

The Marshall-Olkin Half Logistic-G Family of Distributions With Applications

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Abstract

Attempts have been made to define new classes of distributions that provide more flexibility for modeling data that is skewed in nature. In this work, we propose a new family of distributions namely the Marshall-Olkin Half Logistic-G (MO-HL-G) based on the generator pioneered by [Marshall and Olkin, 1997]. This new family of distributions allows for a flexible fit to real data from several fields, such as engineering, hydrology, and survival analysis. The structural properties of these distributions are studied and its model parameters are obtained through the maximum likelihood method. We finally demonstrate the effectiveness of these models via simulation experiments.

Keywords: Marshall-Olkin-G, Half-Logistic-G, maximum likelihood estimation

1. Introduction

More work has recently been done by various authors on the development of new families of distributions through the extension of other existing continuous distributions. These new families have found wider applicability in modeling data in different areas such as engineering, economics, biological studies, environmental sciences, just to mention these few. The main objectives of generalizing this new family of distributions are as follows: achieve skewness for symmetrical models; develop special models exhibiting various shapes of hazard rate function; develop heavy-tailed distributions useful in many real data sets; attain more flexible kurtosis compared to that of the baseline distribution; generate skewed, symmetric, J-shaped or reversed-J shaped distributions with better fits than other generalized distributions having the same underlying model.

[Marshall and Olkin, 1997], proposed a new distribution with cumulative distribution function (cdf) and probability density function (pdf) given by

$$F_{MO-G}(x; \delta, \xi) = 1 - \frac{\delta \bar{G}(x; \xi)}{1 - \bar{\delta} \bar{G}(x; \xi)} \quad (1)$$

and

$$f_{MO-G}(x; \delta, \xi) = \frac{\delta g(x; \xi)}{[1 - \bar{\delta} \bar{G}(x; \xi)]^2}, \quad (2)$$

respectively, where δ denotes the tilt parameter and $G(x; \xi)$ is the baseline cdf. The distribution is more flexible compared to the exponential, Weibull and gamma distributions.

Furthermore, [Cordeiro et al., 2016], developed the type 1 half-logistic family of distributions with the cdf and pdf given by

$$\begin{aligned} F(x; \lambda, \xi) &= \int_0^{-\ln(1-G(x;\xi))} \frac{2\lambda \exp\{-\lambda x\}}{(1 + \exp\{-\lambda x\})^2} dx \\ &= \frac{1 - [1 - G(x; \xi)]^\lambda}{1 + [1 - G(x; \xi)]^\lambda}, \end{aligned} \quad (3)$$

where $G(x; \xi)$ is the cdf of the baseline distribution and $\lambda > 0$ is the shape parameter. We obtain a special case, namely, half-logistic-G (HL-G) model, with cdf

$$F_{HL-G}(x; \xi) = \frac{G(x; \xi)}{1 + \bar{G}(x; \xi)}, \quad (4)$$

if we set $\lambda = 1$ in equation (3). The corresponding pdf of the HL-G model is given by

$$f_{HL-G}(x; \xi) = \frac{2g(x; \xi)}{(1 + \bar{G}(x; \xi))^2} \tag{5}$$

The Marshall-Olkin distribution has been generalized from several distributions to construct many family of distributions achieving some of the above characteristics. The examples of these include; Marshall-Olkin-G (MO-G) family by [Marshall and Olkin , 1997], beta Marshall-Olkin-G (BMO-G) by [Alizadeh et al. , 2015a], Kumaraswamy Marshall-Olkin-G (KwMO-G) by [Alizadeh et al. , 2015b], among others. The proposed distribution gives a new flexible parametric model useful in modeling various forms of data in reliability studies, survival analysis, statistical mechanics, quality control, economics, bio medical studies, etc. [Lepetu et al. , 2017], introduced a new class of distributions called the Marshall-Olkin Log-logistic Extended Weibull (MOLLEW) family of distributions. Their work employs the Marshall-Olkin transformation to the Log-logistic Weibull distribution to obtain more new flexible models suitable for reliability data.

Marshall- Olkin transformation was applied to various well-known distributions: Weibull by [Ghitany et al. , 2005], [Zhang and Xie , 2007]. Lately, the general results have been addressed by [Barreto-Souza et al. , 2013]. [Santos-Neo et al. , 2014], introduces a new class of models called the Marshall-Olkin extended Weibull family of distributions which defines at least twenty-one special models. [Chakraborty and Handique , 2017], presented the generalized Marshall-Olkin Kumaraswamy-G distribution. [Lazhar , 2017] developed and studied the properties of the Marshall-Olkin extended generalized Gompertz distribution. [Kumar , 2016] discussed the ratio and inverse moments of Marshall-Olkin extended Burr Type III distribution from lower generalized order statistics.

On the similar note, the Half-Logistic (HL) distribution has not received much attention from researchers in terms of generalization. [Afify et al. , 2017] proposed a new flexible family of distributions called the Odd exponentiated half logistic-G (OEHL-G) family of distributions using the HL distribution as the generator and studied its mathematical properties. Their results utilized the flexibility of the baseline distribution in modeling various forms of data.

Other generally known families are the beta-G by [Eugene et al. , 2002], odd log-logistic-G by [Gleaton and Lynch , 2004], the transmuted-G by [Shaw and Buckley , 2009], the gamma-G by [Zografos and Balakrishnan , 2009], the Kumaraswamy-G by [Cordeiro and de Castro , 2011], the logistic-G by [Torabi and Montazeri , 2014], exponentiated generalized-G by [Cordeiro et al. , 2013], the McDonald-G by [Alexander et al. , 2012], T-X family by [Alzaatreh et al. , 2013], the exponentiated half-logistic generated family by [Cordeiro et al. , 2014], the beta odd log-logistic generalized by [Cordeiro et al. , 2015], generalized transmuted-G by [Nofal et al. , 2017], generalized odd log-logistic-G by [Cordeiro et al. , 2017]. Various structural properties of the extended distributions may be easily explored using mixture forms of exponentiated-G (exp-G) distributions. It is in the same vein that we develop the new family of distributions called Marshall-Olkin Half Logistic-G (MO-HL-G) family of distributions.

The layout of this paper is as follows; Section 2, presents the new generalized family of distributions with some of its structural properties. The Maximum likelihood estimates are derived under Section 3. Some of the special cases of this new generalized distributions are presented in Section 4. Section 5, presents results from Monte Carlo simulation study. In Section 6. we present applications of the proposed model to real data sets and finally give concluding remarks under Section 7.

2. The Model and Statistical Properties

2.1 The Model

We develop the MO-HL-G distribution using the generalization proposed by [Marshall and Olkin , 1997], and taking the baseline distribution to be the HL-G distribution by [Cordeiro et al. , 2016]. The cdf and pdf of the MO-HL-G family of distributions is given by

$$F_{MO-HL-G}(x; \delta, \xi) = \frac{[\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}]}{1 - \bar{\delta}[1 - \frac{G(x;\xi)}{1+\bar{G}(x;\xi)}]} \tag{6}$$

and

$$f_{MO-HL-G}(x; \delta, \xi) = \frac{2\delta g(x; \xi)}{[1 + \bar{G}(x; \xi)]^2 [1 - \bar{\delta}(1 - \frac{G(x;\xi)}{1+\bar{G}(x;\xi)})]^2}, \tag{7}$$

respectively, for $\delta > 0$, $\bar{\delta} = 1 - \delta$ and ξ is a vector of parameters from the baseline distribution function $G(\cdot)$.

