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Bayesian Prediction Based on Generalized Order Statistics from a Mixture of Two Exponentiated Weibull Distribution Via MCMC Sumulation

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Abstract

This paper is concerned with the problem of obtaining the maximum likelihood prediction (point and interval) and Bayesian prediction (point and interval) for a future observation from mixture of two exponentiated Weibull (MTEW) distributions based on generalized order statistics (GOS). We consider one-sample and two-sample prediction schemes using the Markov chain Monte Carlo (MCMC) algorithm. The conjugate prior is used to carry out the Bayesian analysis. The results are specialized to upper record values.

Keywords: generalized order statistics, record values, MCMC, exponentiated Weibull, mixture distributions

1. Introduction

In life testing, reliability and quality control problems, mixed failure populations are sometimes encountered. Mixture distributions comprise a finite or infinite number of components, possibly of different distributional types, that can describe different features of data. In recent years, the finite mixture of life distributions have to be of considerable interest in terms of their practical applications in a variety of disciplines such as physics, biology, geology, medicine, engineering and economics, among others. Some of the most important references that discussed different types of mixtures of distributions are Everitt and Hand (1981), Titterington et al. (1985), McLachlan and Basford (1988), Jaheen (2005), Jaheen and Mohammed (2011) and AL-Hussaini and Hussein (2012).

Mudholkar and Sirvastava (1993) introduced a new distribution, called exponentiated Weibull (EW) distribution as an extension of the Weibull family of distributions by adding an additional shape parameter. This distribution is very important distribution in lifetime tests. The probability density function (PDF), cumulative distribution function (CDF) and reliability function (RF) are given below

$$f(t) = \alpha\theta t^{\alpha-1} e^{-t^{\alpha}} \eta^{\theta-1},\ t > 0,\ (\alpha > 0),\ (\theta > 0), \tag{1}$$

$$F(t) = \eta^{\theta},\ t > 0, \tag{2}$$

$$R(t) = 1 - \eta^{\theta}, \tag{3}$$

where α and θ are the shape parameters of the model. Also, the hazard rate function (HRF) takes the form

$$H(t) = \alpha\theta t^{\alpha-1}\ e^{-t^{\alpha}} \eta^{\theta-1}[1 - \eta^{\theta}]^{-1},\ t > 0, \tag{4}$$

where $\eta = 1 - e^{-t^{\alpha}}$ and $H(\cdot) = \frac{f(\cdot)}{R(\cdot)}$.

The EW distribution includes a number of distributions as particular cases: if the shape parameter $\theta = 1$, then the PDF is that Weibull distribution, when $\alpha = 1$ then the PDF is that Exponentiated Exponential distribution, if $\alpha = 1$ and $\theta = 1$ then the PDF is that Exponential distribution and if $\alpha = 2$ then the PDF is that One parameters Burr-X distribution. Mudholkar and Hutson (1996) showed that the density function of the EW distribution is decreasing when $\alpha\theta \leq 1$ and unimodal when $\alpha\theta > 1$. Some statistical properties of this distribution are discussed by Jiang and Murthy (1999) and Nassar and Eissa (2003). Nassar and Eissa (2004) obtained Bayes estimators for the two shape parameters, the reliability, and the failure rate function of the EW distribution based on type-II censored and complete samples. Choudhury (2005) proposed a simple derivation for the moments of the EW distribution. Singh et al. (2005a, b) derived the maximum likelihood (ML) and Bayes estimates of the two and three parameters of the EW distribution based on type-II censored samples. Pal et al. (2006) introduced many properties and obtained some inferences for the three-parameter EW distribution. Kim et al. (2009) obtained the ML and Bayes estimators for the two shape parameters and reliability function of the EW distribution based on progressive type-II censored sample.

The PDF, CDF, RF and HRF of the MTEW distribution are given, respectively, by

$$f(t) = p_1 f_1(t) + p_2 f_2(t), \tag{5}$$

$$F(t) = p_1 F_1(t) + p_2 F_2(t), \tag{6}$$

$$R(t) = p_1 R_1(t) + p_2 R_2(t), \tag{7}$$

$$H(t) = \frac{f(t)}{R(t)}, \tag{8}$$

where, for $j = 1, 2$, the mixing proportions p_j are such that $0 \leq p_j \leq 1$, $p_1 + p_2 = 1$ and $f_j(t)$, $F_j(t)$, $R_j(t)$ are given from (1), (2), (3) after using (θ_j, α_j) instead of (θ, α).

The property of identifiability is an important consideration on estimating the parameters in a mixture of distributions. A mixture is identifiable if there exists a one-to-one correspondence between the mixing distribution and a resulting mixture. That is, there is a unique characterization of the mixture. Therefore, a mixture of Exponentiated Weibull components is identifiable. Identifiability of mixtures has been discussed by several authors, including Teicher (1963), AL-Hussaini and Ahmad (1981) and Ahmad (1988).

Recently, there has been a considerable amount of interest in the Bayesian approach which allows prior subjective knowledge on lifetime parameters, as well as experimental data, to be incorporated into the inferential procedure. For the Bayesian approach, a loss function must be specified.

Several authors have predicted future order statistics and records from homogeneous and heterogeneous populations that can be represented by single component distribution and finite mixtures of distributions, respectively. For more details, see AL-Hussaini and Ahmad (2003), Ali Mousa (2003) and AL-Hussaini (2004). Recently, a few of authors utilized the GOS's in Bayesian inference. Such authors are AL-Hussaini and ahmad (2003), Jaheen (2002; 2005) and Ateya and Ahmad (2011). Bayesian inferences based on finite mixture distribution have been discussed by several authors such that: Papadapoulos and Padgett (1986), Attia (1993), Ahmad et al. (1997), Jaheen (2005b), Soliman (2006), Saleem and Aslam (2008a; 2008b) and Saleem and Irfan (2010).

A wide variety of loss functions have been developed in the literature to describe various types of loss structures. The balanced loss function was introduced by Zellner (1994). Jozani et al. (2006) introduced an extended class of the balanced loss function of the form

$$L_{\Phi,\Omega,\delta_o}(\Psi(\theta), \delta) = \Omega \Upsilon(\theta) \Phi(\delta_o, \delta) + (1 - \Omega) \Upsilon(\theta) \Phi(\Psi(\theta), \delta), \tag{9}$$

where $\Upsilon(\cdot)$ is a suitable positive weight function and $\Phi(\Psi(\theta), \delta)$ is an arbitrary loss function when estimating $\Psi(\theta)$ by δ. The parameter δ_o is a chosen prior estimator of $\Psi(\theta)$, obtained for instance from the criterion of ML, least squares or unbiasedness among others. They give a general Bayesian connection between the case of $\Omega > 0$ and $\Omega = 0$ where $0 \leq \Omega < 1$.

