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The RS Generalized Lambda Distribution Based Calibration Model

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Abstract

We propose a flexible linear calibration model with errors from RS (Ramberg & Schmeiser, 1974) generalized lambda distribution ($G\lambda D$). We demonstrate the derivation of the maximum likelihood estimates of RS $G\lambda D$ parameters and examine the estimation performance using a simulation study for sample sizes ranging from 30 to 200. The use of RS $G\lambda D$ calibration model not only provides statistical modeller with a richer range of distributional shapes, but can also provide more precise parameter estimates compared to the standard Normal calibration model or skewed Normal calibration model proposed by Figueiredoa, Bolfarinea, Sandovala and Limab (2010).

Keywords: generalized lambda distribution, linear calibration model, skew normal distribution, maximum likelihood estimation

1. Introduction

The statistical calibration model is a reverse regression technique, where we use the response variable to predict the corresponding explanatory variable. There are number of applications of this technique in science. For example, we may use radiometric dating to ascertain the age of a tree and further verify our result using tree rings. Our aim, however, is to use radiometric dating to estimate age of new trees, and the problem is whether we should minimize errors in the observation or minimize errors in age determination. There are many similar problems in substance concentration determination in biology and chemistry, physical quantities determination in physics and blood pressure/cholsterol level measurement in medicine. The literature on calibration problem has a long history, and one of the earliest works can be found in Eisenhart (1939).

The usual calibration experiment is a two stage process involving two random variables X and Y. The first stage is known as the calibration trial, where we observe the n values of the response variable $y_1, \cdots, y_n$ from a given set of explanatory values $x_1, \cdots, x_n$ and we can estimate the link function between X and Y. The second stage is known as the calibration experiment, where we observe $k \geq 1$ value(s) of the response variable Y as $y_{01}, \cdots, y_{0k}$ which are mapped from some unknown value x_0 from the explanatory variable X. We can express these two stages by the following equations.

$$y_i = \alpha + \beta x_i + \varepsilon_i, \quad i = 1, \cdots, n;$$
$$y_{0j} = \alpha + \beta x_0 + \varepsilon_{0j}, \quad j = 1, \cdots, k, \tag{1.1}$$

We usually assume that the errors $\varepsilon_1, \cdots, \varepsilon_n, \varepsilon_{01}, \cdots, \varepsilon_{0k}$ are i.i.d and Normally distributed with mean 0 and variance σ^2. Also, $x_1, \cdots, x_n$ are known and α, β, x_0 and σ^2 are unknown parameters which we need to estimate.

As an extension to Normal distribution, Azzalini (1985) introduced the skewed Normal distribution. The skewed Normal distribution is defined as

$$g(x; \xi, \omega, \lambda) = \frac{2}{\omega}\phi\left(\frac{x-\xi}{\omega}\right)\Phi\left(\lambda\left(\frac{x-\xi}{\omega}\right)\right), \tag{1.2}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the p.d.f. and c.d.f. of a standard normal distribution respectively. Specially, when $\xi = 0$ and $\omega = 1$, we obtain the standard skewed Normal distribution.

Based on (1.2), Figueiredoa et al. (2010) defined a skew-normal calibration model by assuming that ε_i and ε_{0j} are i.i.d. and follow a skewed Normal distribution with $\xi = 0$ denoted by $SN(0, \omega, \lambda)$. This gives us the following calibration model:

$$y_i|x_i \sim SN(\alpha + \beta x_i; \omega; \lambda), \quad i = 1.\cdots, n,$$
$$y_{0j}|x_0 \sim SN(\alpha + \beta x_i; \omega; \lambda), \quad j = 1, \cdots, k. \tag{1.3}$$

In (1.3), the conditional distribution of y_i given x_i and y_{0j} given x_0 are governed by skewed Normal distributions. This skewed Normal calibration model allows the modeller to cope with some degree of skewness in the error distribution. However, this is still limited as the skewed Normal distribution have limited range of shapes. The skewed Normal distribution still cannot handle heavy tailed, U shape, uniform, triangular or exponential upward/downward patterns. These shapes however, can be captured using $G\lambda D$ (generalized lambda distributions), and we propose a further extension to the calibration model by using RS $G\lambda D$.

