International Journal of Statistics and Probability; Vol. 2, No. 1; 2013
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

A Marshall-Olkin Power Log-normal Distribution and Its Applications to Survival Data

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Received: December 4, 2012 Accepted: December 23, 2012 Online Published: January 16, 2013

doi:10.5539/ijsp.v2n1p63 URL: http://dx.doi.org/10.5539/ijsp.v2n1p63

Abstract

In this paper, using Marshall-Olkin transformation, a new class of Extended Power Log-normal distribution which includes the Power Log-normal and Log-normal distributions as special cases is introduced. Its characterization and statistical properties are studied. A real survival dataset is analyzed and the results show that the proposed model is flexible and appropriate.

Keywords: power log-normal distribution, Marshall-Olkin transformation, survival analysis, maximum likelihood

1. Introduction

A Log-normal distribution is a well known continuous probability distribution of a random variable whose logarithm is normally distributed. In survival analysis, the lognormal distribution is extensively used in applications, for example, see Gupta et al. (1997), Royston (2001), Rutqvist (1985) and Johnson et al. (1996) etc. The density and cumulative distribution functions of a Log-normal random variable denoted by $X \sim LN(\mu, \sigma)$ are given by, for $-\infty < \mu < \infty, \sigma > 0, x > 0$,

$$f(x) = \frac{1}{x\sigma}\phi(\frac{\ln(x) - \mu}{\sigma}), \quad F(x) = \Phi(\frac{\ln(x) - \mu}{\sigma}), \tag{1}$$

where ϕ and Φ are the density and cumulative distribution functions of the standard normal distribution.

Nelson and Dognanksoy (1992) extended the Log-normal distribution and introduced the Power Log-normal distribution whose density and cumulative distribution functions are given by,

$$f(x) = \frac{p}{x\sigma}\phi(\frac{\mu - \ln(x)}{\sigma})[\Phi(\frac{\mu - \ln(x)}{\sigma})]^{p-1}, \quad F(x) = 1 - [\Phi(\frac{\mu - \ln(x)}{\sigma})]^p, \tag{2}$$

for $-\infty < \mu < \infty, \sigma > 0, p > 0, x > 0$. We denote it as $X \sim PLN(\mu, \sigma, p)$.

They fitted it to the life or strength data from specimens of various sizes. They presented that such a model arises when any specimen can be regarded as a series system of smaller portions, where portions of a certain size have a normal life (or strength) distribution. The statistical analysis can also be found in Nelson and Doganaksoy (1995). Szyszkowicz and Yanikomeroglu (2009) and Liu et al. (2008) proposed the use of power lognormal distributions to approximate lognormal sum distributions.

On the other hand, by adding a new parameter $\alpha > 0$ to an existing distribution, Marshall and Olkin (1997) proposed a new family of survival functions. The new parameter results in flexibility in the distribution. Let $\bar{F}(x) = 1 - F(x)$ be the survival function of a random variable X. Then

$$\bar{G}(x) = \frac{\alpha\bar{F}(x)}{1 - (1 - \alpha)\bar{F}(x)} \tag{3}$$

is a proper survival function. $\bar{G}(x)$ is called Marshall-Olkin family of distributions. If $\alpha = 1$, we have that $G = F$. The density function corresponding to (3) is given by

$$g(x) = \frac{\alpha f(x)}{[1 - (1 - \alpha)\bar{F}(x)]^2},$$

and the hazard rate function is given by

$$h(x) = \frac{h_F(x)}{1 - (1-\alpha)\bar{F}(x)},$$

where $h_F(x)$ is the hazard rate function of the original model with distribution F.

Using the Marshall-Olkin transformation (3), several researchers have studied various distribution extensions. Marshall and Olkin (1997) generalized the exponential and Weibull distributions. Alice and Jose (2003) introduced Marshall-Olkin extended semi Pareto model for Pareto type III and estabilished its geometric extreme stability. Semi-Weibull distribution and generalized Weibull distributions are discussed by Alice and Jose (2005). Ghitany et al. (2005) studied the Marshall-Olkin Weibull distribution, that can be obtained as a compound distribution mixing with exponential distribution, and applied it to model censored data. Marshall-Olkin Extended Lomax Distribution was introduced by Ghitany et al. (2007). Jose et al. (2010) investigated Marshall-Olkin q-Weibull distribution and its max-min processes. García et al. (2011) generalized the standard Log-normal distribution.

In this paper, we use the Marshall-Olkin transformation to define a new model, so-called the Marshall-Olkin Power Log-normal distribution (MPLN), which generalizes the Power Log-normal, the Log-normal model. We aim to reveal some statistical properties of the proposed model and apply it to survival analysis.

The rest of this article is organized as follows: in Section 2, we introduce the new defined distribution and investigate its basic properties, including the shape properties of its density function and the hazard rate function, stochastic orderings and representation, moments and measurements based on the moments. Section 3 discusses the estimation of parameters by the method of maximum likelihood. An application of the MPLN model to real survival data is illustrated in Section 4. Our work is concluded in Section 5.