Generalized order statistics (GOS) concept was introduced by Kamps (1995) as unify approach to several models of ordered random variables such as ordinary order statistics, ordinary record values, progressive Type-II censored order statistics and sequential order statistics, among others. Ahsanullah (1996; 2000), Kamps and Gather (1997), Cramer and Kamps (2000), Habibullah and Ahsanullah (2000), Jaheen (2002; 2005a), AL-Hussaini and Ahmad (2003), AL-Hussaini (2004), Ahmad (2007; 2008), Aboeleneen (2010), Ahmad and Abushal (2006-2010), Ahmad (2011), Abu El Fotouh (2011) and Ateya and Ahmad (2011) among others, utilized the (GOS's) in their Works.

Suppose that $T_{1;n,\widetilde{m},k}, T_{2;n,\widetilde{m},k}, ..., T_{r;n,\widetilde{m},k}$, $k > 0$, $\widetilde{m} = (m_1, ..., m_{r-1}) \in \Re^{r-1}, m_1, ..., m_{r-1} \in \Re$, are the first r (out of n) GOS drawn from the MTEW distribution. The likelihood function (LF) is given in (Kamps, 1995), for

$-\infty < t_1 < ... < t_r < \infty$, by

$$L(\theta|\mathbf{t}) = C_{r-1}\{\prod_{i=1}^{r-1}[R(t_i)]^{m_i} f(t_i)\}[R(t_r)]^{\gamma_r-1} f(t_r), \tag{10}$$

where $\mathbf{t} = (t_1, ..., t_r)$, $\theta \in \Theta$, Θ is the parameter space, and

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i,\ \gamma_i = k + n - i + M_i > 0, M_i = \sum_{\nu=i}^{n-1} m_\nu. \tag{11}$$

where $f(t_i)$ and $R(t_i)$ are given, respectively, by (5) and (7).

The purpose of this paper is to obtain the maximum likelihood and the Bayes prediction point and interval in the case of one-sample scheme and two-sample scheme. The point predictors are obtained based on balanced square error loss (BSEL) function and the balanced LINEX (BLINEX) loss function. We used ML to estimate the parameters, α and θ of the MTEW distribution based on GOS. The conjugate prior is assumed to carry out the Bayesian analysis. The results are specialized to record values.

2. Maximum Likelihood Estimation

Assuming that the parameters θ_j and α_j are unknown and p_j is known, the likelihood equations are given, for $j = 1, 2$, by

$$\left.\begin{aligned}
\frac{\partial \ell}{\partial \theta_j} &= -p_j \sum_{i=1}^{r-1} \frac{m_i \upsilon_j^{\theta_j}(t_i) \ln \upsilon_j(t_i)}{R(t_i)} + p_j \alpha_j \sum_{i=1}^{r} \frac{\omega_j(t_i)\upsilon_j^{\theta_j}(t_i)[1 + \theta_j \ln \upsilon_j(t_i)]}{f(t_i)} \\
&- p_j(\gamma_r - 1)\frac{\upsilon_j^{\theta_j}(t_r) \ln \upsilon_j(t_r)}{R(t_r)} = 0, \\
\frac{\partial \ell}{\partial \alpha_j} &= -p_j\theta_j \sum_{i=1}^{r-1} \frac{m_i \upsilon_j^{\theta_j-1}(t_i)\xi_j(t_i)}{R(t_i)} + p_j\theta_j \sum_{i=1}^{r} \frac{\upsilon_j^{\theta_j}(t_i)[\omega_j(t_i) + \alpha_j \upsilon_j^{-1}(t_i)\xi_j(t_i)(\theta_j\omega_j(t_i) + \varrho_j(t_i))]}{f(t_i)} \\
&- p_j\theta_j(\gamma_r - 1)\frac{\upsilon_j^{\theta_j-1}(t_r)\xi_j(t_r)}{R(t_r)} = 0,
\end{aligned}\right\} \tag{12}$$

where, $\ell \equiv \ln L(\theta \mid \mathbf{t})$ and for $j = 1, 2$

$$\left.\begin{aligned}
\upsilon_j(t_i) &= 1 - e^{-t_i^{\alpha_j}}, & \xi_j(t_i) &= \frac{\partial \upsilon_j(t_i)}{\partial \alpha_j} = e^{-t_i^{\alpha_j}} t_i^{\alpha_j} \ln t_i, \\
\omega_j(t_i) &= \frac{e^{-t_i^{\alpha_j}} t_i^{\alpha_j}}{t_i \upsilon_j(t_i)}, & \varrho_j(t_i) &= \frac{1 - t_i^{\alpha_j}}{t_i} - \omega_j(t_i)
\end{aligned}\right\}. \tag{13}$$

Equations (12) do not yield explicit solutions for θ_j and α_j, $j = 1, 2$, and have to be solved numerically to obtain the ML estimates of the four parameters, Newton-Raphson iteration is employed to solve (12).

Remark:

The parameters of the components are assumed to be distinct, so that the mixture is identifiable. For the concept of identifiability of finite mixtures and examples, see Everitt and Hand (1981), AL-Hussaini and Ahmad (1981) and Ahmad and AL-Hussaini (1982).

This section deals with studying the maximum likelihood and the Bayes prediction point and interval in the case of one-sample scheme and two-sample scheme.

3. Prediction in Case of One-sample Scheme

Based on the informative $T_{1;n,\widetilde{m},k}, T_{2;n,\widetilde{m},k}, ..., T_{r;n,\widetilde{m},k}$, GOS's from the MTEW distribution with two parameters, for the remaining $(n - r)$ components, let $T_{s;n,\widetilde{m},k}$, $s = r + 1, r + 2, ..., n$ denote the future lifetime of the s^{th} component to fail, $1 \leq s \leq (n - r)$, the maximum Likelihood prediction (point MLPP and interval MLPI), Bayesian prediction (point BPP and interval BPI) can be obtained.

The conditional PDF of $T_s \equiv T_{s;n,\widetilde{m},k}$ given that the $T_r \equiv T_{r;n,\widetilde{m},k}$ components that had already failed are

$$f^{\star}(t_s|t_r) = \begin{cases} \frac{k^{s-r}}{(s-r-1)!}[lnR(t_r) - lnR(t_s)]^{s-r-1}[R(t_s)]^{k-1}[R(t_r)]^{-k} f(t_s), & m = -1, \\ \frac{C_{s-1}}{(m+1)^{s-r-1}(s-r-1)!C_{r-1}}[R(t_r)^{m+1} - R(t_s)^{m+1}]^{s-r-1}[R(t_s)]^{\gamma_s - 1} \times [R(t_r)]^{-\gamma_{r+1}} f(t_s), & m \neq -1. \end{cases} \tag{14}$$

In the case when $m = -1$, substituting (5) and (7) in (14), the conditional PDF takes the form

$$f_1^{\star}(t_s|\theta_j) \propto [ln[p_1R_1(t_r) + p_2R_2(t_r)] - ln[p_1R_1(t_s) + p_2R_2(t_s)]]^{s-r-1}[p_1R_1(t_s) + p_2R_2(t_s)]^{k-1}$$

$$\times[p_1R_1(t_r) + p_2R_2(t_r)]^{-k}[p_1f_1(t_s) + p_2f_2(t_s)], \; t_s > t_r. \tag{15}$$

In the case when $m \neq -1$, substituting (5) and (7) in (14), the conditional PDF takes the form

$$f_2^{\star}(t_s|\theta_j) \propto [[p_1R_1(t_r) + p_2R_2(t_r)]^{m+1} - [p_1R_1(t_s) + p_2R_2(t_s)]^{m+1}]^{s-r-1}[p_1R_1(t_s) + p_2R_2(t_s)]^{\gamma_s - 1}$$

$$\times[p_1R_1(t_r) + p_2R_2(t_r)]^{-\gamma_{r+1}}[p_1f_1(t_s) + p_2f_2(t_s)], \; t_s > t_r. \tag{16}$$

Now, we shall study two cases: the first case is when the parameter α is known and the second is when the parameters θ and α are unknown.

