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Ruin Probability in a Generalized Risk Process Under Interest Force With Homogenous Markov Chain Premiums

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Abstract

The aim of this paper is to give recursive and integral equations for ruin probabilities for generalized risk processes under interest force with homogenous markov chain premiums. Inequalities for ruin probabilities are derived by using recursive technique. We give recursive equations for finite-time probability and an integral equation for ultimate ruin probability in Theorem 2.1 and Theorem 2.2. Using these equations, we can derive probability inequalities for finite-time probabilities and ultimate ruin probability in Theorem 3.1 and Theorem 3.2. These Theorems give upper bounds for finite-time probabilities and ultimate ruin probability.

Keywords: integral equation, recursive equation, ruin probability, homogeneous Markov chain

1. Introduction

Ruin probability is a main area in risk theory (see Asmussen, 2000). Ruin probabilies in discrete time models have been considered in many papers. In classical risk model, no investment incomes were considered there. Recently, the models with stochastic interest rates have received increasingly a large amount of attention. Kalashnikov and Norberg (2002) assumed that the surplus of an insurance company was invested in a risk asset and obtained the upper bound and lower bound for ruin probability. Paulsen (1998) considered a diffusion risk models with stochastic investment incomes. Yang and Zhang (2003) studied the model in Browers et al. (1997) by using an autoregression process to model both the premiums and the claims, and they also included investment incomes in their model. Both exponential and non exponential upper bounds for the ruin probability were obtained. Cai (2002a, 2002b) and Cai and Dickson (2004) studied the problems of ruin probabilies in discrete time models with random interest rates. In Cai (2002a, 2002b), the author assumed that the interest rates formed a sequence of independent and identically distributed random variables and an autoregressive time series models respectively. In Cai and Dickson (2004), interest rates followed a Markov chain.

In this paper, we study the models considered by Cai and Dickson (2004) to the case homogenous markov chain premiums, independent claims and independent interests. The main difference between the model in our paper and the one in Cai and Dickson (2004) is that premiums in our model are assumed to follow a homogeneous Markov chain. In this paper, we established recursive equations for finite time ruin probabilities and an integral equation for ultimate ruin probability, an exponential upper bound is given for both finite time ruin probabilities and ultimate ruin probability by integrating the inductive method and the recursive equation.

To establish probability inequalities for ruin probabilities of these models, we study two styles of premium collections. On the one hand of the premiums are collected at the beginning of each period then the surplus process $\left\{U_n^{(1)}\right\}_{n\geq 1}$ with initial u can be written as

$$U_n^{(1)} = (U_{n-1}^{(1)} + X_n)(1 + I_n) - Y_n \tag{1}$$

which is equivalent to

$$U_n^{(1)} = u.\prod_{k=1}^{n}(1 + I_k) + \sum_{k=1}^{n}\left[X_k(1 + I_k) - Y_k\right]\prod_{j=k+1}^{n}(1 + I_j), \tag{2}$$

On the other hand, if the premiums are collected at the end of each period, then the surplus process $\left\{U_n^{(2)}\right\}_{n\geq 1}$ with initial u can be written as

$$U_n^{(2)} = U_{n-1}^{(2)}(1 + I_n) + X_n - Y_n \tag{3}$$

which is equivalent to

$$U_n^{(2)} = u \cdot \prod_{k=1}^{n}(1 + I_k) + \sum_{k=1}^{n}(X_k - Y_k) \prod_{j=k+1}^{n} (1 + I_j). \tag{4}$$

where throughout this paper, we denote $\prod_{t=a}^{b} x_t = 1$ and $\sum_{t=a}^{b} x_t = 0$ if $a > b$.

We assume that:

Assumption 1. $U_o^{(1)} = U_o^{(2)} = u > 0$.

Assumption 2. $X = \{X_n\}_{n\geq 0}$ is a homogeneous Markov chain, X_n take values in a finite set of non-negative numbers $E = \{x_1,\ x_2,\ ...,\ x_M\}$ with $X_o = x_i$ and

$$p_{ij} = P\left[X_{m+1} = x_j \middle| X_m = x_i\right], \quad (m \in N);\ \ x_i, x_j \in E \ \text{ where } \begin{cases} 0 \leq p_{ij} \leq 1 \\ \sum_{j=1}^{M} p_{ij} = 1. \end{cases}$$

Assumption 3. $Y = \{Y_n\}_{n\geq 0}$ is sequence of independent and identically distributed non-negative random variables with the same distributive function $F(y) = P(Y_0 \leq y)$.

