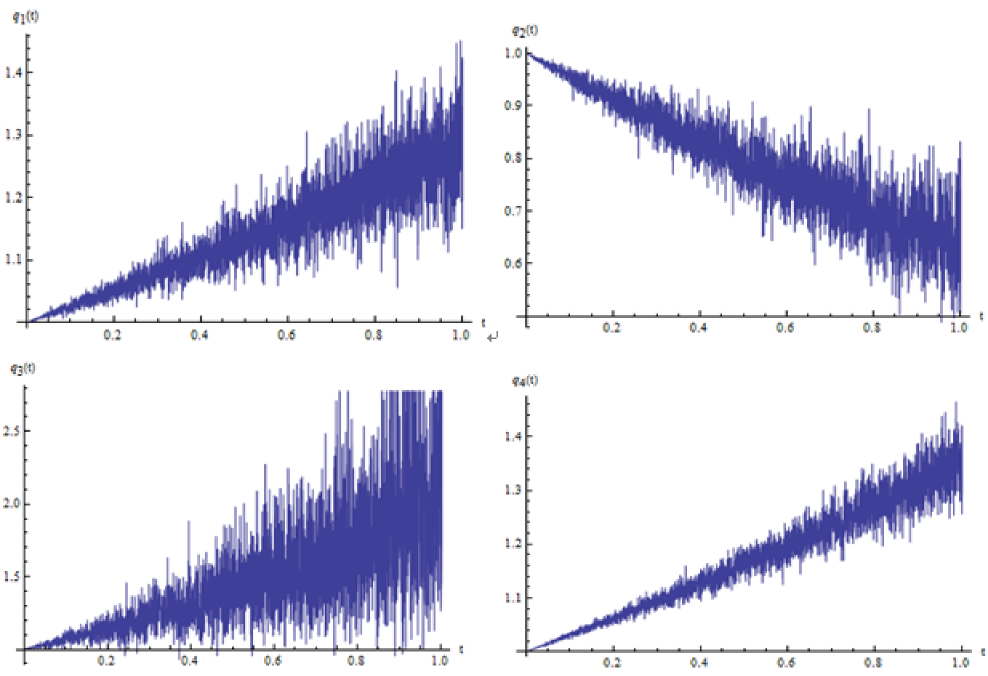

Figure 2. Some realizations of numbers of units processes $q_1(t)$, $q_2(t)$, $q_3(t)$ and $q_4(t)$

Some realization of the average return rate $R_P(0,t)$ for $t \in [0,1]$ is presented in Figure 3. The generated values of index $R_P(0,1)$ for each of i^{th} realization of price and quantity processes ($R_{Pi}(0,1)$: $i = 1,2,...,100$) are presented in Figure 4.

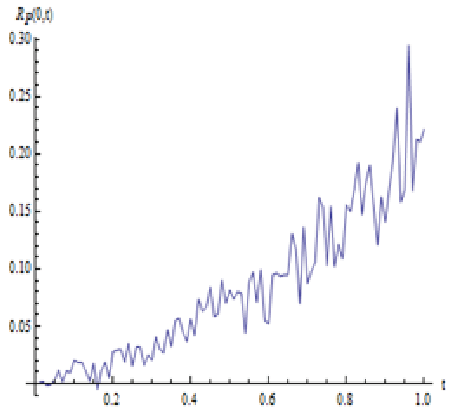

Figure 3. Realization of $R_P(0,t)$ process

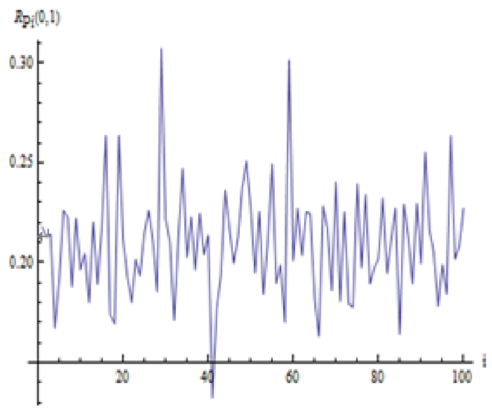

Figure 4. Generated values of index $R_P(0,1)$

The stochastic version of the average rate of return of funds $R_P(0,1)$ is then compared with average rates of return $\bar{r}_{GK}(0,1)$ and $\bar{r}_B(0,1)$ for which we divide the time interval $[0,1]$ into ten subintervals of the same length. It is found that, for $n = 10000$ generated realizations of prices of units and numbers of units processes are presented in Table 1 (to read more about estimation of mean value and variance and the bias of this estimation see Żądło (2006), Małecka (2011) or Papież and Śmiech (2013).