3.1 Prediction When α_j Is Known

Suppose that the mixing proportion, p_j, $j = 1, 2$ and α_j, $j = 1, 2$ are known.

3.1.1 Maximum Likelihood Prediction

Maximum likelihood prediction can be obtained using (15) and (16) by replacing the shape parameters θ_1 and θ_2 by $\widehat{\theta_1}_{(ML)}$ and $\widehat{\theta_2}_{(ML)}$ which is obtained from (12).

1- Interval prediction:

The MLPI for any future observation t_s, $s = r+1, r+2, ..., n$ can be obtained by

$$Pr[t_s \geq \upsilon|\mathbf{t}] = \int_{\upsilon}^{\infty} f_1^{\star}(t_s|\widehat{\theta_1}_{(ML)}, \widehat{\theta_2}_{(ML)})dt_s, \; m = -1, \tag{17}$$

$$= \int_{\upsilon}^{\infty} f_2^{\star}(t_s|\widehat{\theta_1}_{(ML)}, \widehat{\theta_2}_{(ML)})dt_s, \; m \neq -1.$$

A $(1-\tau)\times 100$ % MLPI (L, U) of the future observation t_s is given by solving the following two nonlinear equations

$$Pr[t_s \geq L(t) \mid \mathbf{t}] = 1 - \frac{\tau}{2}, \; Pr[t_s \geq U(t) \mid \mathbf{t}] = \frac{\tau}{2}. \tag{18}$$

2- Point prediction:

The MLPP for any future observation t_s, $s = r+1, r+2, ..., n$ can be obtained by replacing the shape parameters θ_1 and θ_2 by $\widehat{\theta_1}_{(ML)}$ and $\widehat{\theta_2}_{(ML)}$ which, obtained from (12).

$$\widehat{t_s}_{(ML)} = E(t_s) = \int_{t_r}^{\infty} t_s f_1^{\star}(t_s|\widehat{\theta_1}_{(ML)}, \widehat{\theta_2}_{(ML)})dt_s, \; m = -1, \tag{19}$$

$$= \int_{t_r}^{\infty} t_s f_2^{\star}(t_s|\widehat{\theta_1}_{(ML)}, \widehat{\theta_2}_{(ML)})dt_s, \; m \neq -1.$$

3.1.2 Bayesian Prediction

Suppose that the mixing proportion, p_j and α_j are known. Let θ_j, $j = 1, 2$, have a gamma prior distribution with PDF

$$\pi(\theta_j) = \frac{1}{\Gamma(\nu_j)}(\beta_j)^{\nu_j}\theta_j^{\nu_j - 1}\exp[-\beta_j\theta_j], \quad (\theta_j, \nu_j, \beta_j > 0). \tag{20}$$

These are chosen since they are the conjugate priors for the individual parameters. The joint prior density function of $\theta = (\theta_1, \theta_2)$ is given by

$$\pi(\theta) = \pi_1(\theta_1)\pi_2(\theta_2),$$

$$\pi(\theta) \propto \prod_{j=1}^{2} \theta_j^{\nu_j - 1} \exp[-\sum_{j=1}^{2} \beta_j \theta_j], \tag{21}$$

where $j = 1, 2$ $\theta_j > 0$, $(\nu_j, \beta_j) > 0$.

It follows, from (10) and (21), that the joint posterior density function is given by

$$\pi_1^*(\theta|\mathbf{t}) = A_1 \prod_{j=1}^{2} \theta_j^{\nu_j - 1} \exp[-\sum_{j=1}^{2} \beta_j \theta_j] \prod_{i=1}^{r-1} [p_1 R_1(t_i) + p_2 R_2(t_i)]^{m_i}$$

$$\times \prod_{i=1}^{r} [p_1 f_1(t_i) + p_2 f_2(t_i)][p_1 R_1(t_r) + p_2 R_2(t_r)]^{\gamma_r - 1}, \tag{22}$$

where

$$A_1^{-1} = \int_{\theta} \pi(\theta) L(\theta|\mathbf{t})\, d\theta. \tag{23}$$

The Bayes predictive density function can be obtained using (15), (16) and (22) as follows:

$$h_1^{\star}(t_s|\mathbf{t}) \propto \int_0^{\infty} \pi_1^*(\theta|\mathbf{t}) f_1^{\star}(t_s|\theta_j) d\theta, \;\; m = -1,$$

$$\int_0^{\infty} \pi_1^*(\theta|\mathbf{t}) f_2^{\star}(t_s|\theta_j) d\theta, \;\; m \neq -1. \tag{24}$$

1- Interval prediction:

Bayesian prediction interval, for the future observation $T_{s;n,\widetilde{m},k}$, $s = r+1, r+2, ..., n$, can be computed by approximated $h_1^*(t_s|\mathbf{t})$ using the MCMC algorithm, see Ahmad et al. (2011) and Ateya (2011), using the form

$$h_1^*(t_s|\mathbf{t}) \cong \frac{\sum_{i=1}^{\mu} f^{\star}(t_s|\theta_j^i)}{\sum_{i=1}^{\mu} \int_{t_r}^{\infty} f^{\star}(t_s|\theta_j^i) dt_s}, \tag{25}$$

where μ is the number of generated parameters and θ_j^i, $i = 1, 2, 3, ..., \mu$, they are generated from the posterior density function (22) using Gibbs sampler and Metropolis-Hastings techniques, for more details see Press (2003).

A $(1 - \tau) \times 100$ % BPI (L, U) of the future observation t_s is given by solving the following two nonlinear equations

$$\frac{\sum_{i=1}^{\mu} \int_{L}^{\infty} f^{\star}(t_s|\theta_j^i) dt_s}{\sum_{i=1}^{\mu} \int_{t_r}^{\infty} f^{\star}(t_s|\theta_j^i) dt_s} = 1 - \frac{\tau}{2}, \tag{26}$$

$$\frac{\sum_{i=1}^{\mu} \int_{U}^{\infty} f^{\star}(t_s|\theta_j^i) dt_s}{\sum_{i=1}^{\mu} \int_{t_r}^{\infty} f^{\star}(t_s|\theta_j^i) dt_s} = \frac{\tau}{2}. \tag{27}$$

Numerical methods are generally necessary to solve the above two equations to obtain L and U for a given τ.