2.2 Quantile Function

The quantile function for the MO-HL-G family of distributions is derived by solving the non-linear equation

$$F_{MO-HL-G}(x; \delta, \xi) = \frac{\left[\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right]}{1 - \bar{\delta} \left[1 - \frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right]} = u$$

for $0 \leq u \leq 1$, that is,

$$\begin{aligned} \left(\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right) &= \left(\frac{u(1 - \bar{\delta})}{1 - u\bar{\delta}} \right) \\ G(x; \xi) &= 2 \left(\left(\frac{u(1 - \bar{\delta})}{1 - u\bar{\delta}} \right)^{-1} + 1 \right)^{-1}. \end{aligned}$$

Therefore, the quantiles of the MO-HL-G family of distributions may be obtained by solving the equation

$$x(u) = G^{-1} \left[2 \left(\left(\frac{u(1 - \bar{\delta})}{1 - u\bar{\delta}} \right)^{-1} + 1 \right)^{-1} \right] \tag{8}$$

via iterative methods in R or Matlab software.

2.3 Expansion of Density

This sub-section presents statistical properties of the MO-HL-G family of distributions as derived from the general results for the Marshall and Olkin’s family of distributions by [Barreto-Souza et al. , 2013]. Considering

$$f_{MO-HL-G}(x; \delta, \xi) = \frac{\alpha f_{HL-G}(x; \xi)}{(1 - \bar{\delta} \bar{F}_{HL-G}(x; \xi))^2}, \tag{9}$$

we can write equation (7) as

$$f_{MO-HL-G}(x; \delta, \xi) = \frac{f_{HL-G}(x; \xi)}{\delta \left[1 - \frac{\delta - 1}{\delta} F_{HL-G}(x; \xi) \right]^2}, \tag{10}$$

where $f_{HL-G}(x; \xi)$ and $F_{HL-G}(x; \xi)$ are as given in equations (5) and (4), respectively. We apply the series expansion

$$(1 - z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k + j)}{\Gamma(k)j!} z^j, \tag{11}$$

which is valid for $|z| < 1$ and $k > 0$. If $\delta \in (0, 1)$, we can obtain

$$f_{MO-HL-G}(x; \delta, \xi) = f_{HL-G}(x; \xi) \sum_{j=0}^{\infty} \sum_{k=0}^j w_{jk} F_{HL-G}(x; \xi)^{j-k}, \tag{12}$$

where $w_{j,k} = w_{j,k}(\delta) = \delta(j + 1)(1 - \delta)^j (-1)^{j-k} \binom{j}{k}$. For $\delta > 1$, we have

$$f_{MO-HL-G}(x; \delta, \xi) = f_{HL-G}(x; \xi) \sum_{j=0}^{\infty} v_j F_{HL-G}^j(x; \xi), \tag{13}$$

where $v_j = v_j(\delta) = \frac{(j+1)(1-1/\delta)}{\delta}$. For $\delta \in (0, 1)$, equation (7) becomes

$$\begin{aligned} f_{MO-HL-G}(x; \delta, \xi) &= \frac{2g(x; \xi)}{(1 + \bar{G}(x; \xi))^2} \sum_{j=0}^{\infty} \sum_{k=0}^j w_{jk} \left[\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right]^{j-k} \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^j 2w_{jk} g(x, \xi) \frac{[G(x; \xi)]^{j-k}}{[1 + \bar{G}(x; \xi)]^{j-k+2}}. \end{aligned}$$

Using the series expansion

$$[1 + \bar{G}(x; \xi)]^{-(j-k+2)} = \sum_{m=0}^{\infty} (-1)^m \binom{-(j-k+2)}{m} \bar{G}^m(x; \xi)$$

and

$$\bar{G}^m(x; \xi) = [1 - G(x; \xi)]^m = \sum_{l=0}^{\infty} (-1)^l \binom{m}{l} G^l(x; \xi),$$

we can write

$$\begin{aligned}
 f_{MO-HL-G}(x; \delta, \xi) &= \sum_{j,m,l=0}^{\infty} \sum_{k=0}^j \frac{2(-1)^{m+l}}{l+j-k+1} w_{j,k} \binom{-(j-k+2)}{m} \binom{m}{l} \\
 &\times (l+j-k+1)g(x; \xi)[G(x; \xi)]^{l+j-k} \\
 &= \sum_{j,m,l=0}^{\infty} w_{j,m,l}^* g_{l+j-k}(x; \xi).
 \end{aligned} \tag{14}$$

It follows that for $\delta \in (0, 1)$, the MO-HL-G family of distributions can be expressed as an infinite linear combination of the exponentiated-G (Exp-G) distribution with power parameter $(l + j - k)$ and linear component

$$w_{j,m,l}^* = \sum_{k=0}^j \frac{2(-1)^{m+l}}{l+j-k+1} w_{j,k} \binom{-(j-k+2)}{m} \binom{m}{l}. \tag{15}$$

Furthermore, for $\delta > 1$ equation (7) can be written as

$$\begin{aligned}
 f_{MO-HL-G}(x; \delta, \xi) &= \frac{2g(x; \xi)}{(1 + \bar{G}(x; \xi))^2} \sum_{j=0}^{\infty} v_j \left[\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right]^j \\
 &= \sum_{j=0}^{\infty} 2v_j \frac{g(x; \xi)[G(x; \xi)]^j}{[1 + \bar{G}(x; \xi)]^{j+2}}.
 \end{aligned}$$

Applying the series expansion

$$[1 + \bar{G}(x; \xi)]^{-(j+2)} = \sum_{m=0}^{\infty} (-1)^m \binom{-(j+2)}{m} \bar{G}^m(x; \xi)$$

and

$$\bar{G}^m(x; \xi) = [1 - G(x; \xi)]^m = \sum_{l=0}^{\infty} (-1)^l \binom{m}{l} G^l(x; \xi),$$

we can write

$$\begin{aligned}
 f_{MO-HL-G}(x; \delta, \xi) &= \sum_{j,m,l=0}^{\infty} \frac{2(-1)^{m+l}}{l+j+1} v_j \binom{-(j+2)}{m} \binom{m}{l} \\
 &\times (l+j+1)g(x; \xi)[G(x; \xi)]^{l+j} \\
 &= \sum_{j,m,l=0}^{\infty} v_{j,m,l}^* g_{l+j}(x; \xi).
 \end{aligned} \tag{16}$$

Therefore, for $\delta > 1$, the MO-HL-G family of distributions can be expressed as a linear combination of the Exp-G distribution with power parameter $(l + j)$ and linear component

$$v_{j,m,l}^* = \frac{2(-1)^{m+l}}{l+j+1} v_j \binom{-(j+2)}{m} \binom{m}{l}. \tag{17}$$

2.4 Distribution of Order Statistics

Suppose that X_1, X_2, \dots, X_n are independent and identically distributed (i.i.d) random variables distributed according to (7). The pdf of the i^{th} order statistic $X_{i:n}$, is given by

$$f_{i:n}(x; \delta, \xi) = \delta n! f_{HL-G}(x; \xi) \sum_{l=0}^{n-i} \frac{(-1)^l}{(i-1)!(n-i)!} \frac{F_{HL-G}^{l+i-1}(x; \xi)}{[1 - \delta F_{HL-G}(x; \xi)]^{l+i-1}}. \tag{18}$$

If $\delta \in (0, 1)$, we have

$$f_{i:n}(x; \delta, \xi) = f_{HL-G}(x; \xi) \sum_{j=0}^{\infty} \sum_{l=0}^{n-i} \sum_{k=0}^j U_{j,l,k} F_{HL-G}^{j+l-k+i-1}(x; \xi), \tag{19}$$

where

$$U_{j,l,k} = U_{j,l,k}(\delta) = \frac{\delta n!(-1)^l(1-\delta)^j(-1)^{j-k} \binom{j}{k} \binom{l+i+j}{j}}{(i-1)!(n-i)!} \tag{20}$$

For $\delta > 1$, we write $1 - \bar{\delta} \bar{F}_{HL-G}(x; \xi) = \delta \{1 - (\delta - 1)F_{HL-G}(x; \xi)/\delta\}$, such that

$$f_{i:n}(x; \delta, \xi) = f_{HL-G}(x; \xi) \sum_{j=0}^{\infty} \sum_{l=0}^{n-i} c_{j,l} F_{HL-G}^{j+l+i-1}(x; \xi), \tag{21}$$

where

$$c_{j,l} = c_{j,l}(\delta) = \frac{(-1)^l(\delta-1)^j n!}{\delta^{l+j+i}(i-1)!(n-i)!} \binom{l+i+j}{j} \tag{22}$$