2. Marshall-Olkin Power Log-normal Distribution and Its Properties

2.1 Density and Hazard Function

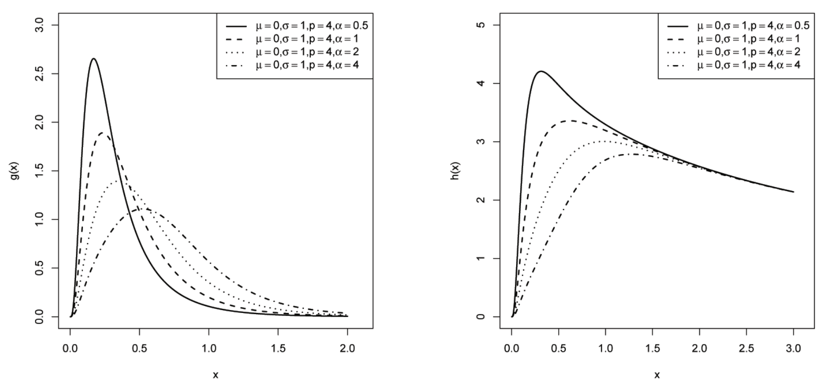
Let X follow the Power Log-normal distribution $PLN(\mu, \sigma, p)$, then its survival function is given by $\bar{F}(x) = 1 - F(x) = [\Phi(\frac{\mu-\ln(x)}{\sigma})]^p$. Substituting it in (3) we obtain a Marshall-Olkin Power Log-normal distribution denoted by $MPLN(\mu, \sigma, p, \alpha)$ with the following survival function

$$\bar{G}(x) = \frac{\alpha[\Phi(\frac{\mu-\ln(x)}{\sigma})]^p}{1-(1-\alpha)[\Phi(\frac{\mu-\ln(x)}{\sigma})]^p}, \quad x > 0. \tag{4}$$

The corresponding density function is given by

$$g(x) = \frac{p\alpha[\Phi(\frac{\mu-\ln(x)}{\sigma})]^{p-1}\phi(\frac{\mu-\ln(x)}{\sigma})}{x\sigma\left((\alpha-1)[\Phi(\frac{\mu-\ln(x)}{\sigma})]^p + 1\right)^2}, \quad x > 0. \tag{5}$$

If $\alpha = 1$, we obtain the Power Log-normal distribution with parameter μ, σ, p. Furthermore, if $p = 1$, it reduces to the Log-normal distribution. This distribution contains the Power Log-normal distribution and Log-normal distribution as particular cases.



(a) Density function (b) Hazard function

Figure 1. Plots of Marshall-Olkin power log-normal density and hazard function for some parameter values

Figure 1(a) shows some density functions of the $MPLN(\mu, \sigma, p, \alpha)$ distribution with various parameters. It indicates that the value of α has a subtantial effect on the tail of the density function.

The hazard rate function of the $MPLN(\mu, \sigma, p, \alpha)$ distribution is given by

$$h(x) = \frac{g(x)}{\bar{G}(x)} = \frac{p\phi(\frac{\mu-\ln(x)}{\sigma})}{x\sigma\Phi(\frac{\mu-\ln(x)}{\sigma})[(\alpha-1)[\Phi(\frac{\mu-\ln(x)}{\sigma})]^p+1]}, \quad x > 0. \tag{6}$$

Figure 1(b) shows some shapes of the $MPLN(\mu, \sigma, p, \alpha)$ hazard function with various parameters.

2.2 Stochastic Orderings

In statistics, a stochastic order measures the concept of one random variable being "larger" than another. It is an important tool to judge the comparative behavior. Here are some basic definitions.

A random variable X is less than Y in the ususal stochastic order (denoted by $X \prec_{st} Y$) if $F_X(x) \geq F_Y(x)$ for all real x. X is less than Y in the hazard rate order (denoted by $X \prec_{hr} Y$) if $h_X(x) \geq h_Y(x)$, for all $x > 0$. X is less than Y in the likelihood ratio order (denoted by $X \prec_{lr} Y$) if $f_Y(x)/f_X(x)$ increases in $x > 0$. It is well known that $X \prec_{lr} Y \Rightarrow X \prec_{hr} Y \Rightarrow X \prec_{st} Y$, see Ramesh and Kirmani (1987).

Proposition 1 *If $X \sim MPLN(\mu, \sigma, p, \alpha_1)$ and $Y \sim MPLN(\mu, \sigma, p, \alpha_2)$, and $\alpha_1 < \alpha_2$, then $X \prec_{lr} Y$, $X \prec_{hr} Y$ and $X \prec_{st} Y$.*

Proof. The density ratio is given by

$$U(x) = \frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1[1-(1-\alpha_2)\,\Phi^p(\frac{\mu-\ln(x)}{\sigma})]^2}{\alpha_2[1-(1-\alpha_1)\,\Phi^p(\frac{\mu-\ln(x)}{\sigma})]^2}.$$

Taking the derivative with respect to x,

$$U'(x) = \frac{2p\alpha_1(\alpha_1-\alpha_2)\Phi^{p-1}(\frac{\mu-\ln(x)}{\sigma})[(\alpha_2-1)\Phi^p(\frac{\mu-\ln(x)}{\sigma})+1]\phi(\frac{\mu-\ln(x)}{\sigma})}{x\sigma\alpha_2[(\alpha_1-1)\Phi^p(\frac{\mu-\ln(x)}{\sigma})+1]^3}.$$

If $\alpha_1 < \alpha_2$, $U'(x) < 0$, $U(x)$ is a decreasing function of x. The results follow.

2.3 Stochastic Representation

Let $\bar{G}_0(x|\lambda), -\infty < x < \infty, -\infty < \lambda < \infty$, be the conditional survival function of a continuous random variable X given a continuous random variable λ. Let Λ be a random variable with probability density function $m(\lambda)$. Then the distribution with survival function

$$\bar{G}(x) = \int_{-\infty}^{\infty} \bar{G}_0(x|\lambda)m(\lambda)d\lambda, \quad -\infty < x < \infty,$$

is called a compounding distribution with mixing density $m(\lambda)$. Compounding distribution provides a useful way to obtain new class of distributions in terms of existing ones. The following result shows that the $MPLN(\mu, \sigma, p, \alpha)$ distribution can be expressed as a compound distribution.

