Abstract
We propose a flexible linear calibration model with errors from RS (Ramberg & Schmeiser, 1974) generalized lambda distribution ($GλD$). We demonstrate the derivation of the maximum likelihood estimates of RS $GλD$ parameters and examine the estimation performance using a simulation study for sample sizes ranging from 30 to 200. The use of RS $GλD$ calibration model not only provides statistical modeller with a richer range of distributional shapes, but can also provide more precise parameter estimates compared to the standard Normal calibration model or skewed Normal calibration model proposed by Figueiredoa, Bolfarinea, Sandovala and Limab (2010).

Keywords: generalized lambda distribution, linear calibration model, skew normal distribution, maximum likelihood estimation

1. Introduction
The statistical calibration model is a reverse regression technique, where we use the response variable to predict the corresponding explanatory variable. There are number of applications of this technique in science. For example, we may use radiometric dating to ascertain the age of a tree and further verify our result using tree rings. Our aim, however, is to use radiometric dating to estimate age of new trees, and the problem is whether we should minimize errors in the observation or minimize errors in age determination. There are many similar problems in substance concentration determination in biology and chemistry, physical quantities determination in physics and blood pressure/cholesterol level measurement in medicine. The literature on calibration problem has a long history, and one of the earliest works can be found in Eisenhart (1939).

The usual calibration experiment is a two stage process involving two random variables $X$ and $Y$. The first stage is known as the calibration trial, where we observe the $n$ values of the response variable $y_1,\ldots,y_n$ from a given set of explanatory values $x_1,\ldots,x_n$ and we can estimate the link function between $X$ and $Y$. The second stage is known as the calibration experiment, where we observe $k \geq 1$ value(s) of the response variable $Y$ as $y_{01},\ldots,y_{0k}$ which are mapped from some unknown value $x_0$ from the explanatory variable $X$. We can express these two stages by the following equations:

$$y_i = \alpha + \beta x_i + \varepsilon_i, \quad i = 1, \ldots, n;$$

$$y_{0j} = \alpha + \beta x_0 + \varepsilon_{0j}, \quad j = 1, \ldots, k,$$

(1.1)

We usually assume that the errors $\varepsilon_1,\ldots,\varepsilon_n,\varepsilon_{01},\ldots,\varepsilon_{0k}$ are i.i.d and Normally distributed with mean 0 and variance $\sigma^2$. Also, $x_1,\ldots,x_n$ are known and $\alpha,\beta, x_0$ and $\sigma^2$ are unknown parameters which we need to estimate.

As an extension to Normal distribution, Azzalini (1985) introduced the skewed Normal distribution. The skewed Normal distribution is defined as

$$g(x;\xi,\omega,\lambda) = \frac{2}{\omega}\phi\left(\frac{x-\xi}{\omega}\right)\Phi\left(\lambda\left(\frac{x-\xi}{\omega}\right)\right),$$

(1.2)

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the p.d.f. and c.d.f. of a standard normal distribution respectively. Specially, when $\xi = 0$ and $\omega = 1$, we obtain the standard skewed Normal distribution.
Based on (1.2), Figueiredo et al. (2010) defined a skew-normal calibration model by assuming that $\epsilon_i$ and $\epsilon_{0j}$ are i.i.d. and follow a skewed Normal distribution with $\xi = 0$ denoted by $SN(0, \omega, \lambda)$. This gives us the following calibration model:

$$
y_{i|\xi} \sim SN(\alpha + \beta x_i; \omega, \lambda), \quad i = 1, \cdots, n,
$$

$$
y_{0j|\xi} \sim SN(\alpha + \beta x_0; \omega, \lambda), \quad j = 1, \cdots, k.
$$

(1.3)

In (1.3), the conditional distribution of $y_i$ given $x_i$ and $y_{0j}$ given $x_0$ are governed by skewed Normal distributions. This skewed Normal calibration model allows the modeller to cope with some degree of skewness in the error distribution. However, this is still limited as the skewed Normal distribution have limited range of shapes. The skewed Normal distribution still cannot handle heavy tailed, U shape, uniform, triangular or exponential upward/downward patterns. These shapes however, can be captured using $GLD$ (generalized lambda distributions), and we propose a further extension to the calibration model by using $RS\ GLD$.