For $\delta \in (0, 1)$, using equation (19) and substituting $f(x)$ by equation (5) and $F(x)$ by equation (4), we get

$$\begin{aligned} f_{i:n}(x; \delta, \xi) &= \frac{2g(x; \xi)}{(1 + \bar{G}(x; \xi))^2} \sum_{j=0}^{\infty} \sum_{l=0}^{n-i} \sum_{k=0}^j U_{j,l,k} \left[\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right]^{j+l-k+i-1} \\ &= \sum_{j=0}^{\infty} \sum_{l=0}^{n-i} \sum_{k=0}^j 2U_{j,l,k} \frac{g(x; \xi)[G(x; \xi)]^{j+l-k+i-1}}{[1 + \bar{G}(x; \xi)]^{j+l-k+i+1}}. \end{aligned}$$

By applying the expansions

$$[1 + \bar{G}(x; \xi)]^{-(j+l-k+i+1)} = \sum_{m=0}^{\infty} (-1)^m \binom{-(j+l-k+i+1)}{m} \bar{G}^m(x; \xi)$$

and

$$\bar{G}^m(x; \xi) = [1 - G(x; \xi)]^m = \sum_{p=0}^{\infty} (-1)^p \binom{m}{p} G^p(x; \xi),$$

we can write

$$\begin{aligned} f_{i:n}(x; \delta, \xi) &= \sum_{j,m,p=0}^{\infty} \sum_{l=0}^{n-i} \sum_{k=0}^j \frac{2(-1)^{m+p}}{p+j+l-k+i} U_{j,l,k} \binom{-(j+l-k+i+1)}{m} \binom{m}{p} \\ &\times (p+j+l-k+i)g(x; \xi)[G(x; \xi)]^{p+j+l-k+i-1} \\ &= \sum_{j,m,p=0}^{\infty} U_{j,m,p}^* g_{p+j+l-k+i}(x; \xi), \end{aligned} \tag{23}$$

where $g_{p+j+l-k+i}(x; \xi) = (p+j+l-k+i)g(x; \xi)[G(x; \xi)]^{p+j+l-k+i-1}$ is an Exp-G distribution with power parameter $(p+j+l-k+i)$ and

$$U_{j,m,p}^* = \sum_{l=0}^{n-i} \sum_{k=0}^j \frac{2(-1)^{m+p}}{p+j+l-k+i} U_{j,l,k} \binom{-(j+l-k+i+1)}{m} \binom{m}{p}. \tag{24}$$

Furthermore, for $\delta > 1$, we get

$$\begin{aligned} f_{i:n}(x; \delta, \xi) &= \frac{2g(x; \xi)}{(1 + \bar{G}(x; \xi))^2} \sum_{j=0}^{\infty} \sum_{i=0}^{n-i} c_{j,l} \left[\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right]^{j+l+i-1} \\ &= \sum_{j=0}^{\infty} \sum_{i=0}^{n-i} 2c_{j,l} \frac{g(x; \xi)[G(x; \xi)]^{j+l+i-1}}{[1 + \bar{G}(x; \xi)]^{j+l+i+1}}. \end{aligned}$$

By applying the binomial expansions

$$[1 + \bar{G}(x; \xi)]^{-(j+l+i+1)} = \sum_{m=0}^{\infty} (-1)^m \binom{-(j+l+i+1)}{m} \bar{G}^m(x; \xi)$$

and

$$\bar{G}^m(x; \xi) = [1 - G(x; \xi)]^m = \sum_{p=0}^{\infty} (-1)^p \binom{m}{p} G^p(x; \xi),$$

we can write

$$\begin{aligned} f_{i;n}(x; \delta, \xi) &= \sum_{j,m,p=0}^{\infty} \sum_{i=0}^{n-i} \frac{2(-1)^{m+p}}{p+j+l+i} c_{j,l} \binom{-(j+l+i+1)}{m} \binom{m}{p} \\ &\times (p+j+l+i)g(x; \xi)[G(x; \xi)]^{p+j+l+i-1} \\ &= \sum_{j,m,p=0}^{\infty} c_{j,m,p}^* g_{p+j+l+i}(x; \xi), \end{aligned} \tag{25}$$

where $g_{p+j+l+i}(x; \xi) = (p+j+l+i)g(x; \xi)[G(x; \xi)]^{p+j+l+i-1}$ is an Exp-G distribution with power parameter $(p+j+l+i)$ and

$$c_{j,m,p}^* = \sum_{i=0}^{n-i} \frac{2(-1)^{m+p}}{p+j+l+i} c_{j,l} \binom{-(j+l+i+1)}{m} \binom{m}{p}. \tag{26}$$

2.5 Entropy

An Entropy gives a measure of variation of uncertainty for a random variable X with pdf $g(x)$. There are two common measures of entropy, namely Shannon entropy by [Shannon , 1951] and Rényi entropy by [Rényi , 1961]. Rényi entropy is defined by

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\int_0^{\infty} g^{\nu}(x) dx \right],$$

where $\nu > 0$ and $\nu \neq 1$. Using expansion (11), for $\delta \in (0, 1)$

$$f_{MO-HL-G}^{\nu}(x; \delta, \xi) = \frac{\delta^{\nu} f_{HL-G}^{\nu}(x; \xi)}{\Gamma(2\nu)} \sum_{j=0}^{\infty} (1 - \delta)^j \Gamma(2\nu + j) \frac{[1 - F_{HL-G}(x; \xi)]^j}{j!}$$

and for $\delta > 1$

$$f_{MO-HL-G}^{\nu}(x; \delta, \xi) = \frac{f_{HL-G}^{\nu}(x)}{\delta^{\nu} \Gamma(2\nu)} \sum_{j=0}^{\infty} (\delta - 1)^j \Gamma(2\nu + j) \frac{F_{HL-G}^j(x; \xi)}{j!}.$$

Thus, Rényi entropy for $\delta \in (0, 1)$ and $\delta > 1$ are given by

$$I_R(\nu) = (1 - \nu)^{-1} \log \left(\sum_{j=0}^{\infty} e_j \int_0^{\infty} f_{HL-G}^{\nu}(x; \xi) (1 - F_{HL-G}(x; \xi))^j dx \right) \tag{27}$$

and

$$I_R(\nu) = (1 - \nu)^{-1} \log \left(\sum_{j=0}^{\infty} h_j \int_0^{\infty} f_{HL-G}^{\nu}(x; \xi) F_{HL-G}^j(x; \xi) dx \right), \tag{28}$$

where

$$e_j = e_j(\delta) = \frac{\delta^{\nu} (1 - \delta)^j \Gamma(2\nu + j)}{\Gamma(2\nu) j!}$$

and

$$h_j = h_j(\delta) = \frac{(\delta - 1)^j \Gamma(2\nu + j)}{\delta^{\nu+j} \Gamma(2\nu) j!}.$$

Now, for $\delta \in (0, 1)$ and using equation (27), we have

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\sum_{j=0}^{\infty} e_j \int_0^{\infty} \frac{2^{\nu} g^{\nu}(x; \xi)}{(1 + \bar{G}(x; \xi))^{2\nu}} \left[1 - \frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right]^j dx \right].$$