Assumption 4. $I = \{I_n\}_{n\geq 0}$ is sequence of independent and identically distributed non-negative random variables with the same distributive function $G(t) = P(I_0 \leq t)$.

Assumption 5. X, Y and I are assumed to be independent.

We define the finite time and ultimate ruin probabilities in model (1) with Assumption 1 to Assumption 5, respectively, by

$$\psi_n^{(1)}(u, x_i) = P\left(\bigcup_{k=1}^{n}(U_k^{(1)} < 0)\middle| U_0^{(1)} = u,\ X_o = x_i\right), \tag{5}$$

$$\psi^{(1)}(u, x_i) = \lim_{n\to\infty} \psi_n^{(1)}(u, x_i) = P\left(\bigcup_{k=1}^{\infty}(U_k^{(1)} < 0)\middle| U_0^{(1)} = u,\ X_o = x_i\right). \tag{6}$$

Similarly, we define the finite time and ultimate ruin probabilities in model (3) with Assumption 1 to Assumption 5, respectively, by

$$\psi_n^{(2)}(u, x_i) = P\left(\bigcup_{k=1}^{n}(U_k^{(2)} < 0)\middle| U_0^{(2)} = u,\ X_o = x_i\right), \tag{7}$$

$$\psi^{(2)}(u, x_i) = \lim_{n\to\infty} \psi_n^{(2)}(u, x_i) = P\left(\bigcup_{k=1}^{\infty}(U_k^{(2)} < 0)\middle| U_o^{(2)} = u,\ X_o = x_i\right). \tag{8}$$

In this paper, we build probability inequalities for $\psi^{(1)}(u, x_i)$ and $\psi^{(2)}(u, x_i)$. The paper is organized as follows; in section 2, we give recursive equations for $\psi_n^{(1)}(u, x_i)$ and $\psi_n^{(2)}(u, x_i)$and integral equations for $\psi^{(1)}(u, x_i)$ and $\psi^{(2)}(u, x_i)$. We give probability inequalities for $\psi^{(1)}(u, x_i)$ and $\psi^{(2)}(u, x_i)$ in section 3 by an inductive approach. Upper bounds of this probability is exponentical function. Finally, we conclude our paper in section 4.

2. Integral Equation for Ruin Probabilities

Throughout this paper, we denote the tail of any distribution function B by $\overline{B}(x) = 1 - B(x)$. We first give recursive equations for $\psi_n^{(1)}(u, x_i)$ and an integral equation for $\psi^{(1)}(u, x_i)$.

Theorem 2.1 *Let model (1) satisfy Assumption 1 to Assumption 5 then for n = 1, 2, ...*

$$\psi_{n+1}^{(1)}(u, x_i) = \sum_{x_j\in E} p_{ij} \left\{\int_0^{+\infty} \int_0^{h_t} \psi_n^{(1)}(h_t - y, x_j) dF(y) dG(t) + \int_0^{+\infty} \overline{F}(h_t) dG(t)\right\}, \tag{9}$$

and

$$\psi^{(1)}(u, x_i) = \sum_{x_j\in E} p_{ij} \left\{\int_0^{+\infty} \int_0^{h_t} \psi^{(1)}(h_t - y, x_j) dF(y) dG(t) + \int_0^{+\infty} \overline{F}(h_t) dG(t)\right\}, \tag{10}$$

where $h_t = (u + x_j)(1 + t)$.

Proof. Let $X_1 = x_j$, $A = \left\{U_o^{(1)} = u, X_o = x_i, X_1 = x_j\right\}$ $(x_j \in E)$, $A_1 = \{Y_1 \leq (u + X_1)(1 + I_1)\}$, $A_2 = \{Y_1 > (u + X_1)(1 + I_1)\}$. From (1), we have

$$U_1^{(1)} = (U_0^{(1)} + X_1)(1 + I_1) - Y_1 = (u + x_j)(1 + I_1) - Y_1.$$

Thus

$$P\left(U_1^{(1)} < 0 \middle| A_2 \cap A\right) = 1 \Rightarrow P\left(\bigcup_{k=1}^{n+1}(U_k^{(1)} < 0)\middle| A_2 \cap A\right) = 1 \tag{11}$$