Our article is organized as follows. In Section 2, we introduce the $GLD$ family. In Section 3, we outline the RS $GLD$ calibration model and discuss possible ways to estimate parameters of the model using maximum likelihood estimation. In Section 4, we demonstrate the estimation performance of our proposed model across a range of different sample sizes from 30 to 200. As a further test to our proposed model to the literature, we compare the performance of RS $GLD$ calibration model against Normal and skewed Normal calibration model with respect to a real life dataset used by Figueiredo et al. (2010) in Section 5. A discussion of our proposed method is given in Section 6.

2. Generalized Lambda Distributions

The RS $GLD$ (Ramberg & Schmeiser, 1974) is an extension of Tukey’s lambda distribution. It is defined by its inverse distribution function:

$$
F^{-1}(u) = \alpha_1 + \frac{u^{\beta_1} - (1 - u)^{\beta_2}}{\alpha_2} \quad 0 \leq u \leq 1
$$

(2.1)

From (2.1), $\alpha_1, \alpha_2, \beta_1, \beta_2$ are respectively the location, inverse scale and shape 1 and shape 2 parameters. Karian and Dudewicz (2000) noted that $GLD$ is defined only if $\lambda_3(1 - u)^{\beta_2} - \lambda_2 u^{\beta_1} \geq 0$ for $0 \leq u \leq 1$. The conditions for which RS $GLD$ is a valid p.d.f. are set out in Karian and Dudewicz (2000) and these are also programmed in GLDEX package in R (Su, 2010, 2007a).

Freimer, Kollia, Mudholkar and Lin (1988) describe another distribution known as FKML $GLD$. The FKML $GLD$ can be written as:

$$
F^{-1}(u) = \alpha_1 + \frac{\alpha_3 u^{\beta_3} - \alpha_4 (1 - u)^{\beta_4}}{\alpha_5} \quad 0 \leq u \leq 1
$$

(2.2)

Under (2.2), $\alpha_1, \alpha_2, \beta_1, \beta_2$ are respectively the location, inverse scale and shape 1 and shape 2 parameters.

The fundamental motivation for the development of FKML $GLD$ is that the distribution is defined over all $\alpha_3$ and $\alpha_4$ (Freimer et al., 1988). The only restriction on FKML $GLD$ is $\alpha_2 > 0$. This is more convenient to deal with computationally than RS $GLD$ and hence it is sometimes the preferred $GLD$ for some researchers.

We restrict our attention in this article to the more difficult problem of fitting RS $GLD$ calibration model to data. Without loss of generality, the method we outlined below can be easily adapted to build FKML $GLD$ calibration model.

3. Statistical Model

3.1 $GLD$ Based Calibration Model

We consider the following usual calibration model:

$$
y_i = \alpha + \beta x_i + \epsilon_i, \quad i = 1, \cdots, n
$$

(3.1)

$$
y_{0j} = \alpha + \beta x_0 + \epsilon_j, \quad j = 1, \cdots, k
$$

(3.2)

We assume that $\epsilon_i$ and $\epsilon_j$ are i.i.d. $GLD(\alpha, \beta, \lambda_3, \lambda_4)$. In general, we consider $x_1, \cdots, x_n$ to be known and fixed and $\alpha, \beta, \lambda_2, \lambda_3$ and $\lambda_4$ are parameters we need to estimate. Our $GLD$ calibration model takes the following form:

$$
y_{i|\xi} \sim GLD(\alpha + \beta x_i, \lambda_2, \lambda_3, \lambda_4),
$$

(3.3)
where given below.

Consequently, the likelihood function for RS \( GADF \) is:

\[
L(\theta, y, y_0) = \prod_{i=1}^{n} \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}} \cdot \prod_{j=1}^{k} \frac{\lambda_2}{\lambda_3 z_j^{\lambda_1} + \lambda_4 (1 - z_j)^{\lambda_1}}.
\]  

(3.5)

where

\[
y_i = (\alpha + \beta x_i) + \frac{z_i^{\lambda_1} - (1 - z_i)^{\lambda_1}}{\lambda_2},
\]

\[
y_{0j} = (\alpha + \beta x_0) + \frac{z_j^{\lambda_1} - (1 - z_j)^{\lambda_1}}{\lambda_2},
\]

and \( 0 \leq z_i, z_j \leq 1, \theta = (\alpha, \beta, x_0, \lambda_2, \lambda_3, \lambda_4). \)