By considering the following expansions

$$\left[1 - \frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right]^j = \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(j+1)}{\Gamma(j+1-m) m!} \left[\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right]^m,$$

$$(1 + \bar{G}(x; \xi))^{-(2\nu+m)} = \sum_{p=0}^{\infty} (-1)^p \binom{-2\nu+m}{p} \bar{G}^p(x; \xi)$$

and

$$[\bar{G}(x; \xi)]^p = [1 - G(x; \xi)]^p = \sum_{q=0}^{\infty} (-1)^q \binom{p}{q} G^q(x; \xi),$$

we can write

$$\begin{aligned} I_R(\nu) &= (1 - \nu)^{-1} \log \left[\sum_{j,m,p,q=0}^{\infty} e_j \frac{2^\nu (-1)^{p+m+q} \Gamma(j+1)}{\Gamma(j+1-m)m!} \binom{-2\nu+m}{p} \binom{p}{q} \right. \\ &\times \left. \frac{1}{\left(\frac{m+q}{\nu} + 1\right)^\nu} \int_0^\infty \left(\frac{m+q}{\nu} + 1\right) g(x; \xi) [G(x; \xi)]^{\frac{m+q}{\nu}} dx \right] \\ &= (1 - \nu)^{-1} \log \left[\sum_{j,m,p,q=0}^{\infty} e_{j,m,p,q}^* \exp(1 - \nu) I_{REG} \right], \end{aligned} \tag{29}$$

where

$$e_{j,m,p,q}^* = e_j \frac{2^\nu (-1)^{p+m+q} \Gamma(j+1)}{\Gamma(j+1-m)m!} \binom{-2\nu}{p} \binom{p-m}{q} \frac{1}{\left(\frac{m+q}{\nu} + 1\right)^\nu} \tag{30}$$

and $I_{REG} = \int_0^\infty \left(\frac{m+q}{\nu} + 1\right) g(x; \xi) [G(x; \xi)]^{\frac{m+q}{\nu}} dx$ is the Rényi entropy of the Exp-G distribution with power parameter $\frac{m+q}{\nu}$. Furthermore, for $\delta > 1$

$$I_R(\nu) = (1 - \nu)^{-1} \log \left[\sum_{j=0}^{\infty} h_j \int_0^\infty \frac{2^\nu g^\nu(x; \xi)}{(1 + \bar{G}(x; \xi))^{2\nu}} \left[\frac{G(x; \xi)}{1 + \bar{G}(x; \xi)} \right]^j dx \right].$$

By considering the following expansions

$$(1 + \bar{G}(x; \xi))^{-(2\nu+j)} = \sum_{m=0}^{\infty} (-1)^m \binom{-2\nu+j}{m} \bar{G}^m(x; \xi)$$

and

$$[\bar{G}(x; \xi)]^m = [1 - G(x; \xi)]^m = \sum_{p=0}^{\infty} (-1)^p \binom{m}{p} G^p(x; \xi),$$

we can write

$$\begin{aligned} I_R(\nu) &= (1 - \nu)^{-1} \log \left[\sum_{j,m,p=0}^{\infty} h_j 2^\nu (-1)^{p+m} \binom{-2\nu+j}{m} \binom{m}{p} \frac{1}{\left(\frac{p+j}{\nu} + 1\right)^\nu} \right. \\ &\times \left. \int_0^\infty \left(\frac{p+j}{\nu} + 1\right) g(x; \xi) [G(x; \xi)]^{\frac{p+j}{\nu}} dx \right] \\ &= (1 - \nu)^{-1} \log \left[\sum_{j,m,p=0}^{\infty} h_{j,m,p}^* \exp(1 - \nu) I_{REG} \right], \end{aligned} \tag{31}$$

where

$$h_{j,m,p}^* = h_j 2^\nu (-1)^{p+m} \binom{-2\nu+j}{m} \binom{m}{p} \frac{1}{\left(\frac{p+j}{\nu} + 1\right)^\nu} \tag{32}$$

and $I_{REG} = \int_0^\infty \left(\frac{p+j}{\nu} + 1\right) g(x; \xi) [G(x; \xi)]^{\frac{p+j}{\nu}} dx$ is the Rényi entropy of the Exp-G distribution with power parameter $\frac{p+j}{\nu}$.

2.6 Moments and Generating Functions

Let $X \sim MO - HL - G(\delta, \xi)$, then the r^{th} moment can be obtained from equations (14) and (16). For $\delta \in (0, 1)$,

$$E(X^r) = \sum_{j,m,l=0}^{\infty} w_{j,m,l}^* E(W_{l+j-k}^r),$$

where $w_{j,m,l}^*$ is as defined in equation (15) and $E(W_{l+j-k}^r)$ denotes the r^{th} moment of W_{l+j-k} which follows an Exp-G distribution with power parameter $(l + j - k)$. For $\delta > 1$

$$E(X^r) = \sum_{j,m,l=0}^{\infty} v_{j,m,l}^* E(W_{l+j}^r), \tag{33}$$

where $v_{j,m,l}^*$ is as defined in equation (17) and $E(W_{l+j}^r)$ denotes the r^{th} moment of W_{l+j} which follows an Exp-G distribution with power parameter $(l + j)$. The incomplete moments can be obtained as follows:

For $\delta \in (0, 1)$

$$I_X(t) = \int_0^t x^s f_{MO-HL-G}(x; \xi) dx = \sum_{j,m,l=0}^{\infty} w_{j,m,l}^* I_{l+j-k}(t),$$

where $I_{l+j-k}(t) = \int_0^t x^r g_{l+j-k}(x; \xi) dx$ and $w_{j,m,l}^*$ is as defined in equation (15). Also, For $\delta > 1$

$$I_X(t) = \int_0^t x^s f_{MO-HL-G}(x; \xi) dx = \sum_{j,m,l=0}^{\infty} v_{j,m,l}^* I_{l+j}(t),$$

where $I_{l+j}(t) = \int_0^t x^r g_{l+j}(x; \xi) dx$ and $v_{j,m,l}^*$ is as defined in equation (17). The moment generating function (mgf) of X is as follows:

For $\delta \in (0, 1)$

$$M_X(t) = \sum_{j,m,l=0}^{\infty} w_{j,m,l}^* E(e^{tW_{l+j-k}}),$$

where $E(e^{tW_{l+j-k}})$ is the mgf of the Exp-G distribution with power parameter $(l + j - k)$ and $w_{j,m,l}^*$ is as defined in equation (15). For $\delta > 1$

$$M_X(t) = \sum_{j,m,l=0}^{\infty} v_{j,m,l}^* E(e^{tW_{l+j}}),$$

where $E(e^{tW_{l+j}})$ is the mgf of the Exp-G distribution with power parameter $(l + j)$ and $v_{j,m,l}^*$ is as defined in equation (17).

Furthermore, we can get the characteristic function and is given by $\phi(t) = E(e^{itX})$, where $i = \sqrt{-1}$, for $\delta \in (0, 1)$

$$\phi(t) = \sum_{j,m,l=0}^{\infty} w_{j,m,l}^* \phi_{l+j-k}(t),$$

where $\phi_{l+j-k}(t)$ is the characteristic function of Exp-G distribution with power parameter $(l + j - k)$ and $w_{j,m,l}^*$ is as defined in equation (15). For $\delta > 1$

$$\phi(t) = \sum_{j,m,l=0}^{\infty} v_{j,m,l}^* \phi_{l+j}(t),$$

where $\phi_{l+j}(t)$ is the characteristic function of Exp-G distribution with power parameter $(l + j)$ and $v_{j,m,l}^*$ is as defined in equation (17).

3. Maximum Likelihood Estimation

If $X_i \sim MO - HL - W(\delta; \xi)$ with the parameter vector $\Delta = (\delta; \xi)^T$. The total log-likelihood $\ell = \ell(\Delta)$ from a random sample of size n is given by

$$\ell = n \log(2\delta) + \sum_{i=1}^n \log[g(x_i; \xi)] - 2 \sum_{i=1}^n \log[1 + \bar{G}(x_i; \xi)] + 2 \sum_{i=1}^n \log \left[1 - \bar{\delta} \left(1 - \frac{G(x_i; \xi)}{1 + \bar{G}(x_i; \xi)} \right) \right].$$

The elements of a score vector, $U = (\frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial \xi_k})$ are given by:

$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} - 2 \sum_{i=1}^n \frac{1 - \frac{G(x_i; \xi)}{1 + \bar{G}(x_i; \xi)}}{[1 - \bar{\delta}(1 - \frac{G(x_i; \xi)}{1 + \bar{G}(x_i; \xi)})]}$$

and

$$\begin{aligned} \frac{\partial \ell}{\partial \xi_k} &= \sum_{i=1}^n \frac{1}{g(x_i; \xi)} \frac{\partial g(x_i; \xi)}{\partial \xi_k} - 2 \sum_{i=1}^n \frac{1}{[1 + \bar{G}(x_i; \xi)]} \frac{\partial [1 + \bar{G}(x_i; \xi)]}{\partial \xi_k} \\ &- 2 \sum_{i=1}^n \frac{1}{[1 - \bar{\delta}(1 - \frac{G(x_i; \xi)}{1 + \bar{G}(x_i; \xi)})]} \frac{\partial [1 - \bar{\delta}(1 - \frac{G(x_i; \xi)}{1 + \bar{G}(x_i; \xi)})]}{\partial \xi_k}, \end{aligned}$$

respectively. These partial derivatives does not have closed form and can be solved using numerical methods such as Newton-Raphson procedure..