Our article is organized as follows. In Section 2, we introduce the $G\lambda D$ family. In Section 3, we outline the RS $G\lambda D$ calibration model and discuss possible ways to estimate parameters of the model using maximum likelihood estimation. In Section 4, we demonstrate the estimation performance of our proposed model across a range of different sample sizes from 30 to 200. As a further test to our proposed model to the literature, we compare the performance of RS $G\lambda D$ calibration model against Normal and skewed Normal calibration model with respect to a real life dataset used by Figueiredoa et al. (2010) in Section 5. A discussion of our proposed method is given in Section 6.

2. Generalized Lambda Distributions

The RS $G\lambda D$ (Ramberg & Schmeiser, 1974) is an extension of Tukey's lambda distribution. It is defined by its inverse distribution function:

$$F^{-1}(u) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2} \qquad 0 \le u \le 1 \tag{2.1}$$

From (2.1), $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are respectively the location, inverse scale and shape 1 and shape 2 parameters. Karian and Dudewicz (2000) noted that $G\lambda D$ is defined only if $\dfrac{\lambda_2}{\lambda_3 u^{\lambda_3-1} + \lambda_4(1-u)^{\lambda_4-1}} \ge 0$ for $0 \le u \le 1$. The conditions for which RS $G\lambda D$ is a valid p.d.f. are set out in Karian and Dudewicz (2000) and these are also programmed in GLDEX package in R (Su, 2010, 2007a).

Freimer, Kollia, Mudholkar and Lin (1988) describe another distribution known as FKML $G\lambda D$. The FKML $G\lambda D$ can be written as:

$$F^{-1}(u) = \lambda_1 + \frac{\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}}{\lambda_2} \qquad 0 \le u \le 1 \tag{2.2}$$

Under (2.2), $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are respectively the location, inverse scale and shape 1 and shape 2 parameters.

The fundamental motivation for the development of FKML $G\lambda D$ is that the distribution is defined over all λ_3 and λ_4 (Freimer et al., 1988). The only restriction on FKML $G\lambda D$ is $\lambda_2 > 0$. This is more convenient to deal with computationally than RS $G\lambda D$ and hence it is sometimes the preferred $G\lambda D$ for some researchers.

We restrict our attention in this article to the more difficult problem of fitting RS $G\lambda D$ calibration model to data. Without loss of generality, the method we outlined below can be easily adapted to build FKML $G\lambda D$ calibration model.

3. Statistical Model

3.1 GλD Based Calibration Model

We consider the following usual calibration model:

$$y_i = \alpha + \beta x_i + \epsilon_i, \quad i = 1, \cdots, n, \tag{3.1}$$

$$y_{0j} = \alpha + \beta x_0 + \epsilon_j, \quad j = 1, \cdots, k. \tag{3.2}$$

We assume that ϵ_i and ϵ_j are i.i.d. $G\lambda D(0, \lambda_2, \lambda_3, \lambda_4)$. In general, we consider $x_1, \cdots, x_n$ to be known and fixed and $\alpha, \beta, \lambda_2, \lambda_3$ and λ_4 are parameters we need to estimate. Our $G\lambda D$ calibration model takes the following form:

$$y_i|x_i \sim G\lambda D(\alpha + \beta x_i, \lambda_2, \lambda_3, \lambda_4), \tag{3.3}$$

$$y_{0j}|x_0 \sim G\lambda D(\alpha + \beta x_0, \lambda_2, \lambda_3, \lambda_4). \tag{3.4}$$