and

$$P\left(U_1^{(1)} < 0\middle| A_1 \cap A\right) = 0. \tag{12}$$

Let $\left\{\tilde{X}_n\right\}_{n\geq 0}$, $\left\{\tilde{Y}_n\right\}_{n\geq 0}$, $\left\{\tilde{I}_n\right\}_{n\geq 0}$ be independent copies of $\{X_n\}_{n\geq 0}$, $\{Y_n\}_{n\geq 0}$, $\{I_n\}_{n\geq 0}$ respectively such that $\tilde{X}_o = X_1 = x_j$, $\tilde{Y}_o = Y_1$, $\tilde{I}_o = I_1$.

Combining (12) and (2) imply that,

$$\begin{aligned}
& P\left(\bigcup_{k=1}^{n+1}(U_k^{(1)} < 0)\middle| A_1 \cap A\right) \\
= \; & P\left(\bigcup_{k=2}^{n+1}(U_k^{(1)} < 0)\middle| A_1 \cap A\right) \\
= \; & P\left(\bigcup_{k=2}^{n+1}\left\{\left((u + x_j)(1 + I_1) - Y_1\right)\prod_{j=2}^{k}(1 + I_j) + \sum_{j=2}^{k}(X_j(1 + I_j) - Y_j)\prod_{p=j+1}^{k}(1 + I_p) < 0\right\}\middle| A_1 \cap A\right) \\
= \; & P\left(\bigcup_{k=1}^{n}\left\{\tilde{U}_o^{(1)}\prod_{j=1}^{k}(1 + \tilde{I}_j) + \sum_{j=1}^{k}(\tilde{X}_j(1 + \tilde{I}_j) - \tilde{Y}_j)\prod_{p=j+1}^{k}(1 + \tilde{I}_p) < 0\right\}\middle| \tilde{U}_o^{(1)} = (u + x_j)(1 + I_1) - Y_1,\ \tilde{X}_o = x_j\right)
\end{aligned}$$

Let $Y_1 = y \in R$, $I_1 = t \in R$ and $h_t = (u + x_j)(1 + t)$, then for $0 \leq y \leq h_t$

$$P\left(\bigcup_{k=1}^{n+1}(U_k^{(1)} < 0)\middle| A_1 \cap A\right) = \psi_n^{(1)}(h_t - y, x_j). \tag{13}$$

That, (5) implies

$$\psi_{n+1}^{(1)}(u, x_i) = P\left\{\bigcup_{k=1}^{n+1}(U_k^{(1)} < 0)\middle| U_o^{(1)} = u,\ X_o = x_i\right\}$$

Thus, we get

$$\begin{aligned}
& \psi_{n+1}^{(1)}(u, x_i) = \sum_{x_j\in E} p_{ij}P\left\{\bigcup_{k=1}^{n+1}(U_k^{(1)} < 0)\middle| A\right\} \\
= \; & \sum_{x_j\in E} p_{ij}\left\{\int_0^{+\infty}\int_0^{h_t} P\left\{\bigcup_{k=1}^{n+1}(U_k^{(1)} < 0)\middle| A_1 \cap A\right\}dF(y)dG(t)\right. \\
& \left. + \int_0^{+\infty}\int_{h_t}^{+\infty} P\left\{\bigcup_{k=1}^{n+1}(U_k^{(1)} < 0)\middle| A_2 \cap A\right\}dF(y)dG(t))\right\}.
\end{aligned} \tag{14}$$

Thus, combining (11), (13) and (14), we have

$$\begin{aligned}
\psi_n^{(1)}(u, x_i) &= \sum_{x_j\in E} p_{ij}\left\{\int_0^{+\infty}\int_0^{h_t}\psi_n^{(1)}(h_t - y, x_j)dF(y)dG(t) + \int_0^{+\infty}\int_{h_t}^{+\infty}dF(y)dG(t)\right\} \\
&= \sum_{x_j\in E} p_{ij}\left\{\int_0^{+\infty}\int_0^{h_t}\psi_n^{(1)}(h_t - y, x_j)dF(y)dG(t) + \int_0^{+\infty}\overline{F}(h_t)dG(t)\right\}.
\end{aligned} \tag{15}$$

Thus, the integaral equation for $\psi^{(1)}(u, x_i)$ in Theorem 2.1 follows immediately from the dominated convergence theorem by letting $n \to \infty$ in (15).