4. Some Special Cases

In this Section, we present some special cases of the MO-HL-G family of distributions. We considered cases when the baseline distributions are log-logistic, Weibull and normal distributions.

4.1 Marshall-Olkin-Half Logistic-Log-Logistic Distribution

Consider the log-logistic distribution as the baseline distribution. The log-logistic distribution has pdf and cdf given by $g(x) = cx^{c-1}(1+x^c)^{-2}$ and $G(x) = 1 - (1+x^c)^{-1}$, for $c > 0$, respectively. Therefore, the Marshall-Olkin-Half Logistic-Log-Logistic (MO-HL-LLo) distribution have the cdf and pdf given by

$$F_{MO-HL-LLo}(x; \delta, c) = \frac{1 - (1+x^c)^{-1}}{[1 + (1+x^c)^{-1}][1 - \bar{\delta}(1 - \frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}})]}$$

and

$$f_{MO-HL-LLo}(x; \delta, c) = \frac{2\delta cx^{c-1}(1+x^c)^{-2}}{[1 + (1+x^c)^{-1}]^2 [1 - \bar{\delta}(1 - \frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}})]^2},$$

respectively, for $\delta, c > 0$.

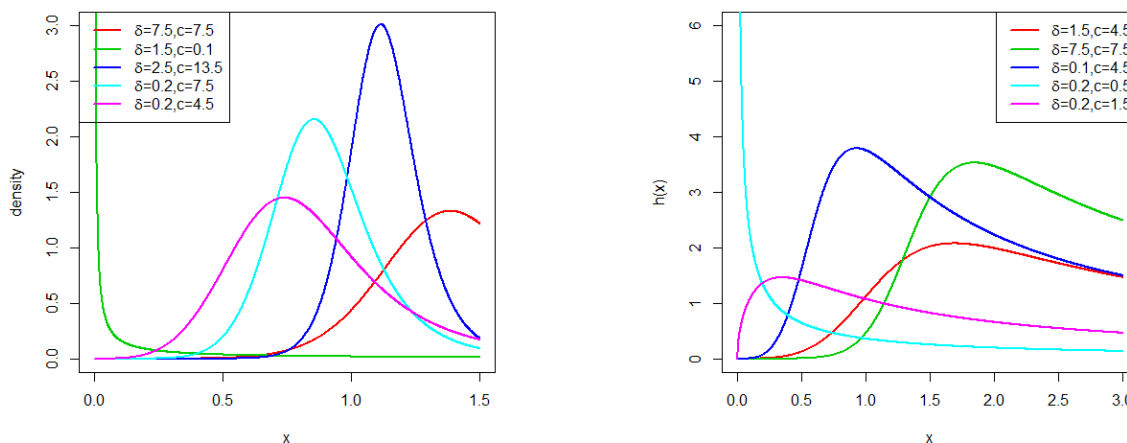


Figure 1. Plots of the pdf and hrf for the MO-HL-LLo distribution

Figure 1 shows the plots of the pdfs and hrfs of the MO-HL-LLo distribution for selected parameters values. The pdf can take various shapes including reverse-J, unimodal, left and right skewed. The MO-HL-LLo distribution exhibits both monotonic and non-monotonic hazard rate functions.

4.2 Marshall-Olkin Half Logistic-Weibull Distribution

Consider the Weibull distribution as the baseline distribution with pdf and cdf given by $g(x; \lambda, \omega) = \lambda\omega x^{\omega-1} e^{-\lambda x^\omega}$ and $G(x; \lambda, \omega) = 1 - e^{-\lambda x^\omega}$, respectively, for $\lambda, \omega > 0$. The cdf and pdf of the Marshall-Olkin-half-logistic-Weibull (MO-HL-W) distribution are given by

$$F_{MO-HL-W}(x; \delta, \lambda, \omega) = \frac{1 - e^{-\lambda x^\omega}}{(1 + e^{-\lambda x^\omega}) \left[1 - \bar{\delta} \left(1 - \frac{1 - e^{-\lambda x^\omega}}{1 + e^{-\lambda x^\omega}} \right) \right]}$$

and

$$f_{MO-HL-W}(x; \delta, \lambda, \omega) = \frac{2\delta\lambda\omega x^{\omega-1} e^{-\lambda x^\omega}}{(1 + e^{-\lambda x^\omega})^2 \left[1 - \bar{\delta} \left(1 - \frac{1 - e^{-\lambda x^\omega}}{1 + e^{-\lambda x^\omega}} \right) \right]^2},$$

respectively, for $\delta, \lambda, \omega > 0$.

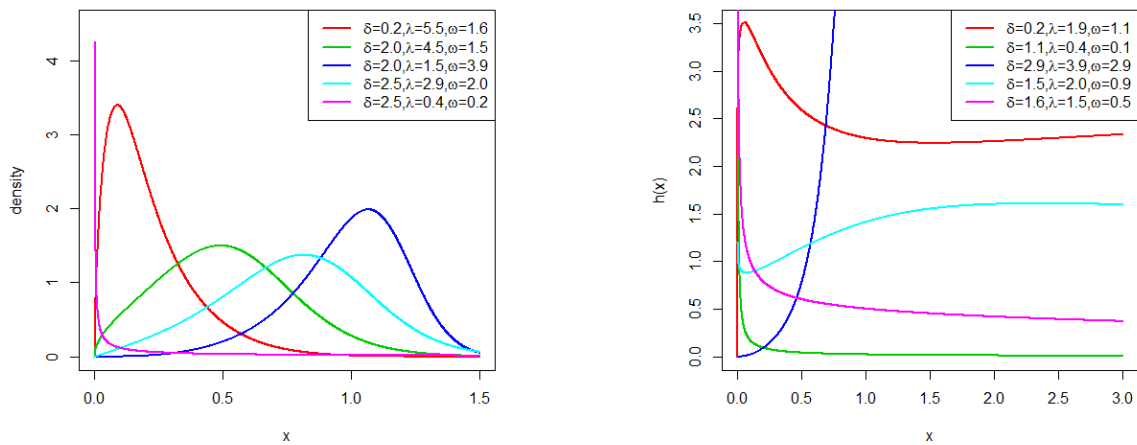


Figure 2. Plots of the pdf and hrf for the MO-HL-W distribution

The pdfs shown in Figure 2 of the MO-HL-W distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. Furthermore, the MO-HL-W distribution exhibit increasing, decreasing, reverse-J, bathtub and upside bathtub shapes for the hazard rate function.