2- Point prediction:

1- BPP for the future observation t_s based on BSEL function can be obtained using

$$\widetilde{t}_{s_{(BS)}} = \Omega \widehat{t}_{s_{(ML)}} + (1 - \Omega) E(t_s|\mathbf{t}), \tag{28}$$

where $\widehat{t}_{s_{(ML)}}$ is the ML prediction for the future observation t_s which can be obtained using (19) and $E(t_s|\mathbf{t})$ can be obtained using

$$E(t_s|\mathbf{t}) = \int_{t_r}^{\infty} t_s h_1^*(t_s|\mathbf{t}) dt_s. \tag{29}$$

2- BPP for the future observation t_s based on BLINX loss function can be obtained using

$$\widetilde{t}_{s_{(BL)}} = -\frac{1}{a} ln[\Omega \exp[-a\widehat{t}_{s_{(ML)}}] + (1 - \Omega) E(e^{-at_s}|\mathbf{t})], \tag{30}$$

where $\widehat{t_{s(ML)}}$ is the ML prediction for the future observation t_s which can be obtained using (19) and $E(e^{-at_s}|\mathbf{t})]$ can be obtained using

$$E(e^{-at_s}|\mathbf{t})] = \int_{t_r}^{\infty} e^{-at_s} h_1^*(t_s|\mathbf{t}) dt_s. \tag{31}$$

3.2 Prediction When α_j and θ_j Are Unknown

Let the mixing proportion, p_j is known and $\theta_j, \alpha_j, j = 1, 2$, are unknown.

3.2.1 Maximum Likelihood Prediction

Maximum likelihood prediction can be obtained using (15) and (16), by replacing the parameters α_1,α_2, θ_1 and θ_2 by $\widehat{\alpha}_{1(ML)}$,$\widehat{\alpha}_{2(ML)}$,$\widehat{\theta}_{1(ML)}$ and $\widehat{\theta}_{2(ML)}$ which we obtained using (12).

1- Interval prediction:

The MLPI for any future observation t_s, $s = r + 1, r + 2, ..., n$ can be obtained by

$$\begin{aligned} Pr[t_s \geq \upsilon|\mathbf{t}] &= \int_{\upsilon}^{\infty} f_1^{\star}(t_s|\widehat{\alpha}_{1(ML)}, \widehat{\alpha}_{2(ML)}, \widehat{\theta}_{1(ML)}, \widehat{\theta}_{2(ML)}) dt_s, \ m = -1, \\ &= \int_{\upsilon}^{\infty} f_2^{\star}(t_s|\widehat{\alpha}_{1(ML)}, \widehat{\alpha}_{2(ML)}, \widehat{\theta}_{1(ML)}, \widehat{\theta}_{2(ML)}) dt_s, \ m \neq -1. \end{aligned} \tag{32}$$

A $(1 - \tau) \times 100$ % MLPI (L, U) of the future observation t_s is given by solving the two nonlinear equations (18).

2- Point prediction:

The MLPP for any future observation t_s, $s = r + 1, r + 2, ..., n$ can be obtained by replacing the parameters α_1,α_2, θ_1 and θ_2 by $\widehat{\alpha}_{1(ML)}$,$\widehat{\alpha}_{2(ML)}$,$\widehat{\theta}_{1(ML)}$ and $\widehat{\theta}_{2(ML)}$ which we obtained using (12).

$$\begin{aligned} \widehat{t_{s(ML)}} = E(t_s) &= \int_{t_r}^{\infty} t_s f_1^{\star}(t_s|\widehat{\alpha}_{1(ML)}, \widehat{\alpha}_{2(ML)}, \widehat{\theta}_{1(ML)}, \widehat{\theta}_{2(ML)}) dt_s, \ m = -1, \\ &= \int_{t_r}^{\infty} t_s f_2^{\star}(t_s|\widehat{\alpha}_{1(ML)}, \widehat{\alpha}_{2(ML)}, \widehat{\theta}_{1(ML)}, \widehat{\theta}_{2(ML)}) dt_s, \ m \neq -1. \end{aligned} \tag{33}$$

3.2.2 Bayesian Prediction

Let the mixing proportion, p_j is known and $\theta_j, \alpha_j, j = 1, 2$, are unknown. Suppose that the prior knowledge about the parameters α and θ is adequately represented by the function $g(\alpha, \theta)$ given by

$$g(\alpha, \theta) = \prod_{j=1}^{2} g(\theta_j|\alpha_j) g(\alpha_j), \tag{34}$$

where $g(\theta_j|\alpha_j)$ is Gamma $(d_j, \frac{1}{\alpha_j})$, $g(\alpha_j)$ is Gamma $(b_j, \frac{1}{c_j})$ with respective densities

$$g(\theta_j|\alpha_j) \propto \alpha_j^{-d_j} \theta_j^{d_j - 1} e^{-\frac{\theta_j}{\alpha_j}}, \alpha_j, \theta_j > 0, (d_j > 0), j = 1, 2, \tag{35}$$

$$g(\alpha_j) \propto \alpha_j^{b_j - 1} e^{-\frac{\alpha_j}{c_j}}, \alpha_j > 0, (b_j, c_j > 0), j = 1, 2. \tag{36}$$

It then follows, by substituting (35) and (36) in (34), that the prior PDF of α_j and θ_j is given by

$$g(\alpha_j, \theta_j) \propto \prod_{j=1}^{2} \alpha_j^{b_j - d_j - 1} \theta_j^{d_j - 1} e^{-\frac{(c_j \theta_j + \alpha_j{}^2)}{\alpha_j c_j}}, \alpha_j, \theta_j > 0, (b_j, c_j, d_j > 0), \tag{37}$$

where b_j, c_j and d_j are the prior parameters (also known as hyperparameters). It follows, from (10) and (37), that the joint posterior density function is given by

$$\pi_2^*(\alpha_j, \theta_j|\mathbf{t}) = A_2 \prod_{j=1}^{2} \alpha_j^{b_j - d_j - 1} \theta_j^{d_j - 1} e^{-\frac{(c_j \theta_j + \alpha_j{}^2)}{\alpha_j c_j}} \prod_{i=1}^{r-1} [p_1 R_1(t_i) + p_2 R_2(t_i)]^{m_i}$$

$$\times\prod_{i=1}^{r}[p_1f_1(t_i)+p_2f_2(t_i)][p_1R_1(t_r)+p_2R_2(t_r)]^{\gamma_r-1}, \tag{38}$$

where

$$A_2^{-1}=\int_0^\infty\int_0^\infty g(\alpha,\theta)L(\theta|\mathbf{t})\,d\theta d\alpha. \tag{39}$$