Table 1. Basic parameters of average return rates

Parameter	$R_P(0,1)$	$\bar{r}_B(0,1)$	$\bar{r}_{GK}(0,1)$
Mean	0,233	0,432	0,479
Standard deviation	0,020	0,216	0,238
Median	**0,232**	**0,241**	**0,273**
Median deviation	0,014	0,141	0,130

5. Conclusions

The form of the R_p measure seems to be proper–in the deterministic case, where $\beta_i(t) = 0$, it can be expressed by using some known chain indices (Divisia, Cobb-Douglas). The known $\bar{r}_{GK}$ and $\bar{r}_B$ measures have the same property in a discrete version. Moreover, the stochastic R_p measure takes into account the volatility of unit prices and number of units (β_i and θ_i parameters). In the simulation study only $R_P(0,1)$ rate has small standard deviation and median deviation. The distribution of $R_P(0,1)$ is quasi-symmetric (mean and median are almost equal), no extreme realizations exist, while in the other two distributions of $\bar{r}_B(0,1)$ and $\bar{r}_{GK}(0,1)$ the mean is higher than the median and thus these distributions are skewed to the right. We conclude that some extreme realizations of price or (and) quantity processes lead to extreme values of the considered chain indices. Using medians, instead of means, for our comparison and thus ruling out these extreme realizations we obtain smaller differences between compared measures (for example: 0,232 in case of $R_P(0,1)$ and 0,241 in case of $\bar{r}_B(0,1)$).

Apart from measuring fund return rates, the proposed formula could be used to measure the mean efficiency of any class of investable assets, and in particular mutual funds. If the class is too broad to be completely enumerated, a representative sample might be chosen according to various criteria of representativeness discussed by Kruskal (1979) and Gamrot (2008).

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Appendix

Theorem

$$R_A(T_1, T_2) + 1 = (R_P(T_1, T_2) + 1)(R_Q(T_1, T_2) + 1)$$

Proof. Let us signify the relative changes of assets, prices and numbers of units as follows

$$\tilde{A} = R_A(T_1, T_2) + 1, \quad \tilde{P} = R_p(T_1, T_2) + 1, \quad \tilde{Q} = R_Q(T_1, T_2) + 1, \tag{A-1}$$

to prove that $\tilde{A} = \tilde{P}\tilde{Q}$.

Let us assume the following:

$$\alpha_i(t) = \alpha_i p_i(t), \quad \beta_i(t) = \beta_i p_i(t), \quad \gamma_i(t) = \gamma_i q_i(t) \quad \text{and} \quad \theta_i(t) = \theta_i q_i(t). \tag{A-2}$$

Thus, from Equations (8) and (10), we get:

$$dp_i(t) = \alpha_i(t)dt + \beta_i(t)dW_i(t), \tag{A-3}$$

$$dq_i(t) = \gamma_i(t)dt + \theta_i(t)dW_i(t). \tag{A-4}$$

Firstly, from the Ito theorem (Ito, 1951) we have:

$$dA(t) = d(\sum_{i=1}^{n} p_i(t)q_i(t)) = \sum_{i=1}^{n} d(p_i(t)q_i(t)) = \sum_{i=1}^{n} p_i(t)dq_i(t) + \sum_{i=1}^{n} q_i(t)dp_i(t) + \sum_{i=1}^{n} \beta_i(t)\theta_i(t)dt. \tag{A-5}$$

From $(A-3)$, $(A-4)$ and $(A-5)$ we obtain:

$$\begin{aligned} dA(t) &= \sum_{i=1}^{n} p_i(t)(\gamma_i(t)dt + \theta_i(t)dW_i(t)) + \sum_{i=1}^{n} q_i(t)(\alpha_i(t)dt + \beta_i(t)dW_i(t)) + \sum_{i=1}^{n} \beta_i(t)\theta_i(t)dt \\ &= (\sum_{i=1}^{n} p_i(t)\gamma_i(t) + \sum_{i=1}^{n} q_i(t)\alpha_i(t) + \sum_{i=1}^{n} \beta_i(t)\theta_i(t))dt + \sum_{i=1}^{n}(p_i(t)\theta_i(t) + q_i(t)\beta_i(t)dW_i(t) \\ &= \Psi(t)dt + \textstyle\sum_{i=1}^{N} B_i(t)dW_i(t), \end{aligned} \tag{A-6}$$

where

$$\Psi(t) = \sum_{i=1}^{N} p_i(t)\gamma_i(t) + \sum_{i=1}^{N} q_i(t)\alpha_i(t) + \sum_{i=1}^{N} \beta_i(t)\theta_i(t), \tag{A-7}$$

and,

$$B_i(t) = p_i(t)\theta_i(t) + q_i(t)\beta_i(t). \tag{A-8}$$

Using the Ito theorem for the function $f(t, x) = \ln x$ and formula $(A-6)$ we get

$$d(\ln A(t)) = (\frac{\Psi(t)}{A(t)} - \frac{\sum_{i=1}^{n} B_i^2(t)}{2A^2(t)})dt + \frac{\sum_{i=1}^{n} B_i(t)dW_i(t)}{A(t)}. \tag{A-9}$$