Consequently, the likelihood function for RS $G\lambda D$ is:

$$L(\boldsymbol{\theta}, \boldsymbol{y}, \boldsymbol{y}_0) = \prod_{i=1}^{n} \frac{\lambda_2}{\lambda_3 z_i^{\lambda_3 - 1} + \lambda_4 (1 - z_i)^{\lambda_4 - 1}} \cdot \prod_{j=1}^{k} \frac{\lambda_2}{\lambda_3 z_j^{\lambda_3 - 1} + \lambda_4 (1 - z_j)^{\lambda_4 - 1}}, \tag{3.5}$$

where

$$y_i = (\alpha + \beta x_i) + \frac{z_i^{\lambda_3} - (1 - z_i)^{\lambda_4}}{\lambda_2},$$
$$y_{0j} = (\alpha + \beta x_0) + \frac{z_j^{\lambda_3} - (1 - z_j)^{\lambda_4}}{\lambda_2},$$

and $0 \leq z_i, z_j \leq 1, \boldsymbol{\theta} = (\alpha, \beta, x_0, \lambda_2, \lambda_3, \lambda_4)$.

3.2 Estimation of Parameters

From (3.5), we obtain the following log likelihood function:

$$\log L(\boldsymbol{\theta}, \boldsymbol{y}, \boldsymbol{y_0}) = \sum_{i=1}^{n} \log\left(f_1(\boldsymbol{\theta}, y_i)\right) + \sum_{j=1}^{k} \log\left(f_2(\boldsymbol{\theta}, y_{0j})\right) \tag{3.6}$$

where

$$f_1(\boldsymbol{\theta}, y_i) = \frac{\lambda_2}{\lambda_3 z_i^{\lambda_3 - 1} + \lambda_4 (1 - z_i)^{\lambda_4 - 1}},$$
$$f_2(\boldsymbol{\theta}, y_{0j}) = \frac{\lambda_2}{\lambda_3 z_j^{\lambda_3 - 1} + \lambda_4 (1 - z_j)^{\lambda_4 - 1}}$$

Taking the derivative of (3.6), we obtain the following:

$$\frac{\partial \log L(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \sum_{i=1}^{n} \frac{1}{f_1} \frac{\partial f_1}{\partial \boldsymbol{\theta}} + \sum_{j=1}^{k} \frac{1}{f_2} \frac{\partial f_2}{\partial \boldsymbol{\theta}}, \tag{3.7}$$

where $\boldsymbol{\theta} = (\alpha, \beta, x_0, \lambda_2, \lambda_3, \lambda_4)$.

Theoretically, the MLE of $\boldsymbol{\theta}$ is the solution of (3.7) when it is set to be equal to 0. The derivatives $\frac{\partial f_1}{\partial \boldsymbol{\theta}}$ and $\frac{\partial f_2}{\partial \boldsymbol{\theta}}$ are given below.