This completes the proof. □

Similarly, the following recursive equations for $\psi_n^{(2)}(u, x_i)$ and integral equation for $\psi^{(2)}(u, x_i)$ hold.

Theorem 2.2 *Let model (3) satisfy Assumption 1 to Assumption 5 then for n = 1, 2, . . .*

$$\psi_{n+1}^{(2)}(u, x_i) = \sum_{x_j\in E} p_{ij}\left\{\int_0^{+\infty}\int_0^{h_t}\psi_n^{(2)}(h_t - y, x_j)dF(y)dG(t) + \int_0^{+\infty}\overline{F}(h_t)dG(t)\right\}, \tag{16}$$

and

$$\psi^{(2)}(u,i)=\sum_{x_j\in E}p_{ij}\left\{\int_0^{+\infty}\int_0^{h_t}\psi^{(2)}(h_t-y,x_j)dF(y)dG(t)+\int_0^{+\infty}\overline{F}(h_t)dG(t)\right\},\tag{17}$$

where $h_t=u(1+t)+x_j$.

Proof. Let $X_1=x_j, A=\left\{U_o^{(2)}=u, X_o=x_i, X_1=x_j\right\}(x_j\in E), A_1=\{Y_1\le u(1+I_1)+X_1\}, A_2=\{Y_1>u(1+I_1)+X_1\}$. From (3), we have

$$U_1^{(2)}=U_0^{(2)}(1+I_1)+X_1-Y_1=u(1+I_1)+x_j-Y_1.$$

Thus

$$P\left(U_1^{(2)}<0\middle|A_2\cap A\right)=1\Rightarrow P\left(\bigcup_{k=1}^{n+1}(U_k^{(2)}<0)\middle|A_2\cap A\right)=1,\tag{18}$$

and

$$P\left(U_1^{(2)}<0\middle|A_1\cap A\right)=0.\tag{19}$$

Let $\left\{\tilde{X}_n\right\}_{n\ge0}, \left\{\tilde{Y}_n\right\}_{n\ge0}, \left\{\tilde{I}_n\right\}_{n\ge0}$ be independent copies of $\{X_n\}_{n\ge0}$, $\{Y_n\}_{n\ge0}$, $\{I_n\}_{n\ge0}$ respectively such that $\tilde{X}_o=X_1=x_j$, $\tilde{Y}_o=Y_1$, $\tilde{I}_o=I_1$.

Combining (19) and (4) imply that

$$\begin{aligned}
&P\left(\bigcup_{k=1}^{n+1}(U_k^{(2)}<0)\middle|A_1\cap A\right)=P\left(\bigcup_{k=2}^{n+1}(U_k^{(2)}<0)\middle|A_1\cap A\right)\\
=\;&P\left(\bigcup_{k=2}^{n+1}\left\{\left(u(1+I_1)+x_j-Y_1\right)\prod_{j=2}^{k}(1+I_j)+\sum_{j=2}^{k}(X_j-Y_j)\prod_{p=j+1}^{k}(1+I_p)<0\right\}\middle|A_1\cap A\right)\\
=\;&P\left(\bigcup_{k=1}^{n}\left\{\tilde{U}_o^{(2)}\prod_{j=1}^{k}(1+\tilde{I}_j)+\sum_{j=1}^{k}(\tilde{X}_j-\tilde{Y}_j)\prod_{p=j+1}^{k}(1+\tilde{I}_p)<0\right\}\middle|\tilde{U}_o^{(2)}=u(1+I_1)+x_j-Y_1,\tilde{X}_o=x_j\right).
\end{aligned}$$

Let $Y_1=y\in R, I_1=t\in R$ and $h_t=u(1+t)+x_j$, then for $0\le y\le h_t$

$$P\left(\bigcup_{k=1}^{n+1}(U_k^{(2)}<0)\middle|A_1\cap A\right)=\psi_n^{(2)}(h_t-y,x_j).\tag{20}$$

That, (7) implies

$$\psi_{n+1}^{(2)}(u,x_i)=P\left\{\bigcup_{k=1}^{n+1}(U_k^{(2)}<0)\middle|U_o^{(2)}=u,\ X_o=x_i\right\}.$$