3.2 Estimation of Parameters

From (3.5), we obtain the following log likelihood function:

\[
\log L(\theta, y, y_0) = \sum_{i=1}^{n} \log (f_1(\theta, y_i)) + \sum_{j=1}^{k} \log \left( f_2(\theta, y_{0j}) \right)
\]

(3.6)

where

\[
f_1(\theta, y_i) = \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}},
\]

\[
f_2(\theta, y_{0j}) = \frac{\lambda_2}{\lambda_3 z_j^{\lambda_1} + \lambda_4 (1 - z_j)^{\lambda_1}}
\]

Taking the derivative of (3.6), we obtain the following:

\[
\frac{\partial \log L(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{1}{f_1} \frac{\partial f_1}{\partial \theta} + \sum_{j=1}^{k} \frac{1}{f_2} \frac{\partial f_2}{\partial \theta},
\]

(3.7)

where \( \theta = (\alpha, \beta, x_0, \lambda_2, \lambda_3, \lambda_4). \)

Theoretically, the MLE of \( \theta \) is the solution of (3.7) when it is set to be equal to 0. The derivatives \( \frac{\partial f_1}{\partial \theta} \) and \( \frac{\partial f_2}{\partial \theta} \) are given below.

\[
\frac{\partial f_1}{\partial \lambda_2} = \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}},
\]

\[
\frac{\partial f_2}{\partial \lambda_2} = \frac{\lambda_2}{\lambda_3 z_j^{\lambda_1} + \lambda_4 (1 - z_j)^{\lambda_1}}
\]

\[
\frac{\partial f_1}{\partial \lambda_3} = \left( -\lambda_2 \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}} \right) \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}} \cdot \left( \frac{z_i^{\lambda_1} - (1 - z_i)^{\lambda_1}}{\lambda_2^2} \right)
\]

\[
\frac{\partial f_2}{\partial \lambda_3} = \left( -\lambda_2 \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}} \right) \frac{\lambda_2}{\lambda_3 z_j^{\lambda_1} + \lambda_4 (1 - z_j)^{\lambda_1}}
\]

\[
\frac{\partial f_1}{\partial \lambda_4} = \left( -\lambda_2 \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}} \right) \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}} \cdot \left( \frac{z_i^{\lambda_1} - (1 - z_i)^{\lambda_1}}{\lambda_2^2} \right)
\]

\[
\frac{\partial f_2}{\partial \lambda_4} = \left( -\lambda_2 \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}} \right) \frac{\lambda_2}{\lambda_3 z_j^{\lambda_1} + \lambda_4 (1 - z_j)^{\lambda_1}}
\]

\[
\frac{\partial f_1}{\partial \alpha} = \left( -\lambda_2 \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}} \right) \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}} \cdot \left( \frac{z_i^{\lambda_1} - (1 - z_i)^{\lambda_1}}{\lambda_2^2} \right)
\]

\[
\frac{\partial f_2}{\partial \alpha} = \left( -\lambda_2 \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}} \right) \frac{\lambda_2}{\lambda_3 z_j^{\lambda_1} + \lambda_4 (1 - z_j)^{\lambda_1}}
\]

\[
\frac{\partial f_1}{\partial \beta} = \left( -\lambda_2 \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}} \right) \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}} \cdot \left( \frac{z_i^{\lambda_1} - (1 - z_i)^{\lambda_1}}{\lambda_2^2} \right)
\]

\[
\frac{\partial f_2}{\partial \beta} = \left( -\lambda_2 \frac{\lambda_2}{\lambda_3 z_i^{\lambda_1} + \lambda_4 (1 - z_i)^{\lambda_1}} \right) \frac{\lambda_2}{\lambda_3 z_j^{\lambda_1} + \lambda_4 (1 - z_j)^{\lambda_1}}
\]
\[
\frac{\partial f_2}{\partial \lambda_1} = (-\lambda_2) \frac{[A_3(\lambda_3 - 1)z_j^{\lambda_1-2} - A_4(\lambda_4 - 1)(1 - z_j)^{\lambda_4-2}](z_j^j \log z_j)}{(A_3z_j^{\lambda_1-1} + A_4(1 - z_j)^{\lambda_4-1})^3}
\]