$$\begin{aligned}
\frac{\partial f_1}{\partial \lambda_2} &= \frac{\partial f_1}{\partial z_i} \cdot \frac{\partial z_i}{\partial y_i} \cdot \frac{\partial y_i}{\partial \lambda_2} \\
&= \left(\lambda_2 \frac{-\lambda_3(\lambda_3 - 1) z_i^{\lambda_3 - 2} + \lambda_4(\lambda_4 - 1)(1 - z_i)^{\lambda_4 - 2}}{(\lambda_3 z_i^{\lambda_3 - 1} + \lambda_4(1 - z_i)^{\lambda_4 - 1})^2}\right) \cdot \left(\frac{\lambda_2}{\lambda_3 z_i^{\lambda_3 - 1} + \lambda_4(1 - z_i)^{\lambda_4 - 1}}\right) \cdot \left(-\frac{z_i^{\lambda_3} - (1 - z_i)^{\lambda_4}}{\lambda_2^2}\right) \\
&= \frac{[\lambda_3(\lambda_3 - 1) z_i^{\lambda_3 - 2} - \lambda_4(\lambda_4 - 1)(1 - z_i)^{\lambda_4 - 2}](z_i^{\lambda_3} - (1 - z_i)^{\lambda_4})}{(\lambda_3 z_i^{\lambda_3 - 1} + \lambda_4(1 - z_i)^{\lambda_4 - 1})^3} \\
\frac{\partial f_1}{\partial \lambda_3} &= (-\lambda_2) \frac{[\lambda_3(\lambda_3 - 1) z_i^{\lambda_3 - 2} - \lambda_4(\lambda_4 - 1)(1 - z_i)^{\lambda_4 - 2}](z_i^{\lambda_3} \log z_i)}{(\lambda_3 z_i^{\lambda_3 - 1} + \lambda_4(1 - z_i)^{\lambda_4 - 1})^3} \\
\frac{\partial f_1}{\partial \lambda_4} &= \lambda_2 \frac{[\lambda_3(\lambda_3 - 1) z_i^{\lambda_3 - 2} - \lambda_4(\lambda_4 - 1)(1 - z_i)^{\lambda_4 - 2}]((1 - z_i)^{\lambda_3} \log(1 - z_i))}{(\lambda_3 z_i^{\lambda_3 - 1} + \lambda_4(1 - z_i)^{\lambda_4 - 1})^3} \\
\frac{\partial f_1}{\partial \alpha} &= (-\lambda_2^2) \frac{[\lambda_3(\lambda_3 - 1) z_i^{\lambda_3 - 2} - \lambda_4(\lambda_4 - 1)(1 - z_i)^{\lambda_4 - 2}]}{(\lambda_3 z_i^{\lambda_3 - 1} + \lambda_4(1 - z_i)^{\lambda_4 - 1})^3} \\
\frac{\partial f_1}{\partial \beta} &= (-\lambda_2^2) \frac{[\lambda_3(\lambda_3 - 1) z_i^{\lambda_3 - 2} - \lambda_4(\lambda_4 - 1)(1 - z_i)^{\lambda_4 - 2}] \cdot x_i}{(\lambda_3 z_i^{\lambda_3 - 1} + \lambda_4(1 - z_i)^{\lambda_4 - 1})^3} \\
\frac{\partial f_2}{\partial \lambda_2} &= \frac{[\lambda_3(\lambda_3 - 1) z_j^{\lambda_3 - 2} - \lambda_4(\lambda_4 - 1)(1 - z_j)^{\lambda_4 - 2}](z_j^{\lambda_3} - (1 - z_j)^{\lambda_4})}{(\lambda_3 z_i^{\lambda_3 - 1} + \lambda_4(1 - z_j)^{\lambda_4 - 1})^3}
\end{aligned}$$

$$\frac{\partial f_2}{\partial \lambda_3} = (-\lambda_2)\frac{[\lambda_3(\lambda_3-1)z_j^{\lambda_3-2} - \lambda_4(\lambda_4-1)(1-z_j)^{\lambda_4-2}](z_j^{\lambda_3}\log z_j)}{(\lambda_3 z_i^{\lambda_3-1} + \lambda_4(1-z_i)^{\lambda_4-1})^3}$$

$$\frac{\partial f_1}{\partial \lambda_4} = \lambda_2\frac{[\lambda_3(\lambda_3-1)z_j^{\lambda_3-2} - \lambda_4(\lambda_4-1)(1-z_j)^{\lambda_4-2}]((1-z_j)^{\lambda_3}\log(1-z_j))}{(\lambda_3 z_j^{\lambda_3-1} + \lambda_4(1-z_j)^{\lambda_4-1})^3}$$

$$\frac{\partial f_2}{\partial \alpha} = (-\lambda_2^2)\frac{[\lambda_3(\lambda_3-1)z_j^{\lambda_3-2} - \lambda_4(\lambda_4-1)(1-z_j)^{\lambda_4-2}]}{(\lambda_3 z_j^{\lambda_3-1} + \lambda_4(1-z_j)^{\lambda_4-1})^3}$$