Thus, we get

$$\begin{aligned}
&\psi_{n+1}^{(2)}(u,x_i)=\sum_{x_j\in E}p_{ij}P\left\{\bigcup_{k=1}^{n+1}(U_k^{(2)}<0)\middle|A\right\}\\
=\;&\sum_{x_j\in E}p_{ij}\left\{\int_0^{+\infty}\int_0^{h_t}P\left\{\bigcup_{k=1}^{n+1}(U_k^{(2)}<0)\middle|A_1\cap A\right\}dF(y)dG(t)\right.\\
&\left.+\int_0^{+\infty}\int_{h_t}^{+\infty}P\left\{\bigcup_{k=1}^{n+1}(U_k^{(2)}<0)\middle|A_2\cap A\right\}dF(y)dG(t))\right\}.
\end{aligned}\tag{21}$$

Thus, combining (18), (20) and (21), we have

$$\begin{aligned}
\psi_n^{(2)}(u,x_i)\;&=\;\sum_{x_j\in E}p_{ij}\left\{\int_0^{+\infty}\int_0^{h_t}\psi_n^{(2)}(h_t-y,x_j)dF(y)dG(t)+\int_0^{+\infty}\int_{h_t}^{+\infty}dF(y)dG(t)\right\}\\
&=\;\sum_{x_j\in E}p_{ij}\left\{\int_0^{+\infty}\int_0^{h_t}\psi_n^{(2)}(h_t-y,x_j)dF(y)dG(t)+\int_0^{+\infty}\overline{F}(h_t)dG(t)\right\}.
\end{aligned}\tag{22}$$

Thus, the integaral equation for $\psi^{(2)}(u,x_i)$ in Theorem 2.2 follows immediately from the dominated convergence theorem by letting $n\to\infty$ in (22).

This completes the proof. □

Next, we establish probability inequalities for ruin probabilities of model (1) and model (3).

3. Probability Inequalities for Ruin Probabilities

To establish probability inequalities for ruin probabilities of model (1), we proof following Lemma.

Lemma 3.1 *Let model (1) satisfy Assumption 1 to Assumption 5. Any $x_i \in E$, if*

$$E(Y_1) < E\left(X_1 | X_o = x_i\right) \text{ and } P\left((Y_1 - X_1(1 + I_1)) > 0 | X_o = x_i\right) > 0, \tag{23}$$

then, there exists a unique positive constant R_i satisfying:

$$E\left(e^{R_i(Y_1 - X_1(1+I_1))} \Big| X_o = x_i\right) = 1. \tag{24}$$

Proof. Define

$$f_i(t) = E\left\{e^{t(Y_1 - X_1(1+I_1))} \Big| X_o = x_i\right\} - 1;\ t \in (0,\ +\infty).$$

Then

$$f_i^{'}(t) = E\left\{(Y_1 - X_1(1 + I_1))\ e^{t(Y_1 - X_1(1+I_1))} \Big| X_o = x_i\right\}$$

$$f_i^{''}(t) = E\left\{(Y_1 - X_1(1 + I_1))^2\ e^{t(Y_1 - X_1(1+I_1))} \Big| X_o = x_i\right\} \geq 0. \tag{25}$$

From (25) implies that

$$f_i\,(t) \text{ is a convex function with } f(0) = 0 \tag{26}$$

and

$$f_i^{'}(0) = E\left\{(Y_1 - X_1(1 + I_1)) | X_o = x_i\right\} \leq EY_1 - E(X_1 | X_o = x_i) < 0. \tag{27}$$

By $P\left((Y_1 - X_1(1 + I_1)) > 0 | X_o = x_i\right) > 0$, we can find some constant $\delta > 0$ satisfies

$$P\left((Y_1 - X_1(1 + I_1)) > \delta > 0 | X_o = x_i\right) > 0.$$

Then, we have

$$\begin{aligned} f_i(t) &= E\left\{e^{t(Y_1 - X_1(1+I_1)}\Big| X_o = x_i\right\} - 1 \\ &\geq E\left\{e^{t(Y_1 - X_1(1+I_1))}\Big| X_o = x_i\right\} \cdot 1_{\{(Y_1 - X_1(1+I_1))>\delta | X_o = x_i\}} - 1 \\ &\geq e^{t\delta} P(\{(Y - X_1(1 + I_1)) > \delta | X_o = x_i\} - 1. \end{aligned}$$

Implies

$$\lim_{t\to+\infty} f(t) = +\infty. \tag{28}$$

Combining (26), (27) and (28), there exists a unique positive constant R_i satisfying (24).