Proposition 2 *Suppose that the conditional survival function of a continuous random variable X given $\Lambda = \lambda$ is given by*

$$\bar{G}_0(x|\lambda) = \exp\left[-\lambda\Phi^{-p}(\frac{\mu-\ln(x)}{\sigma})+\lambda\right], \quad x > 0. \tag{7}$$

Let Λ have an exponential distribution with density function

$$m(\lambda) = \alpha e^{-\alpha\lambda}, \quad \alpha > 0, \lambda > 0.$$

Then the random variable X has the $MPLN(\mu, \sigma, p, \alpha)$ distribution.

Proof. For $x > 0$, the survival function of X is given by

$$\bar{G}(x) = \int_0^{\infty} \bar{G}_0(x|\lambda)m(\lambda)d\lambda = \alpha\int_0^{\infty} e^{-\lambda\Phi^{-p}(\frac{\mu-\ln(x)}{\sigma})+\lambda}e^{-\alpha\lambda}d\lambda = \frac{\alpha[\Phi(\frac{\mu-\ln(x)}{\sigma})]^p}{1-(1-\alpha)[\Phi(\frac{\mu-\ln(x)}{\sigma})]^p},$$

which is the survival function of the $MPLN(\mu, \sigma, p, \alpha)$ distribution.

For $\lambda > 0$, $\bar{G}_0(x|\lambda)$ defines a class of non-standard distributions. Compounding a distribution belonging to this class with an exponential distribution for λ leads to a certain $MPLN(\mu, \sigma, p, \alpha)$ distribution. Next we will present another stochastic representation of the $MPLN(\mu, \sigma, p, \alpha)$ distribution.

Proposition 3 *Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables with a Power Log-normal distribution $PLN(\mu, \sigma, p)$. Let N be a geometric random variable with parameter $0 < \alpha < 1$ such that $P(N = n) = \alpha(1 - \alpha)^{n-1}, n = 1, 2, \ldots,$ which is independent of $\{X_i, i \geq 1\}$. Then,*

(1) $\min(X_1, \ldots, X_N)$ *has a Marshall-Olkin Power Log-normal distribution $MPLN(\mu, \sigma, p, \alpha)$.*

(2) $\max(X_1, \ldots, X_N)$ *has a Marshall-Olkin Power Log-normal distribution $MPLN(\mu, \sigma, p, 1/\alpha)$.*

Proof. The survival function of $\min(X_1, \ldots, X_N)$ is

$$\begin{aligned} P(\min(X_1, \ldots, X_N) > x) &= \sum_{n=1}^{\infty} P(X_1 > x, \ldots, X_n > x) P(N = n) \\ &= \sum_{n=1}^{\infty} [\bar{F}(x)]^n \alpha(1 - \alpha)^{n-1} \\ &= \frac{\alpha \bar{F}(x)}{1 - (1 - \alpha)\bar{F}(x)}, \end{aligned}$$

which is the survival function of the Marshall-Olkin Power Log-normal distribution $MPLN(\mu, \sigma, p, \alpha)$.

The survival function of $\max(X_1, \ldots, X_N)$ is

$$\begin{aligned} P(\max(X_1, \ldots, X_N) > x) &= 1 - P(\max(X_1, \ldots, X_N) \leq x) \\ &= 1 - \sum_{n=1}^{\infty} P(X_1 \leq x, \ldots, X_n \leq x) P(N = n) \\ &= 1 - \sum_{n=1}^{\infty} [F(x)]^n \alpha(1 - \alpha)^{n-1} \\ &= \frac{\frac{1}{\alpha}\bar{F}(x)}{1 - (1 - \frac{1}{\alpha})\bar{F}(x)}, \end{aligned}$$

which is the survival function of the Marshall-Olkin Log-Logistic distribution $MPLN(\mu, \sigma, p, 1/\alpha)$.

2.4 Moments and Quantiles

Proposition 4 *Let $X \sim MPLN(\mu, \sigma, p, \alpha)$, for $k = 1, 2, \ldots,$ Then the kth non-central moment is given by*

$$\mu_k = E(X^k) = p\alpha e^{k\mu} \int_0^1 e^{-k\sigma\Phi^{-1}(h)} \frac{h^{p-1}}{[(\alpha - 1)h^p + 1]^2} dh, \tag{8}$$

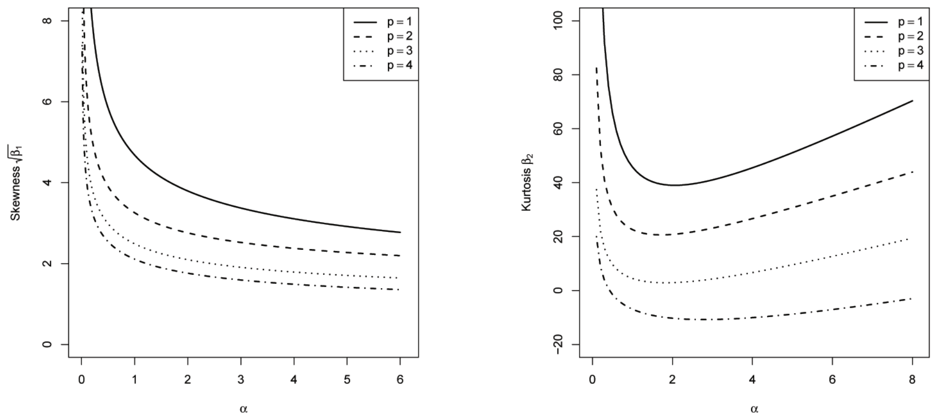
where Φ^{-1} is the inverse (quantile function) of the normal cumulative distribution function.