$$\frac{\partial f_2}{\partial \beta} = (-\lambda_2^2)\frac{[\lambda_3(\lambda_3-1)z_j^{\lambda_3-2} - \lambda_4(\lambda_4-1)(1-z_j)^{\lambda_4-2}]\cdot x_0}{(\lambda_3 z_j^{\lambda_3-1} + \lambda_4(1-z_j)^{\lambda_4-1})^3}$$

$$\frac{\partial f_2}{\partial x_0} = (-\lambda_2^2)\frac{[\lambda_3(\lambda_3-1)z_j^{\lambda_3-2} - \lambda_4(\lambda_4-1)(1-z_j)^{\lambda_4-2}]\cdot \beta}{(\lambda_3 z_j^{\lambda_3-1} + \lambda_4(1-z_j)^{\lambda_4-1})^3}$$

It is difficult to obtain the exact solutions of setting (3.7) to zero using the above formulations, owing to the fact that RS $G\lambda D$ is defined by its inverse quantile function and there is a high degree of complexity involved in solving the above equations. As an alternative, we carry out the maximum likelihood estimation by maximising (3.6) directly using Nelder-Mead optimisation algorithm as is customary done for maximum likelihood estimation problems involving $G\lambda D$ (see Su, 2010, 2007a, 2007b). This is a preferred and more reliable method of estimation as opposed to trying to satisfy the exact conditions to which all of the above equations equal to zero. The GLDEX package in R (Su, 2010, 2007a) facilitates the Nelder-Mead optimisation algorithm for $G\lambda D$.

Our algorithm is as follows:

1) Generate a set of initial values for $\alpha, \beta, x_0, \lambda_2, \lambda_3, \lambda_4$. There are a number of strategies that can be used to determine the best set of initial values. One strategy is to generate initial values α, β, x_0 using Normal or skewed Normal calibration model and then generate some low discrepancy quasi random numbers for $\lambda_2, \lambda_3, \lambda_4$ over a range of values and select the set of initial values that maximises (3.6). Alternatively all initial values can be randomly generated using low discrepancy quasi random numbers.

2) Set $\lambda_1 = \alpha + \beta x_0$.

3) Check that $G\lambda D(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is a valid statistical distribution, this can be done using GLDEX package in R.

4) Check the minimal support of $G\lambda D(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is lower or equal to the lowest value of y_0. Similarly, check that the maximum support of $G\lambda D(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is greater or equal to the largest value of y_0. This is to ensure that the fitted $G\lambda D$ will span the entire dataset. If these conditons are not met, choose another set of initial values and repeat from 2).

5) Conduct Nelder Mead optimisation by maximising (3.6) directly using the above initial values to obtain the required estimates.

4. Simulations

We conduct simulations to illustrate the performance of our RS $G\lambda D$ calibration model for sample size $n = 30, 50, 100$ and 200 with $\alpha = 3, \beta = 1.5, x_0 = 15$ or $40, \lambda_3 = 10, \lambda_4 = 1$, and $\lambda_2 = 2, 5, 10$. We further generate $x_1, x_2, \cdots, x_n$ from $Uniform(10, 30)$, and we set $k = 1$. We use the true parameters as our initial values to kick start the optimisation process to obtain our MLE estimate for x_0.