The Bayes prediction density function of $T_s \equiv T(s,n,m,k)$ can be obtained, see Aitchison and Dunsmore (1975), by

$$h_2^*(t_s|\mathbf{t})=\int_0^\infty\int_0^\infty \pi_2^*(\alpha,\theta|\mathbf{t})f_1^\star(t_s|\theta_j,\alpha_j)d\alpha d\theta,\ \ m=-1, \tag{40}$$

$$=\int_0^\infty\int_0^\infty \pi_2^*(\alpha,\theta|\mathbf{t})f_2^\star(t_s|\theta_j,\alpha_j)d\alpha d\theta,\ \ m\neq-1.$$

1-Interval prediction:

Bayesian prediction interval, for the future observation $T_{s;n,\widetilde{m},k}$, $s=r+1,r+2,...,n$, can be computed by $h_2^*(t_s|\mathbf{t})$ which can be approximated using the MCMC algorithm, see Ahmad et al. (2011), by the form

$$h_2^*(t_s|\mathbf{t})\cong\frac{\sum_{i=1}^{\mu}f^\star(t_s|\theta_j^i,\alpha_j^i)}{\sum_{i=1}^{\mu}\int_{t_r}^\infty f^\star(t_s|\theta_j^i,\alpha_j^i)dt_s}, \tag{41}$$

where $\theta_j^i,\alpha_j^i, i=1,2,3,...,\mu$ are generated from the posterior density function (38) using Gibbs sampler and Metropolis-Hastings techniques. A $(1-\tau)\times 100$ % BPI (L,U) of the future observation t_s is given by solving the following two nonlinear equations

$$\frac{\sum_{i=1}^{\mu}\int_L^\infty f^\star(t_s|\theta_j^i,\alpha_j^i)dt_s}{\sum_{i=1}^{\mu}\int_{t_r}^\infty f^\star(t_s|\theta_j^i,\alpha_j^i)dt_s}=1-\frac{\tau}{2}, \tag{42}$$

$$\frac{\sum_{i=1}^{\mu}\int_U^\infty f^\star(t_s|\theta_j^i,\alpha_j^i)dt_s}{\sum_{i=1}^{\mu}\int_{t_r}^\infty f^\star(t_s|\theta_j^i,\alpha_j^i)dt_s}=\frac{\tau}{2}. \tag{43}$$

Numerical methods are generally necessary to solve the above two equations to obtain L and U for a given τ.

2- Point prediction:

1- BPP for the future observation t_s based on BSEL function can be obtained using

$$\widetilde{t}_{s_{(BS)}}=\Omega\widehat{t}_{s_{(ML)}}+(1-\Omega)E(t_s|\mathbf{t}), \tag{44}$$

where $\widehat{t}_{s_{(ML)}}$ is the ML prediction for the future observation t_s which can be obtained using (33) and $E(t_s|\mathbf{t})$ can be obtained using

$$E(t_s|\mathbf{t})=\int_{t_r}^\infty t_s h_2^*(t_s|\mathbf{t})dt_s. \tag{45}$$

2- BPP for the future observation t_s based on BLINX loss function can be obtained using

$$\widetilde{t}_{s_{(BL)}}=-\frac{1}{a}ln[\Omega\exp[-a\widehat{t}_{s_{(ML)}}]+(1-\Omega)E(e^{-at_s}|\mathbf{t})], \tag{46}$$

where $\widehat{t}_{s_{(ML)}}$ is the ML prediction for the future observation t_s which can be obtained using (33) and $E(e^{-at_s}|\mathbf{t})]$ can be obtained using

$$E(e^{-at_s}|\mathbf{t})]=\int_{t_r}^\infty e^{-at_s}h_2^*(t_s|\mathbf{t})dt_s. \tag{47}$$

4. Prediction in Case of Two-sample Scheme

Based on the informative $T_{1;n,\widetilde{m},k},T_{2;n,\widetilde{m},k},...,T_{r;n,\widetilde{m},k}$ GOS drawn from the MTEW distribution and let $Y_1<Y_2<\cdots<Y_N$, where $Y_i\equiv Y_{i;N,M,K}, i=1,2,\ldots,N$, $M>0$, $K>0$ be a second independent generalized ordered random sample (of size N) of future observations from the same distribution. We want to predict any future (unobserved)

GOS $Y_b \equiv Y_{b;N,M,K}, b = 1, 2, \ldots, N$, in the future sample of size N. The PDF of $Y_b, 1 \leq b \leq N$ given the vector of parameters θ, is:

$$g^*(y_b|\theta) \propto \begin{cases} [R(y_b)]^{\gamma_b^\star - 1} f(y_b) \sum_{j=0}^{b-1} \omega_j^b [R(y_b)]^{j(M+1)}, & M \neq -1, \\ [R(y_b)]^{K-1}[\ln R(y_b)]^{b-1} f(y_b), & M = -1, \end{cases} \tag{48}$$

where $\omega_j^b = (-1)^j \binom{b-1}{j}$ and $\gamma_j^\star = K + (N - j)(M + 1)$.

Substituting from (5) and (7) in (48), we have:

$$g_1^*(y_b|\theta) \propto [p_1R_1(y_b) + p_2R_2(y_b)]^{\gamma_b^\star - 1}[p_1 f_1(y_b) + p_2 f_2(y_b)] \sum_{j=0}^{b-1} \omega_j^b [p_1R_1(y_b) + p_2R_2(y_b)]^{j(M+1)},\ M \neq -1, \tag{49}$$

$$g_2^*(y_b|\theta) \propto [p_1R_1(y_b) + p_2R_2(y_b)]^{K-1}[\ln [p_1R_1(y_b) + p_2R_2(y_b)]]^{b-1}[p_1 f_1(y_b) + p_2 f_2(y_b)],\ M = -1. \tag{50}$$

4.1 Prediction When α Is Known

4.1.1 Maximum Likelihood Prediction

Maximum likelihood prediction can be obtained using (49) and (50) by replacing the shape parameters θ_1 and θ_2 by $\widehat{\theta}_{1_{(ML)}}$ and $\widehat{\theta}_{2_{(ML)}}$.