Proof. By definition of the moment,

$$\begin{aligned} \mu_k &= \int_0^{\infty} x^k g(x) dx \\ &= \int_0^{\infty} x^k \frac{p\alpha[\Phi(\frac{\mu - \ln(x)}{\sigma})]^{p-1} \phi(\frac{\mu - \ln(x)}{\sigma})}{x\sigma\left((\alpha - 1)[\Phi(\frac{\mu - \ln(x)}{\sigma})]^p + 1\right)^2} dx \quad let \quad t = (\mu - \ln(x))/\sigma \\ &= \int_{-\infty}^{\infty} e^{k(\mu - \sigma t)} \frac{p\alpha[\Phi(t)]^{p-1}\phi(t)}{((\alpha - 1)[\Phi(t)]^p + 1)^2} dt \quad let \quad h = \Phi(t) \\ &= p\alpha e^{k\mu} \int_0^1 e^{-k\sigma\Phi^{-1}(h)} \frac{h^{p-1}}{[(\alpha - 1)h^p + 1]^2} dh. \end{aligned}$$

The above expression seems to have no compact form. We can compute it with the help of computer. For the standardized skewness coefficient $\sqrt{\beta_1} = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{3/2}}$ and kurtosis coefficient $\beta_2 = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2}$, where

$\mu_1, \mu_2, \mu_3, \mu_4$ are the moments given in (8), Figure 2 shows the skewness and kurtosis coefficients for the Marshall-Olkin Power Log-normal $MPLN(\mu = 0, \sigma = 1, p, \alpha)$ model.



(a) Skewness coefficient (b) Kurtosis coefficient

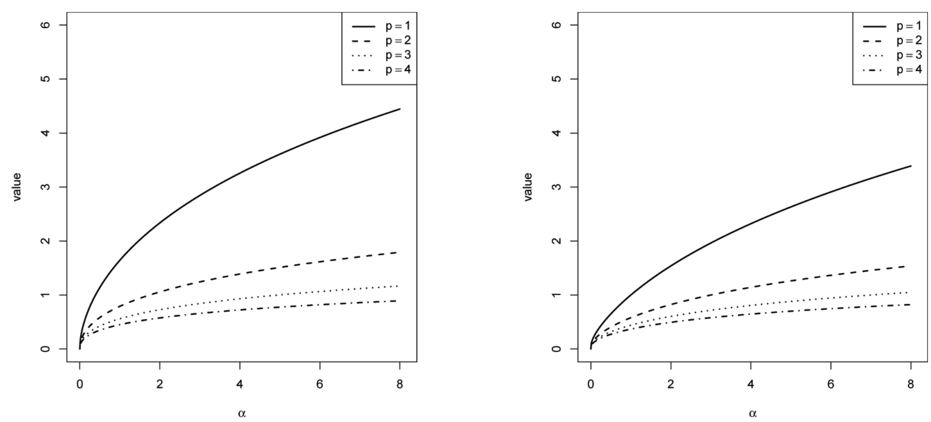
Figure 2. The plots for the skewness $\sqrt{\beta_1}$ and kurtosis coefficient β_2

The qth quantile $x_q = G^{-1}(q)$ of the $MPLN(\mu, \sigma, p, \alpha)$ distribution is given by

$$x_q = e^{\mu - \sigma \Phi^{-1}[(\frac{1-q}{\alpha q - q + 1})^{\frac{1}{p}}]}, \quad 0 \le q \le 1. \tag{9}$$

where $G^{-1}(\cdot)$ is the inverse of distribution function. In particular, the median of the $MPLN(\mu, \sigma, p, \alpha)$ distribution is given by $median(X) = e^{\mu - \sigma \Phi^{-1}[(\frac{1}{\alpha+1})^{\frac{1}{p}}]}$.

Figure 3 displays the measures of central tendency (mean, median) of the $MPLN(\mu = 0, \sigma = 1, p, \alpha)$ distribution. From the figure, it is found that, as expected, the mean is larger than the median. The distribution has a right tail.



(a) Mean value (b) Median value

Figure 3. Plots of mean and median of the $MPLN(\mu = 0, \sigma = 1, p, \alpha)$ distribution

3. Maximum Likelihood Estimation

In this section, we consider the maximum likelihood estimation about the parameters (μ, σ, p, α) of the Marshall-Olkin Power Log-normal model. Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from the Marshall-Olkin Power Log-normal distribution $MPLN(\mu, \sigma, p, \alpha)$. Then the likelihood function is given by

$$\prod_{i=1}^{n} g_X(x_i) = \prod_{i=1}^{n} \frac{p\alpha[\Phi(\frac{\mu - \ln(x_i)}{\sigma})]^{p-1} \phi(\frac{\mu - \ln(x_i)}{\sigma})}{x_i \sigma \left((\alpha - 1)[\Phi(\frac{\mu - \ln(x_i)}{\sigma})]^p + 1\right)^2}, \tag{10}$$

and the log-likelihood function is given by

$$\begin{aligned} l &= \ln(\prod_{i=1}^{n} g_X(x_i)) \\ &= n\ln(p) + n\ln(\alpha) - n\ln(\sigma) + (p-1)\sum_{i=1}^{n}\ln(\Phi(\frac{\mu - \ln(x_i)}{\sigma})) + \sum_{i=1}^{n}\ln(\phi(\frac{\mu - \ln(x_i)}{\sigma})) \\ &\quad - \sum_{i=1}^{n}\ln(x_i) - 2\sum_{i=1}^{n}\ln[(\alpha - 1)\Phi^p(\frac{\mu - \ln(x_i)}{\sigma}) + 1]. \end{aligned} \tag{11}$$

The estimates of the parameters maximize the likelihood function. Taking the partial derivatives of the log-likelihood function with respect to μ, σ, p, α respectively and equalizing the obtained expressions to zero yield to likelihood equations.

$$\frac{\partial l}{\partial \mu} = 0, \quad \frac{\partial l}{\partial \sigma} = 0, \quad \frac{\partial l}{\partial p} = 0, \quad \frac{\partial l}{\partial \alpha} = 0.$$

However, the equations do not lead to explicit analytical solutions for the parameters. Thus, the estimates must be obtained by means of numerical procedures such as Newton-Raphson method. The program R provides the nonlinear optimization function *optim* for solving such problems.