We repeat this process 1000 times, which give us 1000 $\hat{x}_{0m}$ estimates of x_0. The mean $\hat{x}_0$, Bias(x_0) and MSE(x_0) are calculated as follows:

$$\bar{\hat{x}}_0 = \frac{1}{1000}\sum_{m=1}^{1000}\hat{x}_{0m}$$

$$\text{Bias}(x_0) = \frac{1}{1000}\sum_{m=1}^{1000}(\hat{x}_{0m} - x_0)$$

$$\text{MSE}(x_0) = \frac{1}{1000}\sum_{m=1}^{1000}(\hat{x}_{0m} - x_0)^2$$

The results of above simulations are shown in Tables 1 and 2. As expected, the MSE decreases as we increase the sample size or increase the value of inverse scale parameter λ_2. In terms of bias, we observe that the performance appear to be fairly consistent across sample sizes, this gives confidence in the use of RS $G\lambda D$ calibration model for smaller samples, even though there are are more parameters that need to be estimated from this model. There also appears to be a tendency for RS $G\lambda D$ calibration model to slightly overestimate as nearly all the bias results are positive. Increasing the shape parameter λ_3 does not always result in increase in MSE, this is because the shape parameter spaces of λ_3 and λ_4 for RS $G\lambda D$ are fairly complex.

Table 1. Simulations results with $x_0 = 15, \alpha = 3, \beta = 1.5, \lambda_4 = 1$

		$\lambda_2 = 2$			$\lambda_2 = 5$			$\lambda_2 = 10$		
n	λ_3	$\hat{x}_0$	Bias	MSE	$\hat{x}_0$	Bias	MSE	$\hat{x}_0$	Bias	MSE
30	10	15.1105	0.1105	0.0263	15.0386	0.0386	0.0042	15.0178	0.0178	0.0010
50	10	15.0944	0.0944	0.0232	15.0352	0.0352	0.0040	15.0172	0.0172	0.0010
100	10	15.0994	0.0994	0.0184	15.0396	0.0396	0.0035	15.0185	0.0185	0.0008
200	10	15.1053	0.1053	0.0166	15.0340	0.0340	0.0030	15.0173	0.0173	0.0007
30	5	15.1430	0.1430	0.0292	15.0578	0.0578	0.0056	15.0285	0.0285	0.0012
50	5	15.1445	0.1445	0.0270	15.0530	0.0530	0.0047	15.0292	0.0292	0.0012
100	5	15.1381	0.1381	0.0214	15.0531	0.0531	0.0043	15.0264	0.0264	0.0010
200	5	15.1429	0.1429	0.0187	15.0534	0.0534	0.0038	15.0227	0.0227	0.0009
30	1	15.0271	0.0271	0.0244	15.0014	0.0014	0.0061	15.0040	0.0040	0.0017
50	1	15.0367	0.0367	0.0169	15.0030	0.0030	0.0048	14.9993	-0.0007	0.0014
100	1	15.0292	0.0292	0.0084	15.0093	0.0093	0.0030	15.0029	0.0029	0.0010
200	1	15.0262	0.0262	0.0052	15.0130	0.0130	0.0016	15.0022	0.0022	0.0007

Table 2. Simulations results with $x_0 = 40, \alpha = 3, \beta = 1.5, \lambda_4 = 1$

		$\lambda_2 = 2$			$\lambda_2 = 5$			$\lambda_2 = 10$		
n	λ_3	$\hat{x}_0$	Bias	MSE	$\hat{x}_0$	Bias	MSE	$\hat{x}_0$	Bias	MSE
30	10	40.1070	0.1070	0.0259	40.0375	0.0375	0.0049	40.0189	0.0189	0.0012
50	10	40.1051	0.1051	0.0235	40.0388	0.0388	0.0039	40.0177	0.0177	0.0009
100	10	40.1077	0.1077	0.0205	40.0353	0.0353	0.0031	40.0188	0.0188	0.0008
200	10	40.1088	0.1088	0.0169	40.0387	0.0387	0.0028	40.0184	0.0184	0.0008
30	5	40.1339	0.1339	0.0319	40.0557	0.0557	0.0064	40.0288	0.0288	0.0014
50	5	40.1391	0.1391	0.0302	40.0554	0.0554	0.0046	40.0280	0.0280	0.0013
100	5	40.1405	0.1405	0.0232	40.0479	0.0479	0.0039	40.0264	0.0264	0.0010
200	5	40.1538	0.1538	0.0236	40.0474	0.0474	0.0035	40.0205	0.0205	0.0007
30	1	40.0331	0.0331	0.0290	39.9984	-0.0016	0.0058	40.0035	0.0035	0.0016
50	1	40.0348	0.0348	0.0159	40.0031	0.0031	0.0045	40.0022	0.0022	0.0013
100	1	40.0311	0.0311	0.0099	40.0078	0.0078	0.0024	39.9996	-0.0004	0.0009
200	1	40.0217	0.0217	0.0036	40.0114	0.0114	0.0017	40.0031	0.0031	0.0007