This completes the proof. □

Let $R_o = \min\left\{R_i > 0 :\ E\left(e^{R_i(Y_1 - X_1(1+I_1))}\Big| X_o = x_i\right) = 1\ (x_i \in E)\right\}$.

Use Lemma 3.1 and Theorem 2.1, we obtain a probability inequality for $\psi^{(1)}(u, x_i)$ by an inductive approach.

Theorem 3.1 *Let model (1) satisfy Assumption 1 to Assumption 5 and (23) then for any $u > 0$ and $x_i \in E$,*

$$\psi^{(1)}(u, i) \leq \beta_1 \cdot E\left[e^{R_o Y_1}\right] E\left[e^{-R_o(u+X_1)(1+I_1)}\Big| X_o = x_i\right], \tag{29}$$

where

$$\beta_1^{-1} = \inf_{t\geq 0} \frac{\int_t^{+\infty} e^{R_o y} dF(y)}{e^{R_o t} \cdot \overline{F}(t)},\ \beta_1 \leq 1.$$

Proof. Firstly, we have

$$\beta_1^{-1} = \inf_{t\geq 0} \frac{\int_t^{+\infty} e^{R_o y} dF(y)}{e^{R_o t} \cdot \overline{F}(t)} \geq \inf_{t\geq 0} \frac{\int_t^{+\infty} e^{R_o t} dF(y)}{e^{R_o t} \cdot \overline{F}(t)} = \inf_{t\geq 0} \frac{\int_t^{+\infty} dF(y)}{\overline{F}(t)} = 1 \Rightarrow \frac{1}{\beta_1} \geq 1 \Rightarrow \beta_1 \leq 1.$$

For any $t \geq 0$, we have

$$\overline{F}(t) = \left[\frac{\int_t^{+\infty} e^{R_o y} dF(y)}{e^{R_o t} \cdot \overline{F}(t)}\right]^{-1} \cdot e^{-R_o t} \cdot \int_t^{+\infty} e^{R_o y} dF(y) \leq \beta_1 \cdot e^{-R_o t} \cdot \int_t^{+\infty} e^{R_o y} dF(y) \tag{30}$$

$$\leq \beta_1 \cdot e^{-R_o t} \cdot \int_0^{+\infty} e^{R_o y} dF(y) = \beta_1 \cdot e^{-R_o t}.E\left[e^{R_o Y_1}\right]. \tag{31}$$

Then, for $u > 0$ and $x_i \in E$,

$$\psi_1^{(1)}(u, x_i) = P(U_1^{(1)} > 0 \big| U_o^{(1)} = u, X_o = x_i) = \sum_{x_j \in E} p_{ij} \int_0^{+\infty} \overline{F}(h_t) dG(t) \tag{32}$$

Thus, combining (31) and (32), we have

$$\begin{aligned} \psi_1^{(1)}(u, x_i) &= \textstyle\sum_{x_j \in E} p_{ij} \int_0^{+\infty} \overline{F}(h_t) dG(t) \\ &\leq \beta_1 E\left[e^{R_o Y_1}\right] \cdot \textstyle\sum_{x_j \in E} p_{ij} \cdot \int_0^{+\infty} e^{-R_o(u+x_j)(1+t)} dG(t) \\ &\leq \beta_1 E\left[e^{R_o Y_1}\right] \cdot \textstyle\sum_{x_j \in E} p_{ij} \cdot E\left[e^{-R_o(u+x_j)(1+I_1)}\right] \\ &\leq \beta_1 E\left[e^{R_o Y_1}\right] \cdot E\left[e^{-R_o(u+X_1)(1+I_1)} \Big| X_o = x_i\right]. \end{aligned} \tag{33}$$

Under an inductive hypothesis, we assume for any $u > 0$ and $x_i \in E$.

$$\psi_n^{(1)}(u, x_i) \leq \beta_1 E\left[e^{R_o Y_1}\right] \cdot E\left[e^{-R_o(u+X_1)(1+I_1)} \Big| X_o = x_i\right]. \tag{34}$$

Then, (33) implies (34) holds with $n = 1$.