Since (Note 4)

$$\tilde{A} = \frac{A(T_2)}{A(T_1)} = \exp(\int_{T_1}^{T_2} d(\ln A(t)), \tag{A-10}$$

from $(A-10)$ we get

$$\tilde{A} = \exp(\int_{T_1}^{T_2} (\frac{\Psi(t)}{A(t)} - \frac{\sum_{i=1}^{n} B_i^2(t)}{2A^2(t)})dt + \sum_{i=1}^{n} \int_{T_1}^{T_2} \frac{B_i(t)dW_i(t)}{A(t)}). \tag{A-11}$$

From Equations $(A-7)$, $(A-8)$ and $(A-11)$, we obtain

$$
\begin{aligned}
\tilde{A} &= \exp\Big[\int_{T_1}^{T_2}\Big(\frac{\sum_{i=1}^n p_i(t)\gamma_i(t)+\sum_{i=1}^n q_i(t)\alpha_i(t)+\sum_{i=1}^n \beta_i(t)\theta_i(t)}{A(t)}-\\
&\quad -\frac{\sum_{i=1}^n (p_i^2(t)\theta_i^2(t)+q_i^2(t)\beta_i^2(t)+2A_i(t)\theta_i(t)\beta_i(t)}{2A^2(t)}\Big)dt\Big]\times\\
&\quad \times\exp\Big[\sum_{i=1}^n\int_{T_1}^{T_2}\frac{p_i(t)\theta_i(t)}{A(t)}dW_i(t)+\sum_{i=1}^n\int_{T_1}^{T_2}\frac{q_i(t)\beta_i(t)}{A(t)}dW_i(t)\Big]\\
&= \exp\Big(\int_{T_1}^{T_2}\Big(\frac{\sum_{i=1}^n q_i(t)\alpha_i(t)}{A(t)}+\frac{\sum_{i=1}^n \beta_i(t)\theta_i(t)}{2A(t)}-\frac{\sum_{i=1}^n q_i^2(t)\beta_i^2(t)}{2A^2(t)}-\frac{\sum_{i=1}^n A_i(t)\beta_i(t)\theta_i(t)}{2A^2(t)}\Big)dt\\
&\quad +\sum_{i=1}^n\int_{T_1}^{T_2}\frac{q_i(t)\beta_i(t)}{A(t)}dW_i(t)\Big)\times\exp\Big(\int_{T_1}^{T_2}\Big(\frac{\sum_{i=1}^n p_i(t)\gamma_i(t)}{A(t)}+\frac{\sum_{i=1}^n \beta_i(t)\theta_i(t)}{2A(t)}-\frac{\sum_{i=1}^n p_i^2(t)\theta_i^2(t)}{2A^2(t)}-\\
&\quad -\frac{\sum_{i=1}^n A_i(t)\beta_i(t)\theta_i(t)}{2A^2(t)}\Big)dt+\sum_{i=1}^n\int_{T_1}^{T_2}\frac{p_i(t)\theta_i(t)}{A(t)}dW_i(t)\Big).
\end{aligned}
\tag{A - 12}
$$

From $(A-1)$ and $(A-12)$ we get

$$
\begin{aligned}
\tilde{A} &= \exp\Big(\int_{T_1}^{T_2}\Big(\frac{\sum_{i=1}^n A_i(t)\alpha_i}{A(t)}+\frac{\sum_{i=1}^n A_i(t)\beta_i\theta_i}{2A(t)}-\frac{\sum_{i=1}^n A_i^2(t)\beta_i^2}{2A^2(t)}-\frac{\sum_{i=1}^n A_i^2(t)\beta_i\theta_i}{2A^2(t)}\Big)dt+\sum_{i=1}^n\int_{T_1}^{T_2}\frac{A_i(t)\beta_i}{A(t)}dW_i(t)\Big)\times\\
&\quad \times\exp\Big(\int_{T_1}^{T_2}\Big(\frac{\sum_{i=1}^n A_i(t)\gamma_i}{A(t)}+\frac{\sum_{i=1}^n A_i(t)\beta_i\theta_i}{2A(t)}-\frac{\sum_{i=1}^n A_i^2(t)\theta_i^2}{2A^2(t)}-\frac{\sum_{i=1}^n A_i^2(t)\beta_i\theta_i}{2A^2(t)}\Big)dt+\sum_{i=1}^n\int_{T_1}^{T_2}\frac{A_i(t)\theta_i}{A(t)}dW_i(t)\Big)\\
&= \tilde{P}\cdot\tilde{Q}.
\end{aligned}
\tag{A - 13}
$$

Notes

Note 1. In Poland, in 2001 and 2002 Bankowy Fund did not reach the minimal rate of return.

Note 2. Geometric Brownian Motion is used to model stock prices in the BlackCScholes model and is the most widely used model of stock price behavior (Hull, 2009).

Note 3. The chosen parameters of prices of units and numbers of units describe many of funds condition variants: increasing or decreasing prices of units and number of units with small or high volatility. Nevertheless, the simulation study plays a role of some illustration of the presented measures and all conclusions from that study can not be general.

Note 4. In (27) we use Ito integral.

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