4.3 Marshall-Olkin-Half Logistic-Normal Distribution

Consider the normal distribution with pdf $g(x; \mu, \sigma) = \sigma^{-1} \phi(\frac{x-\mu}{\sigma})$ and cdf $G(x; \mu, \sigma) = \Phi(\frac{x-\mu}{\sigma})$, for $\mu \in \mathfrak{R}$ and $\sigma > 0$, as the baseline distribution, we obtain the Marshall-Olkin-half-logistic-normal (MO-HL-N) distribution with cdf and pdf given by

$$F_{MO-HL-N}(x; \delta, \mu, \sigma) = \frac{\Phi(\frac{x-\mu}{\sigma})}{(2 - \Phi(\frac{x-\mu}{\sigma})) \left[1 - \bar{\delta} \left(1 - \frac{\Phi(\frac{x-\mu}{\sigma})}{2 - \Phi(\frac{x-\mu}{\sigma})} \right) \right]}$$

and

$$f_{MO-HL-N}(x; \delta, \mu, \sigma) = \frac{2\delta\sigma^{-1} \phi(\frac{x-\mu}{\sigma})}{(2 - \Phi(\frac{x-\mu}{\sigma}))^2 \left[1 - \bar{\delta} \left(1 - \frac{\Phi(\frac{x-\mu}{\sigma})}{2 - \Phi(\frac{x-\mu}{\sigma})} \right) \right]^2},$$

for $\delta, \sigma > 0$ and $-\infty < \mu < \infty$.

Figure 3 shows that the MO-HL-N distribution can take various shapes for its pdf. Also, the hazard rate function for the MO-HL-N distribution exhibit different shapes for the hazard rate functions for the MO-HL-N distribution.

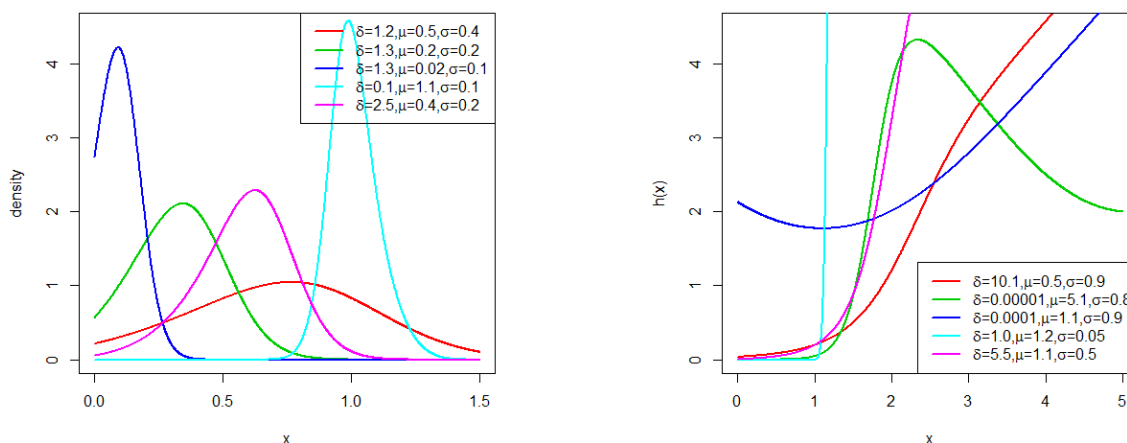


Figure 3. Plots of the pdf and hrf for the MO-HL-N distribution

5. Simulation Study

A simulation study is conducted to evaluate consistency of the maximum likelihood estimators. We simulated for N=1000 times with sample size $n= 50, 100, 200, 400, 800$ and 1000. Simulation results are shown in Table 1. From the Monte Carlo simulation results, we conclude that our model produces consistent results when estimating parameters for the model.

6. Applications

In this Section, three real data examples are used to illustrate the usefulness of the proposed new model compared to other competing known non-nested models. We use the following goodness-of-fit statistics: $-2\log$ likelihood ($-2 \log L$), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramér von Mises (W^*) and Andersen-Darling (A^*) as described by [Chen and Balakrishnan , 1995], Kolmogorov-Sminorv (K-S) and its p-value to assess the best fitting model. The model with the smaller values of these statistics, is deemed to be the better model.

We used the nlm function in R software to estimate the model parameters and the package **AdequacyModel** in **R** software for goodness-of-fit test. Model parameters estimates (standard errors in parenthesis) and the goodness-of-fit-statistics for the three data sets are shown in Tables 2, 3 and 4. Plots of the fitted densities, the histogram of the data and probability plots (Chambers, Cleveland, Kleiner and Tukey (1983)) are also presented to show how well our model fits the observed data sets compared to the selected non-nested models. The plots are shown in Figures 4 (a), 4 (b), 5 (a), 5 (b), 6 (a) and 6 (b).

We compared the MO-HL-W distribution with other competing three parameter non-nested models: the exponentiated-Fréchet (EFr) distribution by [Nadarajah and Kotz , 2003], other two non-nested studied by [Barreto-Souza et al. , 2013], namely, Marshall-Olkin extended Fréchet (MOEFr) and Marshall-Olkin extended generalized exponential (MOEGE) distributions, Marshall-Olkin extended inverse Weibull (IWMO) by [Pakungwati et al. , 2018], exponentiated Weibull by [Pal et al. , 2006] and alpha power Weibull (APW) by [Nassar et al. , 2018] distributions. The pdfs of the non-nested models are given by:

$$f_{IWMO}(x; \alpha, \theta, \lambda) = \frac{\alpha \lambda \theta^{-\lambda} x^{-\lambda-1} e^{-(\theta/x)^{-\lambda}}}{[\alpha - (\alpha - 1)e^{-(\theta/x)^{-\lambda}}]^2},$$

for $\alpha, \theta, \lambda > 0$,

$$f_{EF}(x; \alpha, \lambda, \delta) = \alpha \lambda \delta^\lambda [1 - e^{-(\delta/x)^\lambda}]^{\alpha-1} x^{-(1+\lambda)} e^{-(\lambda+1)(\delta/x)^\lambda},$$

for $\alpha, \lambda, \delta > 0$,

$$f_{MOEF}(x; \alpha, \lambda, \delta) = \frac{\alpha \lambda \delta^\lambda x^{-(\lambda+1)} e^{-(\delta/x)^\lambda}}{[1 - \bar{\alpha}(1 - e^{-(\delta/x)^\lambda})]^2},$$

for $\alpha, \lambda, \delta > 0$,

$$f_{MOEGE}(x; \alpha, \gamma, \lambda) = \frac{\alpha \gamma \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\gamma-1}}{(1 - \bar{\alpha}[1 - e^{-\lambda x}]^\gamma)^2},$$

Table 1. Monte Carlo Simulation Results for MO-HL-W Distribution: Mean, RMSE and Average Bias

parameter	n	$\delta = 0.3, \lambda = 0.02, \omega = 0.3$			$\delta = 0.04, \lambda = 0.04, \omega = 0.7$		
		Mean	RMSE	Bias	Mean	RMSE	Bias
δ	50	1.762498	9.685856	1.462498	0.610298	6.655783	0.570298
	100	0.844508	3.890994	0.544508	0.129558	1.395083	0.089558
	200	0.699785	4.770718	0.399785	0.063553	0.049943	0.023553
	400	0.342086	0.147265	0.042086	0.049088	0.023075	0.009088
	800	0.320848	0.090219	0.020848	0.046309	0.017197	0.006309
	1000	0.314583	0.069455	0.014583	0.044346	0.013374	0.004346
λ	50	0.079121	0.229997	0.059121	0.130856	0.350873	0.090856
	100	0.048857	0.138366	0.028857	0.078191	0.148263	0.038191
	200	0.035782	0.126931	0.015782	0.061161	0.045279	0.021161
	400	0.023250	0.013159	0.003249	0.048323	0.022022	0.008323
	800	0.021462	0.008276	0.001462	0.046057	0.016502	0.006057
	1000	0.020888	0.006446	0.000888	0.044062	0.013070	0.004062
ω	50	0.280205	0.060672	-0.019795	0.672670	0.106492	-0.027330
	100	0.285469	0.044548	-0.014531	0.685898	0.076765	-0.014102
	200	0.291384	0.033146	-0.008616	0.686340	0.055460	-0.013660
	400	0.296389	0.021142	-0.003611	0.694657	0.036994	-0.005343
	800	0.298494	0.015789	-0.001506	0.694497	0.026790	-0.005503
	1000	0.298659	0.012699	-0.001341	0.697050	0.023444	-0.002950
δ	50	0.081044	0.100232	0.041044	0.406615	4.491051	0.356615
	100	0.309052	5.419012	0.269052	0.169623	1.752614	0.119623
	200	0.049754	0.034046	0.009754	0.072667	0.053849	0.022667
	400	0.040278	0.017388	0.000278	0.059569	0.026896	0.009569
	800	0.034577	0.008367	-0.005423	0.055151	0.019439	0.005151
	1000	0.035456	0.007580	-0.004544	0.053182	0.015412	0.003182
λ	50	1.122168	0.849914	0.422168	0.127529	0.280013	0.077529
	100	0.950882	0.645455	0.250882	0.089910	0.175096	0.039910
	200	0.813907	0.396657	0.113907	0.069754	0.047828	0.019754
	400	0.709154	0.229924	0.009154	0.058419	0.025297	0.008419
	800	0.645285	0.098588	-0.054715	0.054782	0.018518	0.004782
	1000	0.668169	0.083547	-0.031831	0.052771	0.014996	0.002771
ω	50	0.712711	0.092397	0.012711	0.583070	0.088722	-0.016930
	100	0.713895	0.078494	0.013895	0.588429	0.066197	-0.011571
	200	0.714713	0.054730	0.014713	0.590984	0.047204	-0.009016
	400	0.719261	0.040985	0.019261	0.595801	0.032625	-0.004199
	800	0.721728	0.036012	0.021728	0.597247	0.023541	-0.002753
	1000	0.723451	0.034163	0.023451	0.599234	0.020848	-0.000766