1- Interval prediction:

The MLPI for any future observation $y_b,\ 1 \leq b \leq N$ can be obtained by

$$Pr[y_b \geq v|\mathbf{t}] = \int_v^\infty g_1^*(y_b|\widehat{\theta}_{1_{(ML)}}, \widehat{\theta}_{2_{(ML)}})dy_b,\ M \neq -1, \tag{51}$$

$$= \int_v^\infty g_2^*(y_b|\widehat{\theta}_{1_{(ML)}}, \widehat{\theta}_{2_{(ML)}})dy_b,\ M = -1.$$

A $(1-\tau)\times 100$ % MLPI (L, U) of the future observation y_b is given by solving the following two nonlinear equations

$$Pr[y_b \geq L(t) \mid \mathbf{t}] = 1 - \frac{\tau}{2},\ Pr[y_b \geq U(t) \mid \mathbf{t}] = \frac{\tau}{2}. \tag{52}$$

2- Point prediction:

The MLPP for any future observation y_b can be obtained by replacing the shape parameters θ_1 and θ_2 by $\widehat{\theta}_{1_{(ML)}}$ and $\widehat{\theta}_{2_{(ML)}}$

$$\widehat{y}_{b_{(ML)}} = E[y_b|\mathbf{t}] = \int_0^\infty y_b\, g_1^*(y_b|\widehat{\theta}_{1_{(ML)}}, \widehat{\theta}_{2_{(ML)}})dy_b,\ M \neq -1, \tag{53}$$

$$= \int_0^\infty y_b\, g_2^*(y_b|\widehat{\theta}_{1_{(ML)}}, \widehat{\theta}_{2_{(ML)}})dy_b,\ M = -1.$$

4.1.2 Bayesian Prediction

The predictive density function of $Y_b, 1 \leq b \leq N$ is given by:

$$\Psi^*(y_b \mid \mathbf{t}) = \int_0^\infty g^*(y_b \mid \theta)\pi_1^*(\theta \mid \mathbf{t})d\theta,\ y_b > 0, \tag{54}$$

where for $M \neq -1$ and $m \neq -1$

$$\Psi_1^*(y_b \mid \mathbf{t}) = \int_0^\infty g_1^*(y_b|\theta)\pi_1^*(\theta \mid \mathbf{t})d\theta \propto \prod_{j=1}^{2} \theta_j^{\nu_j - 1} \exp[-\sum_{j=1}^{2} \beta_j\theta_j][p_1R_1(y_b) + p_2R_2(y_b)]^{\gamma_b^\star - 1}$$

$$\times[p_1 f_1(y_b) + p_2 f_2(y_b)] \sum_{j=0}^{b-1} \omega_j^b [p_1R_1(y_b) + p_2R_2(y_b)]^{j(M+1)} \prod_{i=1}^{r-1}[p_1R_1(t_i) + p_2R_2(t_i)]^{m_i}$$

$$\times \prod_{i=1}^{r}[p_1 f_1(t_i) + p_2 f_2(t_i)][p_1R_1(t_r) + p_2R_2(t_r)]^{\gamma_r - 1}d\theta. \tag{55}$$

Also, when $M = -1$ and $m = -1$

$$\Psi_2^*(y_b|\mathbf{t}) = \int_0^\infty g_2^*(y_b|\theta)\pi_1^*(\theta \mid \mathbf{t})d\theta \propto \prod_{j=1}^{2}\theta_j^{\gamma_j-1}\exp[-\sum_{j=1}^{2}\beta_j\theta_j][p_1f_1(y_b)+p_2f_2(y_b)]$$

$$\times[ln[p_1R_1(y_b)+p_2R_2(y_b)]]^{b-1}[p_1R_1(y_b)+p_2R_2(y_b)]^{K-1}\prod_{i=1}^{r-1}[p_1R_1(t_i)+p_2R_2(t_i)]^{-1}$$

$$\times\prod_{i=1}^{r}[p_1f_1(t_i)+p_2f_2(t_i)][p_1R_1(t_r)+p_2R_2(t_r)]^{\gamma_r-1}d\theta. \tag{56}$$

1-Interval prediction:

Bayesian prediction interval, for the future observation $Y_b, 1 \le b \le N$, can be computed using (55) and (56) which can be approximated using MCMC algorithm by the form

$$\Psi^*(y_b \mid \mathbf{t}) = \frac{\sum_{i=1}^{\mu} g^*(y_b|\theta_j^i)}{\sum_{i=1}^{\mu}\int_0^\infty g^*(y_b|\theta_j^i)dy_b}, \tag{57}$$

where $\theta_j^i, i = 1, 2, ..., \mu$ are generated from the posterior density function (22) using Gibbs sampler and Metropolis-Hastings techniques.

A $(1-\tau)\times 100$ % BPI (L, U) of the future observation y_b is given by solving the following two nonlinear equations

$$\frac{\sum_{i=1}^{\mu}\int_L^\infty g^*(y_b|\theta_j^i)dy_b}{\sum_{i=1}^{\mu}\int_0^\infty g^*(y_b|\theta_j^i)dy_b} = 1-\frac{\tau}{2}, \tag{58}$$

$$\frac{\sum_{i=1}^{\mu}\int_U^\infty g^*(y_b|\theta_j^i)dy_b}{\sum_{i=1}^{\mu}\int_0^\infty g^*(y_b|\theta_j^i)dy_b} = \frac{\tau}{2}. \tag{59}$$

Numerical methods such as Newton-Raphson are generally necessary to solve the above two nonlinear equations (58) and (59), to obtain L and U for a given τ.

2- Point prediction:

1- BPP for the future observation y_b based on BSEL function can be obtained using

$$\widetilde{y}_{b_{(BS)}} = \Omega\widehat{y}_{b_{(ML)}} + (1-\Omega)E(y_b|\mathbf{t}), \tag{60}$$

where $\widehat{y}_{b_{(ML)}}$ is the ML prediction for the future observation y_b which can be obtained using (53) and $E(y_b|\mathbf{t})$ can be obtained using

$$E(y_b|\mathbf{t}) = \int_0^\infty y_b\,\Psi^*(y_b|\mathbf{t})dy_b. \tag{61}$$

2- BPP for the future observation y_b based on BLINX loss function can be obtained using

$$\widetilde{y}_{b_{(BL)}} = -\frac{1}{a}ln[\Omega\exp[-a\widehat{y}_{b_{(ML)}}] + (1-\Omega)E(e^{-ay_b}|\mathbf{t})], \tag{62}$$

where $\widehat{y}_{b_{(ML)}}$ is the ML prediction for the future observation y_b which can be obtained using (53) and $E(e^{-ay_b}|\mathbf{t})]$ can be obtained using

$$E(e^{-ay_b}|\mathbf{t})] = \int_0^\infty e^{-ay_b}\Psi^*(y_b|\mathbf{t})dy_b. \tag{63}$$

4.2 Prediction When α and θ Are Unknown

4.2.1 Maximum Likelihood Prediction

Maximum likelihood prediction can be obtained using (49) and (50) by replacing the parameters θ_1, θ_2, α_1 and α_2 by $\widehat{\theta}_{1_{(ML)}}$,$\widehat{\theta}_{2_{(ML)}}$,$\widehat{\alpha}_{1_{(ML)}}$ and $\widehat{\alpha}_{2_{(ML)}}$.