It is known that under some regular conditions, as the sample size increases, the distribution of the MLE tends to a multivariate normal distribution with mean $\theta = (\mu, \sigma, p, \alpha)^T$ and covariance matrix equal to the inverse of the Fisher information matrix $I^{-1}(\theta)$, see Cox and Hinkley (1979). The score vector and Hessian matrix are given in the Appendix. The multivariate normal distribution can be used to construct approximate confidence intervals for the parameters.

The likelihood ratio test can be used to test if the fit using MPLN model is statistically better than a fit using the PLN model. That is, we can test the hull hypothesis H_0: $\alpha = 1$ against H_1: $\alpha \neq 1$. When H_0 is true, the likelihood ratio statistic $d = 2[l(\hat{\mu}, \hat{\sigma}, \hat{p}, \hat{\alpha}) - l(\hat{\mu}, \hat{\sigma}, \hat{p}, 1)]$ has approximately a chi-square distribution with 1 degree of freedom, see Neyman and Pearson (1928) and Wilks (1938).

4. Application

In this section, we consider a real data set to illustrate the proposed model. The data taken from Davis (1952) are the number of miles to first and succeeding major motor failures of 191 buses operated by a large city bus company. The data is shown in Table 1.

Table 1. Initial bus motor failures

Distance interval(1000 miles)	Observed number of failures
Less than 20	6
20-40	11
40-60	16
60-80	25
80-100	34
100-120	46
120-140	33
140-160	16
160-180	2
180-up	2

We fit the data set with the Log-normal(LN), the Power Log-normal (PLN) and the Marshall-Olkin Power Log-normal(MPLN) distributions, respectively, using maximum likelihood method. The results are reported in Table 2. The usual Akaike information criterion (AIC) introduced by Akaike (1973) and Bayesian information criterion (BIC) proposed by Schwarz (1978) to measure of the goodness of fit are also computed. $AIC = 2k - 2\ln(L)$ and $BIC = k\ln(n) - 2\ln(L)$. where k is the number of parameters in the distribution and L is the maximized value of the likelihood function.

The results show that MPLN model fits best. Figure 4 displays the histogram and fitted models using the MLE estimates.

Table 2. Maximum likelihood parameter estimates (with standard deviation) of the LN, PLN and MPLN models for the initial bus motor failure data

Model	$\hat{\mu}$	$\hat{\sigma}$	$\hat{p}$	$\hat{\alpha}$	loglik	AIC	BIC
LN	4.454	0.566	–	–	−1013.410	2030.820	2037.325
	(0.141)	(0.290)					
PLN	12.312	1.729	19.714	–	−971.501	1949.002	1958.759
	(0.437)	(1.002)	(1.208)				
MPLN	15.435	2.671	10.558	12.454	−960.498	1928.996	1942.005
	(0.529)	(0.408)	(1.604)	(1.109)			

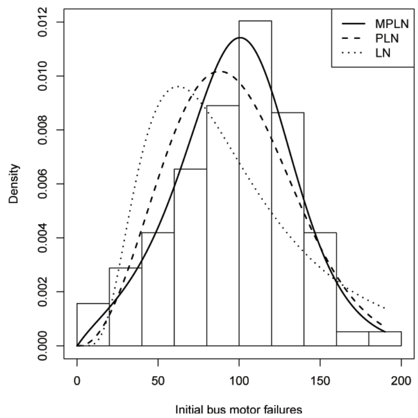


Figure 4. Histogram and fitted curves for the initial bus motor failure data

5. Conclusions

In this paper, the Power Log-normal distribution is generalized by adding an extra parameter. It is achieved by using the well known Marshall-Olkin transformation. The new model, named Marshall-Olkin Power Log-normal distribution, includes the Power Log-normal and Log-normal distributions as special cases.

Its detailed characterization and statistical properties such as stochastic orderings, stochastic representation, the moments and measures based on the moments, are presented. The estimation of parameters is approached by the method of maximum likelihood and the Hessian matrix is derived. A real survival dataset is analyzed and the results show that the proposed model is flexible and appropriate.

6. Appendix: Score Vector and Hessian Matrix

Suppose $x_1, x_2, ..., x_n$ is a random sample from the $MPLN(\mu, \sigma, p, \alpha)$ distribution, then the log-likelihood function is given by (11). The elements of the score vector are obtained by differentiation