Table 3. Simulations results with $x_0 = 15, \alpha = 3, \beta = 1.5$, true error distribution GEV(0.1860, 0.4016, 0.1511) is approximated by RS $G\lambda D$ with $\lambda_1 = 0, \lambda_2 \approx -0.0374, \lambda_3 \approx -0.0027, \lambda_4 \approx -0.0212$

n	$\hat{x}_0$	Bias	MSE
30	15.3140	0.3140	0.2149
50	15.3269	0.3269	0.2154
100	15.2815	0.2815	0.1774
200	15.2860	0.2860	0.1689

We further considered using RS $G\lambda D$ to approximate generalized extreme value distribution (GEV) with location, scale and shape parameters being 0.1860, 0.4016, 0.1511 respectively. We choose RS $G\lambda D$ with $\lambda_1 = 0, \lambda_2 \approx -0.0374, \lambda_3 \approx -0.0027, \lambda_4 \approx -0.0212$ for this demonstration (Figure 1). We then generate simulated data based on GEV and use our approximated RS $G\lambda D$ to estimate x_0 with $\alpha = 3, \beta = 1.5$ and repeat this over 1000 simulation runs. The result of this simulation is given in Table 3. We observe that the RS $G\lambda D$ calibration model tends to overestimate the true x_0 by a small margin, but the bias appears to decrease as sample size increases.

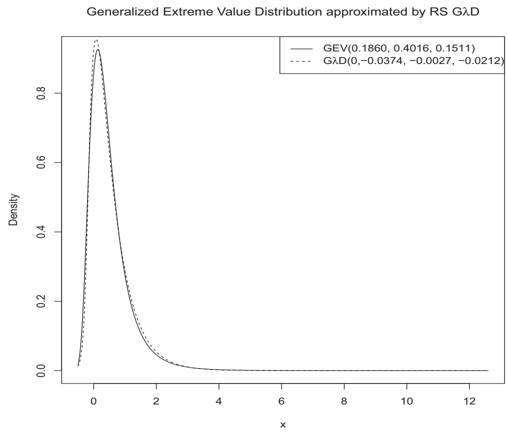

Figure 1. Approximating GEV using RS $G\lambda D$

5. Application

We apply the RS $G\lambda D$ calibration model to a dataset which measures teenager testicular volume (ml^3). This dataset is from Chipkevitch, Nishimura, Tu and Galea-Rajas (1996) and consists of 42 observations. Figueiredoa et al. (2010) considered two measurement methods from Chipkevitch et al. (1996): dimensional measurement with a caliper (DM) and measurement by ultrasonography (US) and the data is given in Table 4. In their paper, Figueiredoa et al. (2010) consider the x_0 value of 16.4, which is observed twice by ultrasonography. They subsequently treated this value as unknown, with corresponding y_{0j} values of $y_{01} = 10.3$ and $y_{02} = 17.3$. Then, they estimate x_0 using their skewed Normal calibration model and compared this with the standard Normal calibration model. We did the same using the RS $G\lambda D$ calibration model and our results are shown in Table 5.