for $\alpha, \gamma, \lambda > 0$,

$$f_{EW}(x; \alpha, \beta, \delta) = \alpha\beta\delta x^{\beta-1} e^{-\alpha x^\beta} (1 - e^{-\alpha x^\beta})^\delta,$$

for $\alpha, \beta, \delta > 0$ and

$$f_{APW}(x; \alpha, \beta, \theta) = \frac{\log(\alpha)}{(\alpha - 1)} \beta \theta x^{\beta-1} e^{-\theta x^\beta} \alpha^{1-e^{-\theta x^\beta}}$$

for $\alpha, \beta, \theta > 0$.

6.1 Strengths of 1.5 cm Glass Fibres Data

The first data set is on strengths of 1.5 cm glass fibres. The data set was also analyzed by [Bourguignon et al. , 2014] and [Smith and Naylor , 1987]. The data are 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

Table 2. Parameter estimates and goodness-of-fit statistics for various models fitted for 1.5 cm glass fibres data set

Model	Estimates			Statistics								
	δ	λ	ω	$-2 \log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value	
MO-HL-W	8.3217 (10.3885)	0.6948 (0.5651)	3.2018 (0.9478)	24.1	30.1	30.5	36.5	0.1057	0.5912	0.0999	0.5547	
MOEGE	α 1.3296 $\times 10^{-3}$ (0.3306)	γ 10.7200 (0.1849)	λ 5.8366 (0.2071)	31.9	37.9	38.4	44.4	0.4410	2.4309	0.9995	$< 2.200 \times 10^{-16}$	
EFr	α 0.04621 (0.0153)	δ 0.4993 (0.0147)	λ 20.1145 (6.1482)	189.1	195.1	195.5	201.6	1.1986	6.3098	0.4279	1.9210×10^{-10}	
MOEFr	α 54074 (3.8277×10^{-8})	δ 0.3858 (6.0532×10^{-2})	λ 7.9253 (0.8731)	45.6	51.6	51.9	58.0	19.2509	122.7666	0.9997	$< 2.200 \times 10^{-16}$	
EW	α 1.8741 (0.5608)	β 1.3803 (0.3216)	δ 15.3624 (9.0935)	-	64.9	70.9	71.3	77.3	0.7088	3.8788	0.2487	0.0008
IWMO	α 52636 (9.7035)	λ 7.9256 (0.1041)	θ 2.5828 (54.9189)	45.6	51.6	51.9	58.0	0.4974	2.7509	0.1536	0.1020	
APW	α 10.8558 (12.7241)	β 4.4836 (0.7632)	θ 0.1948 (0.1083)	26.9	32.9	33.4	39.4	0.1686	0.9272	0.1225	0.3009	

The estimated variance-covariance matrix for MO-HL-W model on 1.5 cm glass fibres data set is given by

$$\begin{bmatrix} 107.9206 & 5.6728 & -9.0626 \\ 5.6728 & 0.3193 & -0.5279 \\ -9.0626 & -0.5279 & 0.8983 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [8.3217 \pm 20.3614]$, $\lambda \in [0.6948 \pm 1.1076]$ and $\omega \in [3.2018 \pm 1.8577]$. Based on the results shown in Table 2 and the plots shown in Figures 4 (a) and 4 (b), we can conclude that the new proposed model performs better than the selected non-nested models on strength of 1.5 cm glass fibres.

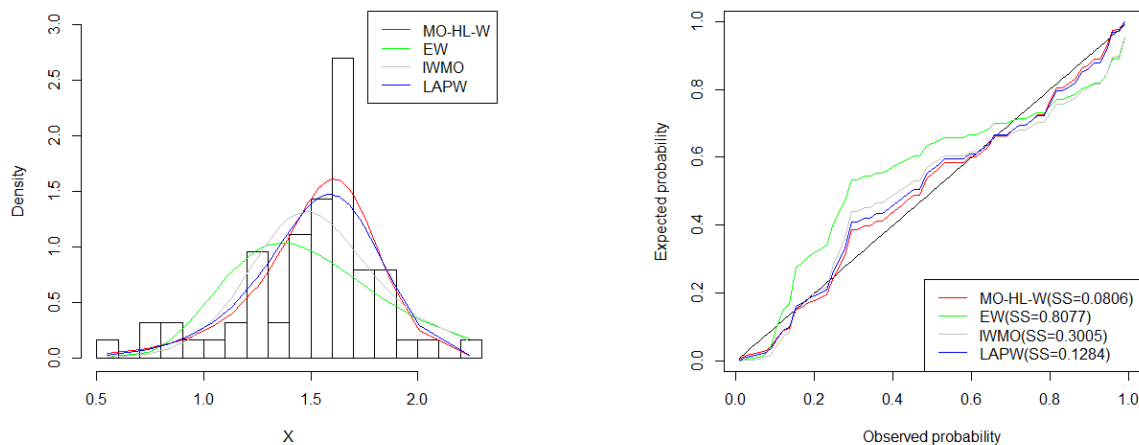


Figure 4. Fitted densities and probability plots for glass fibres data

6.2 Silicon Nitride Data

The second data set is on fracture toughness of silicon nitride measured in $\text{MPa } m^{1/2}$. The data set was also analyzed by [Nadarajah and Kotz, 2007] and also by [Ali et al., 2015]. The data are 5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70, 3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50, 4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80, 5.85, 5.90, 5.75, 6.25, 6.05, 5.90,

3.60, 4.10, 4.50, 5.30, 4.85, 5.30, 5.45, 5.10, 5.30, 5.20, 5.30, 5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60, 3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40, 3.70, 4.40, 4.90, 4.90, 5.00.

The estimated variance-covariance matrix for MO-HL-W model on silicon nitride data set is given by

$$\begin{bmatrix} 21.4266 & 0.1708 & -6.2857 \\ 0.1708 & 0.0013 & -0.0516 \\ -6.2857 & -0.0516 & 1.9136 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [2.8087 \pm 9.0726]$, $\lambda \in [0.0134 \pm 0.0732]$ and $\omega \in [3.3255 \pm 2.7114]$. Results in Table 3 show that the MO-HL-W distribution performs better than the various non-nested models on silicon nitride data. We can also deduce from the fitted densities and probability plots that the MO-HL-W distribution fits the silicon nitride data better than the selected non-nested models.