For $x_j \in E$, $(u + x_j)(1 + t) - y > 0$ and $I_1 \geq 0$, we have

$$\begin{aligned} \psi_n^{(1)}(h_t - y, x_j) &\leq \beta_1^* E\left[e^{R_o^* Y_1}\right] \cdot E\left[e^{-R_o^*\left[((u+x_j)(1+t)-y+X_1)(1+I_1)\right]} \Big| X_o = x_j\right] \\ &= \beta_1^* E\left[e^{R_o^* Y_1}\right] \cdot E\left[e^{-R_o^*\left[(u+x_j)(1+t)-y\right](1+I_1) - R_o^* X_1(1+I_1)} \Big| X_o = x_j\right] \\ &\leq \beta_1^* E\left[e^{R_o^* Y_1}\right] \cdot E\left[e^{-R_o^* X_1(1+I_1)} \Big| X_o = x_j\right] \cdot e^{-R_o^*\left[(u+x_j)(1+t)-y\right]} \\ &= \beta_1^* \cdot e^{-R_o^*\left[(u+x_j)(1+t)-y\right]}. \end{aligned}$$

where $\beta_1^{*-1} = \inf\limits_{t\geq 0} \dfrac{\int_t^{+\infty} e^{R_o^* y} dF(y)}{e^{R_o^* t} \cdot \overline{F}(t)}$, $E\left(e^{R_o^*(Y_1 - X_1(1+I_1))} \Big| X_o = x_j\right) = 1$ and $R_o^* \geq R_o > 0$.

Any $t \geq 0$: $\dfrac{\int_t^{+\infty} e^{R_o y} dF(y)}{e^{R_o t} \cdot \overline{F}(t)} = \dfrac{\int_t^{+\infty} e^{R_o(y-t)} dF(y)}{\overline{F}(t)} \leq \dfrac{\int_t^{+\infty} e^{R_o^*(y-t)} dF(y)}{\overline{F}(t)} = \dfrac{\int_t^{+\infty} e^{R_o^* y} dF(y)}{e^{R_o^* t} \cdot \overline{F}(t)}$, then

$$\beta_1^{-1} = \inf_{t\geq 0} \frac{\int_t^{+\infty} e^{R_o y} dF(y)}{e^{R_o t} \cdot \overline{F}(t)} \leq \beta_1^{*-1} = \inf_{t\geq 0} \frac{\int_t^{+\infty} e^{R_o^* y} dF(y)}{e^{R_o^* t} \cdot \overline{F}(t)} \Leftrightarrow \frac{1}{\beta_1} \leq \frac{1}{\beta_1^*} \Leftrightarrow \beta_1^* \leq \beta_1.$$

We get $R_o^*\left[(u + x_j)(1 + t) - y\right] \geq R_o\left[(u + x_j)(1 + t) - y\right] > 0$, then

$$\psi_n^{(1)}(h_t - y, j) \leq \beta_1 . e^{-R_o\left[(u+x_j)(1+t)-y\right]} (x_j \in E,\ (u + x_j)(1 + t) - y > 0) \tag{35}$$

Therefore, by Lemma 3.1, (9), (30) and (35), we get

$$\begin{aligned} \psi_{n+1}^{(1)}(u, x_i) &= \textstyle\sum_{x_j \in E} p_{ij} \left\{ \int_0^{+\infty} \int_0^{h_t} \psi_n^{(1)}(h_t - y, x_j) dF(y) dG(t) + \int_0^{+\infty} \overline{F}(h_t) dG(t) \right\} \\ &\leq \textstyle\sum_{x_j \in E} p_{ij} \left\{ \int_0^{+\infty} \int_0^{h_t} \beta_1 e^{-R_o\left[(u+x_j)(1+t)-y\right]} dF(y) dG(t) + \int_0^{+\infty} \left(\beta_1 e^{-R_o(u+x_j)(1+t)} \int_{h_t}^{+\infty} e^{R_o y} dF(y) \right) dG(t) \right\} \\ &= \beta_1 \cdot \textstyle\sum_{x_j \in E} p_{ij} \left\{ \int_0^{+\infty} e^{-R_o(u+x_j)(1+t)} dG(t) \cdot \int_0^{h_t} e^{R_o y} dF(y) + \int_0^{+\infty} e^{-R_o(u+x_j)(1+t)} dG(t) \cdot \int_{h_t}^{+\infty} e^{R_o y} dF(y) \right\} \\ &= \beta_1 \cdot \textstyle\sum_{x_j \in E} p_{ij} \int_0^{+\infty} e^{-R_o(u+x_j)(1+t)} dG(t) \cdot \int_0^{+\infty} e^{R_o y} dF(y) \\ &= \beta_1 E\left[e^{R_o Y_1}\right] \cdot E\left[e^{-R_o(u+X_1)(1+I_1)} \Big| X_o = x_i\right]. \end{aligned}$$