1- Interval prediction:

The maximum likelihood Interval prediction (MLIP) for any future observation $y_b, 1 \leq b \leq N$ can be obtained by

$$Pr[y_b \geq \upsilon|\mathbf{t}] = \int_{\upsilon}^{\infty} g_1^*(y_b|\widehat{\theta}_{1(ML)}, \widehat{\theta}_{2(ML)}, \widehat{\alpha}_{1(ML)}, \widehat{\alpha}_{2(ML)})dy_b,\ M \neq -1, \tag{64}$$

$$= \int_{\upsilon}^{\infty} g_2^*(y_b|\widehat{\theta}_{1(ML)}, \widehat{\theta}_{2(ML)}, \widehat{\alpha}_{1(ML)}, \widehat{\alpha}_{2(ML)})dy_b,\ M = -1.$$

A $(1-\tau)\times 100$ % MLIP (L, U) of the future observation y_b is given by solving the following two nonlinear equations

$$Pr[y_b \geq L(t) \mid \mathbf{t}] = 1 - \frac{\tau}{2},\ \ Pr[y_b \geq U(t) \mid \mathbf{t}] = \frac{\tau}{2}. \tag{65}$$

2- Point prediction:

The MLPP for any future observation $y_b, 1 \leq b \leq N$ can be obtained by replacing the parameters θ_1, θ_2, α_1 and α_2 by $\widehat{\theta}_{1(ML)}$,$\widehat{\theta}_{2(ML)}$,$\widehat{\alpha}_{1(ML)}$ and $\widehat{\alpha}_{2(ML)}$

$$\widehat{y}_{b(ML)} = E(y_b) = \int_0^{\infty} y_b\, g_1^*(y_b|\widehat{\theta}_{1(ML)}, \widehat{\theta}_{2(ML)}, \widehat{\alpha}_{1(ML)}, \widehat{\alpha}_{2(ML)})dy_b,\ M \neq -1, \tag{66}$$

$$= \int_0^{\infty} y_b\, g_2^*(y_b|\widehat{\theta}_{1(ML)}, \widehat{\theta}_{2(ML)}, \widehat{\alpha}_{1(ML)}, \widehat{\alpha}_{2(ML)})dy_b,\ M = -1.$$

4.2.2 Bayesian Prediction

The predictive density function of $Y_b, 1 \leq b \leq N$ is given by:

$$\Psi^\star(y_b \mid \mathbf{t}) = \int_0^{\infty}\int_0^{\infty} g^*(y_b \mid \alpha, \theta)\pi_2^*(\alpha, \theta \mid \mathbf{t})d\alpha d\theta,\ y_b > 0, \tag{67}$$

For $M \neq -1$ and $m \neq -1$

$$\Psi_1^\star(y_b \mid \mathbf{t}) = \int_0^{\infty}\int_0^{\infty} g_1^*(y_b|\alpha, \theta)\pi_2^*(\alpha, \theta \mid \mathbf{t})d\alpha d\theta. \tag{68}$$

Also, when $M = -1$ and $m = -1$

$$\Psi_2^\star(y_b|\mathbf{t}) = \int_0^{\infty}\int_0^{\infty} g_2^*(y_b|\alpha, \theta)\pi_2^*(\alpha, \theta \mid \mathbf{t})d\alpha d\theta. \tag{69}$$

1-Interval prediction:

Bayesian prediction interval, for the future observation $Y_b, 1 \leq b \leq N$, can be computed using (68) and (69) which can be approximated using MCMC algorithm by the form

$$\Psi^\star(y_b \mid \mathbf{t}) = \frac{\sum_{i=1}^{\mu} g^*(y_b|\alpha_j^i, \theta_j^i)}{\sum_{i=1}^{\mu}\int_0^{\infty} g^*(y_b|\alpha_j^i, \theta_j^i)dy_b}, \tag{70}$$

where $\alpha_j^i, \theta_j^i, i = 1, 2, ..., \mu$ are generated from the posterior density function (38) using Gibbs sampler and Metropolis-Hastings techniques.

A $(1 - \tau) \times 100$ % BPI (L, U) of the future observation y_b is obtained by solving the following two nonlinear equations

$$\frac{\sum_{i=1}^{\mu}\int_L^{\infty} g^*(y_b|\alpha_j^i, \theta_j^i)dy_b}{\sum_{i=1}^{\mu}\int_0^{\infty} g^*(y_b|\alpha_j^i, \theta_j^i)dy_b} = 1 - \frac{\tau}{2}, \tag{71}$$

$$\frac{\sum_{i=1}^{\mu}\int_U^{\infty} g^*(y_b|\alpha_j^i, \theta_j^i)dy_b}{\sum_{i=1}^{\mu}\int_0^{\infty} g^*(y_b|\alpha_j^i, \theta_j^i)dy_b} = \frac{\tau}{2}. \tag{72}$$

Numerical methods such as Newton-Raphson are necessary to solve the above two nonlinear equations (71) and (72), to obtain L and U for a given τ.

2- Point prediction:

1- BPP for the future observation y_b based on BSEL function can be obtained using

$$\widetilde{y}_{b_{(BS)}} = \Omega\widehat{y}_{b_{(ML)}} + (1-\Omega)E(y_b|\mathbf{t}), \tag{73}$$

where $\widehat{y}_{b_{(ML)}}$ is the ML prediction for the future observation y_b and can be obtained using (66) and $E(y_b|\mathbf{t})$

$$E(y_b|\mathbf{t}) = \int_0^\infty y_b\,\Psi^\star(y_b|\mathbf{t})dy_b. \tag{74}$$

2- BPP for the future observation y_b based on BLINX loss function can be obtained using

$$\widetilde{y}_{b_{(BL)}} = -\frac{1}{a}ln[\Omega\exp[-a\widehat{y}_{b_{(ML)}}] + (1-\Omega)E(e^{-ay_b}|\mathbf{t})], \tag{75}$$

where $\widehat{y}_{b_{(ML)}}$ is the ML prediction for the future observation y_b and can be obtained using (66) and $E(e^{-ay_b}|\mathbf{t})]$

$$E(e^{-ay_b}|\mathbf{t})] = \int_0^\infty e^{-ay_b}\Psi^\star(y_b|\mathbf{t})dy_b. \tag{76}$$

5. Numerical Computations

The upper record values can be obtained from the GOS by taking $m = -1, k = 1$ and $\gamma_r = 1$. In this Section, we will compute point and interval predictors of future upper record values in two cases, one sample and two sample prediction as following:

5.1 One Sample Prediction

The following steps are used to obtain ML prediction (point and interval) and Bayesian prediction (point and interval) for the remaining $(n-r)$ failure times $T_s \equiv T_{s,n,\widetilde{m},k}$, $s = r+1$.

1) For given values of α_1, α_2, θ_1 and θ_2, upper record values of different sizes are generated from the MTEW distribution.

2) Generate θ_1^i, θ_2^i, α_1^i and α_2^i, $i = 1, 2, ..., \mu$, from the posterior PDF using MCMC algorithm.

3) Solving equations (18), numerically, we get the 95% MLPI for unobserved upper record values.

4) The MLPP for the future observation t_{r+1}, is computed using (19) when α_j is known and (33) when α_j and θ_j are unknown.

5) The 95% BPI for unobserved upper record are obtained by solving equations (26) and (27) when α_j is known and (42) and (43) when α_j and θ_j are unknown.