\[
\frac{\partial f_1}{\partial \lambda_4} = \lambda_2 \frac{[A_3(\lambda_3 - 1)z_j^{\lambda_1-2} - A_4(\lambda_4 - 1)(1 - z_j)^{\lambda_4-2}][(1 - z_j)^{\lambda_4-1} \log(1 - z_j))]}{(A_3z_j^{\lambda_1-1} + A_4(1 - z_j)^{\lambda_4-1})^3}
\]

\[
\frac{\partial f_2}{\partial \alpha} = (-\lambda_2^2) \frac{[A_3(\lambda_3 - 1)z_j^{\lambda_1-2} - A_4(\lambda_4 - 1)(1 - z_j)^{\lambda_4-2}]}{(A_3z_j^{\lambda_1-1} + A_4(1 - z_j)^{\lambda_4-1})^3}
\]

\[
\frac{\partial f_2}{\partial \beta} = (-\lambda_2^2) \frac{[A_3(\lambda_3 - 1)z_j^{\lambda_1-2} - A_4(\lambda_4 - 1)(1 - z_j)^{\lambda_4-2}] \cdot x_0}{(A_3z_j^{\lambda_1-1} + A_4(1 - z_j)^{\lambda_4-1})^3}
\]

\[
\frac{\partial f_2}{\partial \lambda_0} = (-\lambda_2^2) \frac{[A_3(\lambda_3 - 1)z_j^{\lambda_1-2} - A_4(\lambda_4 - 1)(1 - z_j)^{\lambda_4-2}] \cdot \beta}{(A_3z_j^{\lambda_1-1} + A_4(1 - z_j)^{\lambda_4-1})^3}
\]

It is difficult to obtain the exact solutions of setting (3.7) to zero using the above formulations, owing to the fact that RS \textit{GID} is defined by its inverse quantile function and there is a high degree of complexity involved in solving the above equations. As an alternative, we carry out the maximum likelihood estimation by maximising (3.6) directly using Nelder-Mead optimisation algorithm as is customary done for maximum likelihood estimation problems involving \textit{GID} (see Su, 2010, 2007a, 2007b). This is a preferred and more reliable method of estimation as opposed to trying to satisfy the exact conditions to which all of the above equations equal to zero. The GLDEX package in R (Su, 2010, 2007a) facilitates the Nelder-Mead optimisation algorithm for \textit{GID}.

Our algorithm is as follows:

1) Generate a set of initial values for \(\alpha, \beta, x_0, \lambda_2, \lambda_3, \lambda_4\). There are a number of strategies that can be used to determine the best set of initial values. One strategy is to generate initial values \(\alpha, \beta, x_0\) using Normal or skewed Normal calibration model and then generate some low discrepancy quasi random numbers for \(\lambda_2, \lambda_3, \lambda_4\) over a range of values and select the set of initial values that maximises (3.6). Alternatively all initial values can be randomly generated using low discrepancy quasi random numbers.

2) Set \(\lambda_1 = \alpha + \beta x_0\).

3) Check that \textit{GID}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) is a valid statistical distribution, this can be done using GLDEX package in R.

4) Check the minimal support of \textit{GID}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) is lower or equal to the lowest value of \(y_0\). Similarly, check that the maximum support of \textit{GID}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) is greater or equal to the largest value of \(y_0\). This is to ensure that the fitted \textit{GID} will span the entire dataset. If these conditions are not met, choose another set of initial values and repeat from 2).

5) Conduct Nelder Mead optimisation by maximising (3.6) directly using the above initial values to obtain the required estimates.

4. Simulations

We conduct simulations to illustrate the performance of our RS \textit{GID} calibration model for sample size \(n = 30, 50, 100\) and 200 with \(\alpha = 3, \beta = 1.5, x_0 = 15\) or 40, \(\lambda_3 = 10, \lambda_4 = 1\), and \(\lambda_2 = 2, 5, 10\). We further generate \(x_1, x_2, \ldots, x_4\) from \textit{Uniform}(10,30), and we set \(k = 1\). We use the true parameters as our initial values to kick start the optimisation process to obtain our MLE estimate for \(x_0\).