Table 4. Measurements obtained by dimensional measurement with a caliper (DM) and by ultrasonography (US) from the right testis for 42 teenagers, in ml^3

DM	US	DM	US	DM	US	DM	US	DM	US	DM	US
5.9	5	**17.3**	**16.4**	7.2	6.7	4.8	5.7	17.3	17.6	5.9	5.3
6.8	7.4	7.9	10	16.3	20	3.1	2.6	4.4	4.1	16.3	18.8
5	5.7	11.4	12.7	12.2	13.9	4.4	6.1	4.1	2.7	10.3	9.4
6	6.2	11.1	10.2	10.8	9.1	8.8	10.4	15.3	16.5	13	14.1
7.9	9.1	3.9	4.5	8.4	9.3	13	14.8	4.5	5.6	22.1	20.9
10.3	**16.4**	9.7	11	10.6	11.5	8.2	9.6	11.3	9.2	9.7	9.7
19.8	15.7	8.8	8.5	11.6	13.7	2	3	6.1	5.4	8.1	8.9

Table 5. A comparison of linear calibration models

	RS $G\lambda D$ model		SN model		Normal model	
Parameter	Estimate	Stdev.	Estimate	Stdev.	Estimate	Stdev.
α	0.014	0.497	-0.69	-	0.32	0.56
β	0.855	0.035	0.86	0.07	0.92	0.05
σ	-	-	2.13	-	1.55	0.17
x_0	12.128	0.963	12.66	1.81	14.58	1.24
λ	-	-	2.16	1.73	-	-
λ_2	0.146	0.355	-	-	-	-
λ_3	-0.030	0.061	-	-	-	-
λ_4	-0.162	0.184	-	-	-	-
AIC	150.36		160.69		163.74	
BIC	160.79		169.38		170.69	
HQ	144.58		156.55		161.15	

The theoretical derivation of the variability of our estimates under RS $G\lambda D$ is not readily tractable as in the cases of skewed Normal and Normal distributions. As we need to numerically derive our calculations, small errors in numerical procedures could accumulate into large errors even if we could evaluate the exact theoretical solution. As a workaround, we adopt the following procedure. Once we obtained the parameters of our model, $\alpha, \beta, x_0, \lambda_2$, λ_3, λ_4, we conduct simulations to estimate the variability of our estimate. We use our estimated parameters from the RS $G\lambda D$ calibration model and x_i (excluding $x_i = 16.4$) from the original data to randomly generate y_{0j} and y_i according to (3.1) and (3.2). We then maximise the likelihood in (3.6) using Nelder Mead Simplex algorithm with initial values being our original estimated parameters. We repeat the process 1000 times and calculate the sample standard deviations of our estimated parameters.

Table 5 lists the estimated parameters and their standard deviations from RS $G\lambda D$, skewed Normal and Normal calibration models. We compute the Akaike, Bayesian and Hannan-Quinn information criterion (AIC, BIC, and HQ) to allow model selection between three models. All three criterion favors the RS $G\lambda D$ calibration model. In addition, the RS $G\lambda D$ model is much more efficient compared to the other models, with the smallest variability in its parameter estimates.

6. Concluding Remarks

We propose a new calibration model with RS $G\lambda D$ errors, which is an extremely flexible model that can cope with a wide range of different error distributions. Our method also lends to the development of FKML $G\lambda D$ calibration model, which may have better properties with regard to numerical convergence. Our simulations studies suggest our proposed model perform well for small sample sizes across a range of inverse scale and shape parameters of RS $G\lambda D$. We further demonstrate that the RS $G\lambda D$ calibration model can outperform skewed Normal or Normal calibration model, with lower AIC, BIC and HQ information criterion and lower variability in our parameter estimates in the context of a real life data. These simulation results are promising and future statistical models should aim to develop statistical technique that are tailored to data, rather than requiring empirical data to satisfy a particular statistical model. One possible extension of our model is the development of a mixture RS $G\lambda D$ calibration model, which would extend the flexibility of our model even further but also present a very challenging problem for data with small samples.

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