Hence, for any $n = 1, 2, \ldots$ (34) holds. Therefore, (29) follows by letting $n \to \infty$ in (34).

This completes the proof. □

Remark 3.1 Let $A(u,x_i) = \beta_1 E\left[e^{R_o Y_1}\right] E\left[e^{-R_o(u+X_1)(1+I_1)}\Big| X_o = x_i\right]$. From $I_1 \geq 0, X_1 \geq 0$ and $\beta_2 \leq 1$, we have

$$\begin{aligned} A(u,x_i) &= \beta_1 E\left[e^{R_o Y_1}\right] E\left[e^{-R_o u(1+I_1)-R_o X_1(1+I_1)}\Big| X_o = x_i\right] \\ &\leq \beta_1 E\left[e^{R_o Y_1}\right] E\left[e^{-R_o u-R_o X_1(1+I_1)}\Big| X_o = x_i\right] \\ &= \beta_1 e^{-R_o u} E\left[e^{R_o[Y_1-X_1(1+I_1)]}\Big| X_o = x_i\right] \\ &= \beta_1 e^{-R_o u} \leq e^{-R_o u}. \end{aligned}$$

Therefore, upper bound for ruin probability in (29) is better than $e^{-R_o u}$.

Similarly, we have Lemma 3.2 and Theorem 3.2 for $\psi^{(2)}(u,x_i)$.

Lemma 3.2 *Let model (3) satisfy Assumption 1 to Assumption 5. Any $x_i \in E$, if*

$$E(Y_1) < E\left(X_1 | X_o = x_i\right) \text{ and } P\left((Y_1 - X_1) > 0 | X_o = x_i\right) > 0 \tag{36}$$

then, there exists a unique positive constant R_i satisfying:

$$E\left(e^{R_i(Y_1-X_1))}\Big| X_o = x_i\right) = 1.$$

Let $\overline{R}_o = \min\left\{R_i > 0 : E\left(e^{R_i(Y_1-X_1)}\Big| X_o = x_i\right) = 1\ (x_i \in E)\right\}$.

Theorem 3.2 *Let model (3) satisfy Assumption 1 to Assumption 5 and (36) then for any $u > 0$ and$x_i \in E$,*

$$\psi^{(2)}(u,x_i) \leq \beta_2 \cdot E\left[e^{-\overline{R}_o u(1+I_1)}\right], \tag{37}$$

where

$$\beta_2^{-1} = \inf_{t\geq 0} \frac{\int_t^{+\infty} e^{\overline{R}_o y} dF(y)}{e^{\overline{R}_o t} \cdot \overline{F}(t)};\ \beta_2 \leq 1.$$

Remark 3.2 Let $B(u,x_i) = \beta_2 \cdot E\left[e^{-\overline{R}_o u(1+I_1)}\right]$. From $I_1 \geq 0$ and $\beta_2 \leq 1$, we have

$$B(u,x_i) \leq \beta_2 \cdot E\left[e^{-\overline{R}_o u}\right] = \beta_2 e^{-\overline{R}_o u} \leq e^{-\overline{R}_o u}.$$

Therefore, upper bound for ruin probability in (37) is better than $e^{-\overline{R}_o u}$.

4. Conclusion

Our main results in this paper are not only, Theorem 2.1 and Theorem 2.2 give recursive equation for $\psi_n^{(1)}(u,x_i)$ and $\psi_n^{(2)}(u,x_i)$and integral equation for $\psi^{(1)}(u,x_i)$and $\psi^{(2)}(u,x_i)$. In addition, Theorem 3.1 and Theorem 3.2 give probability inequalities for $\psi^{(1)}(u,x_i)$ and $\psi^{(2)}(u,x_i)$by an inductive approach.

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