We repeat this process 1000 times, which give us 1000 \(\hat{x}_{0m}\) estimates of \(x_0\). The mean \(\hat{x}_0\), Bias(\(x_0\)) and MSE(\(x_0\)) are calculated as follows:

\[
\hat{x}_0 = \frac{1}{1000} \sum_{m=1}^{1000} \hat{x}_{0m}
\]

Bias(\(x_0\)) = \[\frac{1}{1000} \sum_{m=1}^{1000} (\hat{x}_{0m} - x_0)\]

MSE(\(x_0\)) = \[\frac{1}{1000} \sum_{m=1}^{1000} (\hat{x}_{0m} - x_0)^2\]
The results of above simulations are shown in Tables 1 and 2. As expected, the MSE decreases as we increase the sample size or increase the value of inverse scale parameter $\lambda_2$. In terms of bias, we observe that the performance appear to be fairly consistent across sample sizes, this gives confidence in the use of RS $G\text{AD}$ calibration model for smaller samples, even though there are are more parameters that need to be estimated from this model. There also appears to be a tendency for RS $G\text{AD}$ calibration model to slightly overestimate as nearly all the bias results are positive. Increasing the shape parameter $\lambda_3$ does not always result in increase in MSE, this is because the shape parameter spaces of $\lambda_3$ and $\lambda_4$ for RS $G\text{AD}$ are fairly complex.

Table 1. Simulations results with $x_0 = 15$, $\alpha = 3, \beta = 1.5, \lambda_4 = 1$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\lambda_3$</th>
<th>$\hat{\lambda}_2 = 2$ Bias</th>
<th>MSE</th>
<th>$\hat{\lambda}_3 = 5$ Bias</th>
<th>MSE</th>
<th>$\hat{\lambda}_4 = 10$ Bias</th>
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<tr>
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Table 2. Simulations results with $x_0 = 40$, $\alpha = 3, \beta = 1.5, \lambda_4 = 1$

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<th>$\lambda_3$</th>
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<th>MSE</th>
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<td>40.0114 0.0114 0.0017</td>
<td>40.0031 0.0031 0.0007</td>
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Table 3. Simulations results with $x_0 = 15$, $\alpha = 3, \beta = 1.5$, true error distribution $GEV(0.1860, 0.4016, 0.1511)$ is approximated by RS $G\text{AD}$ with $\lambda_1 = 0, \lambda_2 \approx -0.0374, \lambda_3 \approx -0.0027, \lambda_4 \approx -0.0212$

<table>
<thead>
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<th>$n$</th>
<th>$\hat{x}_0$</th>
<th>Bias</th>
<th>MSE</th>
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<td>0.2815 0.1774</td>
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<tr>
<td>200</td>
<td>15.2860</td>
<td>0.2860 0.1689</td>
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</table>

We further considered using RS $G\text{AD}$ to approximate generalized extreme value distribution (GEV) with location, scale and shape parameters being 0.1860, 0.4016, 0.1511 respectively. We choose RS $G\text{AD}$ with $\lambda_1 = 0, \lambda_2 \approx -0.0374, \lambda_3 \approx -0.0027, \lambda_4 \approx -0.0212$ for this demonstration (Figure 1). We then generate simulated data based on GEV and use our approximated RS $G\text{AD}$ to estimate $x_0$ with $\alpha = 3, \beta = 1.5$ and repeat this over 1000 simulation runs. The result of this simulation is given in Table 3. We observe that the RS $G\text{AD}$ calibration model tends to overestimate the true $x_0$ by a small margin, but the bias appears to decrease as sample size increases.
5. Application

We apply the RS $GAD$ calibration model to a dataset which measures teenager testicular volume ($ml^3$). This dataset is from Chipkevitch, Nishimura, Tu and Galea-Rajas (1996) and consists of 42 observations. Figueiredoa et al. (2010) considered two measurement methods from Chipkevitch et al. (1996): dimensional measurement with a caliper (DM) and measurement by ultrasonography (US) and the data is given in Table 4. In their paper, Figueiredoa et al. (2010) consider the $x_0$ value of 16.4, which is observed twice by ultrasonography. They subsequently treated this value as unknown, with corresponding $y_0$ values of $y_{01} = 10.3$ and $y_{02} = 17.3$. Then, they estimate $x_0$ using their skewed Normal calibration model and compared this with the standard Normal calibration model. We did the same using the RS $GAD$ calibration model and our results are shown in Table 5.