$$\begin{aligned}
l_\mu &= -\sum_{i=1}^{n}\frac{2p(\alpha-1)\phi(\frac{\mu-\ln(x_i)}{\sigma})\Phi^{p-1}(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma[(\alpha-1)\Phi^{p}(\frac{\mu-\ln(x_i)}{\sigma})+1]}+\sum_{i=1}^{n}\frac{\phi'(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma\phi(\frac{\mu-\ln(x_i)}{\sigma})}+\sum_{i=1}^{n}\frac{(p-1)\phi(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma\Phi(\frac{\mu-\ln(x_i)}{\sigma})},\\
l_\sigma &= \sum_{i=1}^{n}\frac{2p(\alpha-1)(\mu-\ln(x_i))\phi(\frac{\mu-\ln(x_i)}{\sigma})\Phi^{p-1}(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma^2[(\alpha-1)\Phi^{p}(\frac{\mu-\ln(x_i)}{\sigma})+1]}-\sum_{i=1}^{n}\frac{(\mu-\ln(x_i))\phi(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma^2\phi(\frac{\mu-\ln(x_i)}{\sigma})}-\frac{n}{\sigma}\\
&\quad -\sum_{i=1}^{n}\frac{(p-1)(\mu-\ln(x_i))\phi(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma^2\Phi(\frac{\mu-\ln(x_i)}{\sigma})},\\
l_p &= -\sum_{i=1}^{n}\frac{2(\alpha-1)\ln[\Phi(\frac{\mu-\ln(x_i)}{\sigma})]\Phi^{p}(\frac{\mu-\ln(x_i)}{\sigma})}{(\alpha-1)\Phi^{p}(\frac{\mu-\ln(x_i)}{\sigma})+1}+\sum_{i=1}^{n}\ln[\Phi(\frac{\mu-\ln(x_i)}{\sigma})]+\frac{n}{p},\\
l_\alpha &= \frac{n}{\alpha}-\sum_{i=1}^{n}\frac{2\Phi^{p}(\frac{\mu-\ln(x_i)}{\sigma})}{(\alpha-1)\Phi^{p}(\frac{\mu-\ln(x_i)}{\sigma})+1}.
\end{aligned}$$

The Hessian matrix, second partial derivatives of the log-likelihood, is given by

$$H(\theta)=\begin{pmatrix} l_{\mu\mu} & l_{\mu\sigma} & l_{\mu p} & l_{\mu\alpha}\\ l_{\sigma\mu} & l_{\sigma\sigma} & l_{\sigma p} & l_{\sigma\alpha}\\ l_{p\mu} & l_{p\sigma} & l_{pp} & l_{p\alpha}\\ l_{\alpha\mu} & l_{\alpha\sigma} & l_{\alpha p} & l_{\alpha\alpha}\end{pmatrix}$$