Table 3. Parameter estimates and goodness-of-fit statistics for various models fitted for silicon nitride data set

Model	Estimates			Statistics							
	δ	λ	ω	$-2 \log L$	<i>AIC</i>	<i>AICC</i>	<i>BIC</i>	W^*	A^*	K-S	p-value
MO-HL-W	2.8087 (4.6289)	0.0134 (0.0374)	3.3255 (1.3834)	335.6	341.6	341.8	349.9	0.0506	0.3047	0.0523	0.9009
MOEGE	1.0718×10^{-2} (7.0715×10^{-3})	20.7600 (3.5685×10^{-5})	1.7321 (0.1434)	340.3	346.3	346.5	354.6	0.2326	1.4637	0.9926	$< 2.2000 \times 10^{-16}$
EFr	0.0555 (0.0244)	1.5719 (0.0359)	18.4091 (7.7522)	586.9	592.9	593.2	601.3	1.3837	7.6851	0.3861	7.772×10^{-16}
MOEFr	2407.7 (7.7867×10^{-6})	1.4344 (0.1296)	7.0579 (0.5495)	356.6	362.6	362.9	370.9	38.2495	235.4782	0.9989	$< 2.2000 \times 10^{-16}$
EW	α 0.7015 (0.2973)	β 1.1441 (0.2012)	δ 23.9388 (12.5289)	- 381.5 -	387.5	387.7	395.8	0.6594	3.8924	0.1676	0.0025
IWMO	α 2407.7 (8.9422×10^{-7})	λ 7.0579 (0.5495)	θ 0.6972 (0.0630)	356.6	362.6	362.9	370.9	0.3596	2.2543	0.0804	0.4241
APW	α 7.4246 (0.0201)	β 3.8975 (0.2965)	θ 0.0039 (0.0019)	335.8	341.8	342.0	350.2	0.0571	0.3591	0.0560	0.8492

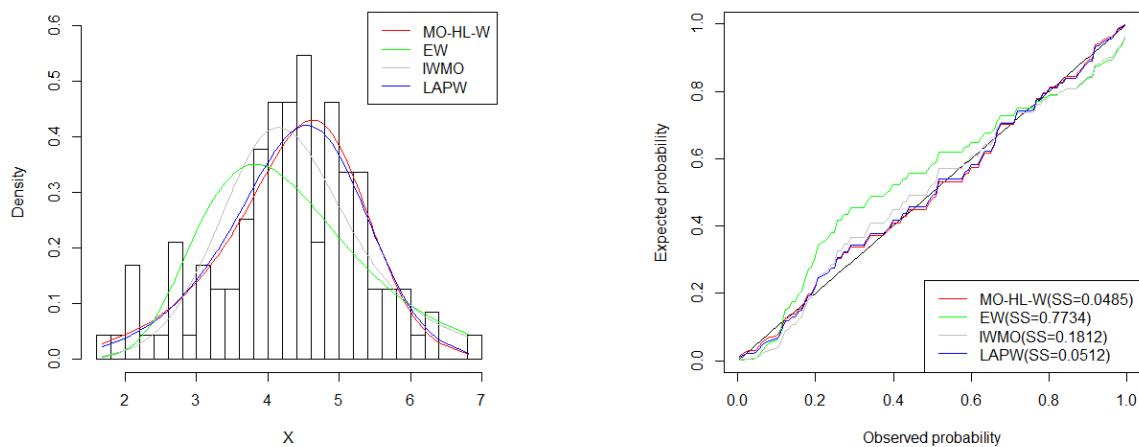


Figure 5. Fitted densities and probability plots for silicon nitride data

6.3 Turbocharger Failure Times Data

The third data set represents failure times ($10^3 h$) of turbocharger of one type of engine as report by [Xu et al. , 2003]. The data are 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

Table 4. Parameters estimates and goodness-of-fit statistics for various models fitted for turbocharger failure times data set

Model	Estimates			Statistics							
	δ	λ	ω	$-2 \log L$	<i>AIC</i>	<i>AICC</i>	<i>BIC</i>	<i>W*</i>	<i>A*</i>	K-S	p-value
MO-HL-W	2.4294 (2.8327)	0.0096 (0.0203)	2.7877 (0.8728)	162.6	168.6	169.3	173.7	0.0496	0.3766	0.0918	0.8889
MOEGE	α 0.116 (0.0280)	γ 3.1108 (0.6155)	λ 0.8755 (0.1246)	167.1	173.1	173.8	178.2	0.1183	0.8382	0.9932	$< 2.2000 \times 10^{-16}$
EF	α 0.0398 (0.0065)	δ 1.4314 (0.0414)	λ 18.2212 (0.6291)	253.7	259.7	260.3	264.7	0.6684	3.7808	0.4145	2.1430×10^{-6}
MOEF	α 4836.2 (1.1593×10^{-5})	δ 1.0782 (0.2678)	λ 4.8391 (0.6512)	177.4	183.4	184.1	188.5	12.8752	78.7089	0.9987	$< 2.2000 \times 10^{-16}$
EW	α 0.7549 (0.6721)	β 0.8389 (0.0400)	δ 17.9156 (22.6988)	- 187.4	193.4	194.1	198.5	0.3195	1.9997	0.1672	0.2134
IWMO	α 4836.3 (8.4139×10^{-6})	λ 4.8391 (0.6512)	θ 0.9274 (0.2303)	177.4	183.4	18401	188.5	0.2153	1.4129	0.1438	0.3796
LATW	α 4.3051 (0.0444)	β 3.3068 (0.4576)	θ 0.0024 (0.0022)	163.8	169.8	170.5	174.9	0.0621	0.4691	0.0986	0.8320

The estimated variance-covariance matrix for MO-HL-W model on turbocharger failure times data set is given by

$$\begin{bmatrix} 8.0241 & 0.0528 & -2.1838 \\ 0.0528 & 0.0004 & -0.0176 \\ -2.1838 & -0.0176 & 0.7617 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [2.4294 \pm 5.5521]$, $\lambda \in [0.0096 \pm 0.0399]$ and $\omega \in [2.7877 \pm 1.7106]$. Results in Table 4 and plots in Figures 6 (a) and 6 (b) show that the MO-HL-W distribution performs better than the various non-nested models on failure times of turbocharger data.

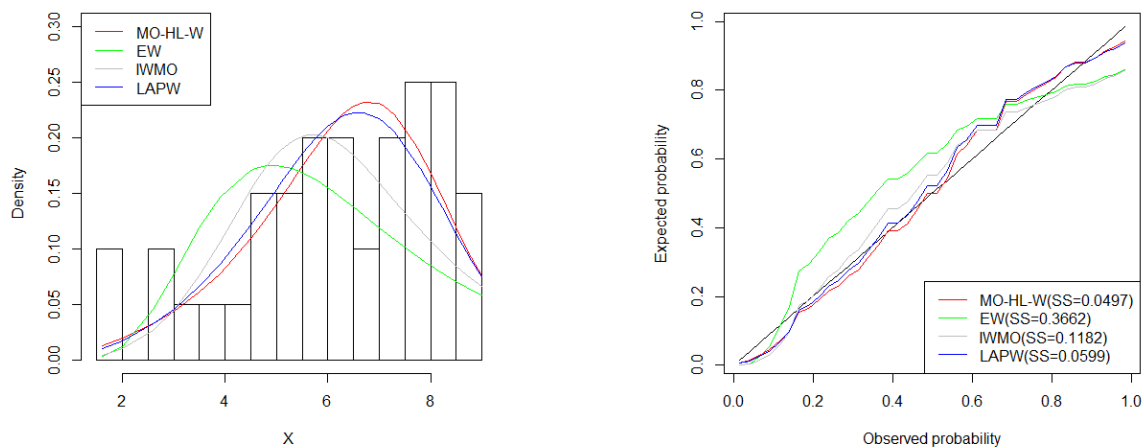


Figure 6. Fitted densities and probability plots for turbocharger failure times data

7. Concluding Remarks

A new model was developed, which is referred to as the Marshall-Olkin-Half Logistic-G family of distributions. The new distribution is a linear combination of the Exp-G distribution. The new proposed distribution can be applied to heavily skewed data and also data that have non-monotonic hazard rate shapes. From real data examples presented in Tables 2, 3 and 4, we conclude that the MO-HL-W distribution performs better than several competing non-nested models.

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