Table 4. Measurements obtained by dimensional measurement with a caliper (DM) and by ultrasonography (US) from the right testis for 42 teenagers, in $ml^3$

<table>
<thead>
<tr>
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<td>10</td>
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<td>6.1</td>
<td>4.1</td>
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<td>2</td>
<td>3</td>
<td>6.1</td>
<td>5.4</td>
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</table>

Table 5. A comparison of linear calibration models

<table>
<thead>
<tr>
<th>Parameter</th>
<th>RS $GAD$ model</th>
<th>$SN$ model</th>
<th>Normal model</th>
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<tr>
<td>$\sigma$</td>
<td>0.014</td>
<td>0.497</td>
<td>-0.69</td>
</tr>
<tr>
<td>$\beta$</td>
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<td>0.035</td>
<td>0.86</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>-</td>
<td>-</td>
<td>2.13</td>
</tr>
<tr>
<td>$x_0$</td>
<td>12.128</td>
<td>0.963</td>
<td>12.66</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>-</td>
<td>-</td>
<td>2.16</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>0.146</td>
<td>0.355</td>
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</tr>
<tr>
<td>$\lambda_3$</td>
<td>-0.030</td>
<td>0.061</td>
<td>-</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>-0.162</td>
<td>0.184</td>
<td>-</td>
</tr>
<tr>
<td>AIC</td>
<td>150.36</td>
<td>160.69</td>
<td>163.74</td>
</tr>
<tr>
<td>BIC</td>
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<td>169.38</td>
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</tr>
<tr>
<td>HQ</td>
<td>144.58</td>
<td>156.55</td>
<td>161.15</td>
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</table>

The theoretical derivation of the variability of our estimates under $RS\ GAD$ is not readily tractable as in the cases of skewed Normal and Normal distributions. As we need to numerically derive our calculations, small errors in...
numerical procedures could accumulate into large errors even if we could evaluate the exact theoretical solution. As a workaround, we adopt the following procedure. Once we obtained the parameters of our model, \( \alpha, \beta, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \), we conduct simulations to estimate the variability of our estimate. We use our estimated parameters from the RS G\( \lambda \)D calibration model and \( x_1 \) (excluding \( x_1 = 16.4 \)) from the original data to randomly generate \( y_{0j} \) and \( y_i \) according to (3.1) and (3.2). We then maximise the likelihood in (3.6) using Nelder Mead Simplex algorithm with initial values being our original estimated parameters. We repeat the process 1000 times and calculate the sample standard deviations of our estimated parameters.

Table 5 lists the estimated parameters and their standard deviations from RS G\( \lambda \)D, skewed Normal and Normal calibration models. We compute the Akaike, Bayesian and Hannan-Quinn information criterion (AIC, BIC, and HQ) to allow model selection between three models. All three criterion favors the RS G\( \lambda \)D calibration model. In addition, the RS G\( \lambda \)D model is much more efficient compared to the other models, with the smallest variability in its parameter estimates.

6. Concluding Remarks

We propose a new calibration model with RS G\( \lambda \)D errors, which is an extremely flexible model that can cope with a wide range of different error distributions. Our method also lends to the development of FFKML G\( \lambda \)D calibration model, which may have better properties with regard to numerical convergence. Our simulations studies suggest our proposed model perform well for small sample sizes across a range of inverse scale and shape parameters of RS G\( \lambda \)D. We further demonstrate that the RS G\( \lambda \)D calibration model can outperform skewed Normal or Normal calibration model, with lower AIC, BIC and HQ information criterion and lower variability in our parameter estimates in the context of a real life data. These simulation results are promising and future statistical models should aim to develop statistical technique that are tailored to data, rather than requiring empirical data to satisfy a particular statistical model. One possible extension of our model is the development of a mixture RS G\( \lambda \)D calibration model, which would extend the flexibility of our model even further but also present a very challenging problem for data with small samples.

References