where

$$\begin{aligned}
l_{\mu\mu} &= -\sum_{i=1}^{n}\frac{2(p-1)p(\alpha-1)\phi(\frac{\mu-\ln(x_i)}{\sigma})^2\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^{p-2}}{\sigma^2\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p+1\right)}-\sum_{i=1}^{n}\frac{2p(\alpha-1)\phi'(\frac{\mu-\ln(x_i)}{\sigma})\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^{p-1}}{\sigma^2\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p+1\right)}\\
&+\sum_{i=1}^{n}\frac{2p^2(\alpha-1)^2\phi(\frac{\mu-\ln(x_i)}{\sigma})^2\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^{2p-2}}{\sigma^2\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p+1\right)^2}+\sum_{i=1}^{n}\frac{\phi''(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma^2\phi(\frac{\mu-\ln(x_i)}{\sigma})}-\sum_{i=1}^{n}\frac{[\phi'(\frac{\mu-\ln(x_i)}{\sigma})]^2}{\sigma^2[\phi(\frac{\mu-\ln(x_i)}{\sigma})]^2}\\
&+\sum_{i=1}^{n}\frac{(p-1)\phi'(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma^2\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)}-\sum_{i=1}^{n}\frac{(p-1)\phi(\frac{\mu-\ln(x_i)}{\sigma})^2}{\sigma^2\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^2},\\
l_{\mu\sigma} &= l_{\sigma\mu}=\sum_{i=1}^{n}\frac{2(p-1)p(\alpha-1)(\mu-\ln(x_i))\phi(\frac{\mu-\ln(x_i)}{\sigma})^2\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{p-2}}{\sigma^3\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p+1\right)}\\
&+\sum_{i=1}^{n}\frac{2p(\alpha-1)\phi(\frac{\mu-\ln(x_i)}{\sigma})\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{p-1}}{\sigma^2\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p+1\right)}+\sum_{i=1}^{n}\frac{2p(\alpha-1)(\mu-\ln(x_i))\phi'\left(\frac{\mu-\ln(x_i)}{\sigma}\right)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{p-1}}{\sigma^3\left((\alpha-1)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^p+1\right)}\\
&-\sum_{i=1}^{n}\frac{2p^2(\alpha-1)^2(\mu-\ln(x_i))\phi(\frac{\mu-\ln(x_i)}{\sigma})^2\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{2p-2}}{\sigma^3\left((\alpha-1)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^p+1\right)^2}+\sum_{i=1}^{n}\frac{(\mu-\ln(x_i))[\phi'\left(\frac{\mu-\ln(x_i)}{\sigma}\right)]^2}{\sigma^3[\phi(\frac{\mu-\ln(x_i)}{\sigma})]^2}\\
&-\sum_{i=1}^{n}\frac{(\mu-\ln(x_i))\phi''\left(\frac{\mu-\ln(x_i)}{\sigma}\right)}{\sigma^3\phi(\frac{\mu-\ln(x_i)}{\sigma})}-\sum_{i=1}^{n}\frac{(p-1)\phi(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma^2\Phi(\frac{\mu-\ln(x_i)}{\sigma})}-\sum_{i=1}^{n}\frac{(p-1)(\mu-\ln(x_i))\phi'\left(\frac{\mu-\ln(x_i)}{\sigma}\right)}{\sigma^3\Phi(\frac{\mu-\ln(x_i)}{\sigma})}\\
&+\sum_{i=1}^{n}\frac{(p-1)\left(\mu-\ln(x_i)\right)\phi(\frac{\mu-\ln(x_i)}{\sigma})^2}{\sigma^3\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^2}-\sum_{i=1}^{n}\frac{\phi'(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma^2\phi(\frac{\mu-\ln(x_i)}{\sigma})},\\
l_{\mu p} &= l_{p\mu}=-\sum_{i=1}^{n}\frac{2p(\alpha-1)\ln\left(\Phi(\frac{\mu-\ln(x_i)}{\sigma})\right)\phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{p-1}}{\sigma\left((\alpha-1)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^p+1\right)}-\sum_{i=1}^{n}\frac{2(\alpha-1)\phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{p-1}}{\sigma\left((\alpha-1)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^p+1\right)}\\
&+\sum_{i=1}^{n}\frac{2p(\alpha-1)^2\ln\left(\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)\right)\phi(\frac{\mu-\ln(x_i)}{\sigma})\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^{2p-1}}{\sigma\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p+1\right)^2}+\sum_{i=1}^{n}\frac{\phi(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)},\\
l_{\mu\alpha} &= l_{\alpha\mu}=\sum_{i=1}^{n}\frac{2p(\alpha-1)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{2p-1}\phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)}{\sigma\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p+1\right)^2}-\sum_{i=1}^{n}\frac{2p\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{p-1}\phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)}{\sigma\left((\alpha-1)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^p+1\right)},\\
l_{\sigma\sigma} &= -\sum_{i=1}^{n}\frac{2(p-1)p(\alpha-1)(\mu-\ln(x_i))^2\phi(\frac{\mu-\ln(x_i)}{\sigma})^2\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{p-2}}{\sigma^4\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p+1\right)}\\
&-\sum_{i=1}^{n}\frac{4p(\alpha-1)(\mu-\ln(x_i))\phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{p-1}}{\sigma^3\left((\alpha-1)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^p+1\right)}-\sum_{i=1}^{n}\frac{2p(\alpha-1)(\mu-\ln(x_i))^2\phi'(\frac{\mu-\ln(x_i)}{\sigma})\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^{p-1}}{\sigma^4\left((\alpha-1)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^p+1\right)}\\
&+\sum_{i=1}^{n}\frac{2p^2(\alpha-1)^2(\mu-\ln(x_i))^2\phi(\frac{\mu-\ln(x_i)}{\sigma})^2\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{2p-2}}{\sigma^4\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p+1\right)^2}+\sum_{i=1}^{n}\frac{2(\mu-\ln(x_i))\phi'\left(\frac{\mu-\ln(x_i)}{\sigma}\right)}{\sigma^3\phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)}\\
&-\sum_{i=1}^{n}\frac{(\mu-\ln(x_i))^2[\phi'\left(\frac{\mu-\ln(x_i)}{\sigma}\right)]^2}{\sigma^4[\phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)]^2}+\sum_{i=1}^{n}\frac{(\mu-\ln(x_i))^2\phi''\left(\frac{\mu-\ln(x_i)}{\sigma}\right)}{\sigma^4\phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)}+\frac{n}{\sigma^2}+\sum_{i=1}^{n}\frac{2(p-1)(\mu-\ln(x_i))\Phi'(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma^3\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)}
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{i=1}^{n} \frac{(p-1)(\mu-\ln(x_i))^2 \phi'(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma^4 \Phi(\frac{\mu-\ln(x_i)}{\sigma})} - \sum_{i=1}^{n} \frac{(p-1)(\mu-\ln(x_i))^2 \Phi'(\frac{\mu-\ln(x_i)}{\sigma})^2}{\sigma^4 \Phi(\frac{\mu-\ln(x_i)}{\sigma})^2}, \\
l_{\sigma p} \quad = \quad & l_{p\sigma} = \sum_{i=1}^{n} \frac{2(\alpha-1)(\mu-\ln(x_i))\phi(\frac{\mu-\ln(x_i)}{\sigma})\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{p-1}}{\sigma^2\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p + 1\right)} \\
& + \sum_{i=1}^{n} \frac{2p(\alpha-1)(\mu-\ln(x_i))\ln\left(\Phi(\frac{\mu-\ln(x_i)}{\sigma})\right)\phi(\frac{\mu-\ln(x_i)}{\sigma})\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{p-1}}{\sigma^2\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p + 1\right)} \\
& - \sum_{i=1}^{n} \frac{2p(\alpha-1)^2(\mu-\ln(x_i))\ln\left(\Phi(\frac{\mu-\ln(x_i)}{\sigma})\right)\phi(\frac{\mu-\ln(x_i)}{\sigma})\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{2p-1}}{\sigma^2\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p + 1\right)^2} - \sum_{i=1}^{n} \frac{(\mu-\ln(x_i))\phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)}{\sigma^2\Phi(\frac{\mu-\ln(x_i)}{\sigma})}, \\
l_{\sigma\alpha} \quad = \quad & l_{\alpha\sigma} = \sum_{i=1}^{n} \frac{2p(\mu-\ln(x_i))\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{p-1}\phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)}{\sigma^2\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p + 1\right)} - \sum_{i=1}^{n} \frac{2p(\alpha-1)(\mu-\ln(x_i))\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^{2p-1}\phi(\frac{\mu-\ln(x_i)}{\sigma})}{\sigma^2\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p + 1\right)^2}, \\
l_{pp} \quad = \quad & -\sum_{i=1}^{n} \frac{2(\alpha-1)\ln^2\left(\Phi(\frac{\mu-\ln(x_i)}{\sigma})\right)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p}{(\alpha-1)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^p + 1} + \sum_{i=1}^{n} \frac{2(\alpha-1)^2\ln^2\left(\Phi(\frac{\mu-\ln(x_i)}{\sigma})\right)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^{2p}}{\left((\alpha-1)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^p + 1\right)^2} - \frac{n}{p^2}, \\
l_{p\alpha} \quad = \quad & l_{\alpha p} = \sum_{i=1}^{n} \frac{2(\alpha-1)\ln\left(\Phi(\frac{\mu-\ln(x_i)}{\sigma})\right)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^{2p}}{\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p + 1\right)^2} - \sum_{i=1}^{n} \frac{2\ln\left(\Phi(\frac{\mu-\ln(x_i)}{\sigma})\right)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p}{(\alpha-1)\Phi(\frac{\mu-\ln(x_i)}{\sigma})^p + 1}, \\
l_{\alpha\alpha} \quad = \quad & \sum_{i=1}^{n} \frac{2\Phi(\frac{\mu-\ln(x_i)}{\sigma})^{2p}}{\left((\alpha-1)\Phi\left(\frac{\mu-\ln(x_i)}{\sigma}\right)^p + 1\right)^2} - \frac{n}{\alpha^2}.
\end{aligned}
$$