6) The BPP for the future observation t_{r+1}, is computed based on BSEL function using (28) when α_j is known and (44) when α_j and θ_j are unknown.

7) The BPP for the future observation t_{r+1}, is computed based on BLINX loss function using (30) when α_j is known and (46) when α_j and θ_j are unknown.

5.2 Two Sample Prediction

The following steps are used to obtain ML prediction (point and interval) and Bayesian prediction (point and interval) for future upper record values $Y_b, b = 1$.

1) For given values of α_1, α_2, θ_1 and θ_2, upper record values of different sizes are generated from the MTEW distribution.

2) Generate θ_1^i, θ_2^i, α_1^i and α_2^i, $i = 1, 2, ..., \mu$, from the posterior PDF using MCMC algorithm.

3) Solving equations (52) when α_j is known and (65) when α_j and θ_j are unknown we get the 95% MLPI for unobserved upper record values.

4) The MLPP for the future observation y_1, is computed using (53) when α_j is known and (66) when α_j and θ_j are unknown.

5) The 95% BPI for unobserved upper record are obtained by solving equations (58) and (59) when α_j is known and (71) and (72) when α_j and θ_j are unknown.

6) The BPP for the future observation y_1, is computed based on BSEL function using (60) when α_j is known and (73) when α_j and θ_j are unknown.

7) The BPP for the future observation y_1, is computed based on BLINX loss function using (62) when α_j is known and (75) when α_j and θ_j are unknown.

8) Generate $10,000$ samples each of size $N = 5$ from a MTEW distribution, then calculate the coverage percentage (CP) of Y_1.

The computational (our) results were computed by using Mathematica 7.0. When α_j is known, the prior parameters chosen as $\nu_1 = 2,\ \nu_2 = 4,\ \beta_1 = 1,\ \beta_2 = 2$ which yield the generated values of $\theta_1 = 2.3008$ and $\theta_2 = 2.4297$. While, in the case of four parameters are unknown the prior parameters $(b_1, b_2, c_1, c_2, d_1, d_2)$ chosen as (1,2,0.5,1.5,2,3) which yield the generated values of $\theta_1 = 1.09192, \theta_2 = 2.09977, \alpha_1 = 1.87056, \alpha_2 = 1.45415$. In Tables (1), (2) and (3), (4) point and 95% interval predictors for the first future upper record value are computed in case of the one- and two sample predictions, respectively.

6. Conclusions

In our study, we obtained the maximum likelihood prediction (point and interval) and Bayesian prediction (point and interval) for the future observation from mixture of two Exponentiated Weibull (MTEW) distributions. From Tables (1-4) we observe the following:

1) Point and 95% interval predictors for future observations are obtained using a one-sample and two-sample schemes based on a MTEW distribution. Our results are specialized to upper record values.

2) It is evident from all tables that, the lengths of the MLPI and BPI decrease as the sample size increase.

3) The percentage coverage improves by use of a large number of observed values.

4) It may be noticed that when $\omega = 1$, we obtain the MLEs while the case $\omega = 0$, yields the Bayes prediction under SE and LINX loss function.

Table 1. Point and 95% interval predictors for the future upper record values T_s^*, $s = r + 1$ when $(\alpha_1 = 2, \alpha_2 = 2, \theta_1 = 2.3008, \theta_2 = 2.4297, p = 0.4, \Omega = 0.5)$

r	point predictions					interval predictions	
	ML	$BSEL$	$BLINEX$			ML	Bayes
			$a = 0.001$	$a = 2$	$a = 3$	(L, U) Length	(L, U) Length
3	1.68679	1.67341	1.6734	1.6422	1.62918	(1.35717,2.38895) 1.03179	(1.3551,2.35294) 0.997841
5	2.07701	2.07633	2.07632	2.05588	2.04768	(1.83841,2.65733) 0.818921	(1.83833,2.65519) 0.816863
7	2.73883	2.73875	2.73874	2.72607	2.72093	(2.56036,3.19704) 0.636678	(2.56035,3.19676) 0.636407

Table 2. Point and 95% interval predictors for the future upper record values T_s^*, $s = r+1$ when $(\alpha_1 = 1.87056, \alpha_2 = 1.45415, \theta_1 = 1.09192, \theta_2 = 2.09977, p = 0.4, \Omega = 0.5)$

r	point predictions					interval predictions	
	ML	$BSEL$	$BLINEX$			ML	Bayes
			$a = 0.001$	$a = 2$	$a = 3$	(L, U) Length	(L, U) Length
3	1.47888	1.47275	1.47273	1.44914	1.43914	(1.19352,2.07271) 0.879189	(1.1927,2.05073) 0.858027
5	1.76718	1.76758	1.76757	1.75595	1.75108	(1.58924,2.18908) 0.599841	(1.58929,2.19013) 0.600841
7	2.66639	2.66649	2.66650	2.65607	2.65179	(2.50679,3.08025) 0.573461	(2.5068,3.08077) 0.57397

Table 3. Point and 95% interval predictors for the future upper record values Y_b^*, $b = 1$ when ($\alpha_1 = 2, \alpha_2 = 2, \theta_1 = 2.3008, \theta_2 = 2.4297, p = 0.4, \Omega = 0.5$)

	point predictions					interval predictions	
r	ML	$BSEL$	$BLINEX$			ML	Bayes
			$a = 0.001$	$a = 2$	$a = 3$	(L, U) Length (CP)	(L, U) Length (CP)
3	0.869292	0.87952	0.933136	0.827488	0.776678	(0.146129,1.90915) 1.76302 (96.82)	(0.00140531,1.97937) 1.97796 (97.88)
5	0.959412	0.919886	0.957384	0.843413	0.782067	(0.221228,1.96968) 1.74845 (97.50)	(0.0027575,1.9742) 1.97144 (97.69)
7	0.945327	0.905571	0.951503	0.841096	0.781509	(0.208555,1.96031) 1.75176 (97.27)	(0.000573547,1.96927) 1.9687 (97.43)

Table 4. Point and 95% interval predictors for the future upper record values Y_b^*, $b = 1$ when ($\alpha_1 = 1.87056, \alpha_2 = 1.45415, \theta_1 = 1.09192, \theta_2 = 2.09977, p = 0.4, \Omega = 0.5$)

	point predictions					interval predictions	
r	ML	$BSEL$	$BLINEX$			ML	Bayes
			$a = 0.001$	$a = 2$	$a = 3$	(L, U) Length (CP)	(L, U) Length (CP)
3	1.10673	1.14704	1.14694	1.01421	0.950383	(0.232142,2.30746) 2.07532 (94.81)	(0.249429,2.5401) 2.29067 (95.77)
5	1.02567	1.06597	1.06586	0.938197	0.882518	(0.224342,2.33226) 2.10791 (95.47)	(0.244942,2.53322) 2.28828 (95.83)
7	0.898788	0.97657	0.976504	0.871424	0.824372	(0.0982443,1.97087) 1.87263 (93.24)	(0.195128,2.18531) 1.99018 (95.04)

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