The Fisher information matrix $I(\theta) = -E(H(\theta))$.

References

Akaike, H. (1973). Information theory and an extension of the maximum likelihood principle. In *International Symposium on Information Theory* (2 nd.), *Tsahkadsor, Armenian SSR*, 267-281. http://dx.doi.org/10.1007/978-1-4612-0919-5_38

Alice, T., & Jose, K. (2003). Marshall-olkin pareto processes. *Far East Journal of Theoretical Statistics, 9*(2), 117-132.

Alice, T., & Jose, K. (2005). Marshall-olkin semi-weibull minification processes. *Recent Advances in Statistical Theory and Applications, I*, 6-17.

Cox, D., & Hinkley, D. (1979). *Theoretical statistics*. Chapman & Hall/CRC.

Davis, D. (1952). An analysis of some failure data. *Journal of the American Statistical Association, 47*(258), 113-150. http://dx.doi.org/10.2307/2280740

García, V., Gómez-Déniz, E., & Vázquez-Polo, F. (2011). A generalization of the log–normal distribution and its applications. *4th Workshop on Risk Management and Insurance*.

Ghitany, M., Al-Awadhi, F., & Alkhalfan, L. (2007). Marshall-olkin extended lomax distribution and its application to censored data. *Communications in Statistics Theory and Methods, 36*(10), 1855-1866. http://dx.doi.org/10.1080/03610920601126571

Ghitany, M., Al-Hussaini, E., & Al-Jarallah, R. (2005). Marshall-olkin extended weibull distribution and its application to censored data. *Journal of Applied Statistics, 32*(10), 1025-1034. http://dx.doi.org/10.1080/02664760500165008

Gupta, R., Kannan, N., & Raychaudhuri, A. (1997). Analysis of lognormal survival data. *Mathematical biosciences, 139*(2), 103-115. http://dx.doi.org/10.1016/S0025-5564(96)00133-2

Johnson, W., Lee, J., Zellner, A., & Johnson, W. (1996). Predictive influence in the lognormal survival model. *Prediction and Modeling in Statistics and Econometrics: Essays in Honor of Seymour Geisser*.

Jose, K., Naik, S., & Ristić, M. (2010). Marshall-olkin q-weibull distribution and max-min processes. *Statistical papers, 51*(4), 837-851. http://dx.doi.org/10.1007/s00362-008-0173-9

Liu, Z., Almhana, J., & McGorman, R. (2008). Approximating lognormal sum distributions with power lognormal distributions. *Vehicular Technology, IEEE Transactions on, 57*(4), 2611-2617. http://dx.doi.org/10.1109/TVT.2007.912338

Marshall, A., & Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and weibull families. *Biometrika, 84*(3), 641-652. http://dx.doi.org/10.1093/biomet/84.3.641

Nelson, W., & Doganaksoy, N. (1995). Statistical analysis of life or strength data from specimens of various sizes using the power-(log) normal model. *Recent Advances in Life-Testing and Reliability*, CRC Press, 377-408.

Nelson, W., & Dognanksoy, N. (1992). Computer program pownor for fitting the power-normal and-lognormal models to life or strength data from specimens of various sizes. *NASA STI/Recon Technical Report N, 92*, 29813.

Neyman, J., & Pearson, E. (1928). On the use and interpretation of certain test criteria for purposes of statistical inference: Part i. *Biometrika*, 175-240. http://dx.doi.org/10.2307/2331945

Ramesh, C., & Kirmani, S. (1987). On order relations between reliability measures. *Stochastic Models, 3*(1), 149-156. http://dx.doi.org/10.1080/15326348708807050

Royston, P. (2001). The lognormal distribution as a model for survival time in cancer, with an emphasis on prognostic factors. *Statistica Neerlandica, 55*(1), 89-104. http://dx.doi.org/10.1111/1467-9574.00158

Rutqvist, L. (1985). On the utility of the lognormal model for analysis of breast cancer survival in sweden 1961-1973. *British journal of cancer, 52*(6), 875-883. http://dx.doi.org/10.1038/bjc.1985.272

Schwarz, G. (1978). Estimating the dimension of a model. *The annals of statistics, 6*(2), 461-464. http://dx.doi.org/10.1214/aos/1176344136

Szyszkowicz, S., & Yanikomeroglu, H. (2009). Modified-power-lognormal approximation to the sum of lognormals distribution. *Global Telecommunications Conference, 2009*, 1-6. http://dx.doi.org/10.1109/GLOCOM.2009.5426029

Wilks, S. (1938). The large-sample distribution of the likelihood ratio for testing composite hypotheses. *The Annals of Mathematical Statistics*, 60-62. http://dx.doi.org/10.1214/aoms/1177732360