

Properties, Inference and Applications of Alpha Power Extended Inverted Weibull Distribution

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Abstract

In this work, we introduce a new generalization of the Inverted Weibull distribution called the alpha power Extended Inverted Weibull distribution using the alpha power transformation method. This approach adds an extra parameter to the baseline distribution. The statistical properties of this distribution including the mean, variance, coefficient of variation, quantile function, median, ordinary and incomplete moments, skewness, kurtosis, moment and moment generating functions, reliability analysis, Lorenz and Bonferroni and curves, Rényi of entropy and order statistics are studied. We consider the method of maximum likelihood for estimating the model parameters and the observed information matrix is derived. Simulation method and three real life data sets are presented to demonstrate the effectiveness of the new model.

Keywords: information matrix, Lorenz and Bonferroni and curves, order statistics, moments, maximum likelihood estimation

1. Introduction

Many studies have been made to introduce new generalized families of distributions. The common feature of these new distributions is that they have more parameters that induce flexibility into the new generated distribution which makes then adaptable to modeling various forms of real life data which failure rate may be increasing, decreasing, non-monotone, bathtub etc. such improved modeling capabilities can be found in the work of: Eugene et al. (2002) and Jones (2004), developed family of beta-generated distributions. Zografos and Balakrishnan (2009) proposed generalized gamma generated G family of distributions (type 1), Cordeiro and de Castro (2011) developed and studied new generated family of distributions based on the Kumaraswamy distribution, the log-gamma G family of distribution was developed by Amini et al. (2014). Torabi and Montazari (2014) proposed and studied the logistic-G family of distribution; Weibull-g family of distribution was developed by Bourguignon et al. (2014). Transformed-Transformer (T-X) family of distribution was studied by Alzaatreh et al. (2013). Exponentiated Transformed transformer was developed by Alzaghal (2013), Rezaei et al. (2017) proposed a new generated family of distributions based on Topp-Leone, Type 1 half logistic family of distribution was developed and studied Cordeiro et al. (2016). Hamedani et al. (2018) introduced a new extended G family of continuous distributions that can be viewed as a mixture representation of the exponentiated G densities.

This focus of this work is to develop another generalization of the Inverted Weibull distribution using Alpha Power Transformation (APT) as given by Mahdavi and Kundu. (2017) called alpha power Extended Inverted Weibull distribution. The method introduced an extra shape parameter into the baseline distribution (Inverted Weibull distribution), thereby incorporating skewness and improves the fit of the baseline distribution. The alpha power transformation is defined as follows:

Let $J(x)$ be the cdf of any continuous random variable X , then cdf of APT family is given as

$$F(x; \mathcal{J}) = \begin{cases} \frac{\alpha^{J(x)-1}}{\alpha-1}, & \text{if } \alpha > 0, \alpha \neq 1 \\ J(x), & \text{if } \alpha = 1 \end{cases} \quad (1)$$

And the associated pdf is given by

$$f(x; \mathcal{Z}) = \begin{cases} \frac{\log \alpha}{\alpha - 1} j(x) \alpha^{J(x)}, & \text{if } \alpha > 0, \alpha \neq 1 \\ j(x), & \text{if } \alpha = 1 \end{cases} \quad (2)$$

The transformation has been applied by different researchers to obtain alpha power transformed distributions. For example, Dey et al. (2017a, 2017b, 2018, 2019) investigated the properties of new extension of generalized exponential distribution with application to ozone data, a new extension of Weibull distribution with application to lifetime data, alpha power transformed inverse Lindley distribution with an upside-down bathtub-shaped hazard function and alpha-power transformed Lindley distribution with application to earthquake data, respectively. Hassan et al. (2018) studied the properties of Alpha power transformed extended exponential distribution, alpha power Weibull distribution was studied by Nassar et al. (2017), Ogunde et al. (2020) studied the properties of Alpha power extended Bur II distribution .

The inverted Weibull (IW) distribution has recently received attention and has been used in modeling life time data. Though it possesses a monotone failure rate and have been generalised to improve its fit so that it can be used to model data from all sphere of life. Some of such modifications include: Exponentiated Inverted Weibull (EIW) distribution by Flair et al. (2012). Hassan et al. (2014) proffer solution to the estimation of population parameters for the EIW distribution based on grouped data. In addition, Bayesian estimation procedures for the parameters of the EIW distribution were studied by Karam (2014). The properties of Transmuted Inverted Weibull distribution, Exponentiated transmuted inverted Weibull distribution by Ogunde et al. (2017a, 2017b).

A random variable X is said to have an Inverted Weibull (IW) distribution if its cdf is given by

$$J(x; c) = e^{-x^{-c}}, \quad x \geq 0, c > 0 \quad (3)$$

And the corresponding pdf is given by

$$j(x; c) = cx^{-c}e^{-x^{-c}}, x \geq 0, c > 0 \quad (4)$$

Where c is a shape parameter.

The focus of this study is to propose a new and more flexible distribution, which, we call Alpha Power Extended Inverted Weibull (APEIW) distribution, by introducing an additional parameter to the conventional Inverted Weibull distribution in other to improve its fit in modeling real life data.

2. Alpha Power Extended Inverted Weibull Distribution

Random variable X is said to have an APEIW distribution with parameters $\mathcal{Z}(\alpha, c)$, if its cdf is of the form

$$F(x; \mathcal{Z}) = \begin{cases} \frac{\alpha e^{-x^{-c}} - 1}{\alpha - 1}, & \text{if } \alpha > 0, \alpha \neq 1 \\ e^{-x^{-c}}, & \text{if } \alpha = 1 \end{cases} \quad (5)$$

And the associated pdf is given by

$$f(x; \mathcal{Z}) = \begin{cases} \frac{\log \alpha}{\alpha - 1} cx^{-c}e^{-x^{-c}} \alpha^{e^{-x^{-c}}}, & \text{if } \alpha > 0, \alpha \neq 1 \\ cx^{-c}e^{-x^{-c}}, & \text{if } \alpha = 1 \end{cases} \quad (6)$$

Where α and c are shape parameters.

Figure 1 drawn below shows the shape of the density function of the APEIW distribution from arbitrary values of the parameters. The graph shows that the pdf of APEIW is unimodal and right skewed with different degrees of kurtosis.

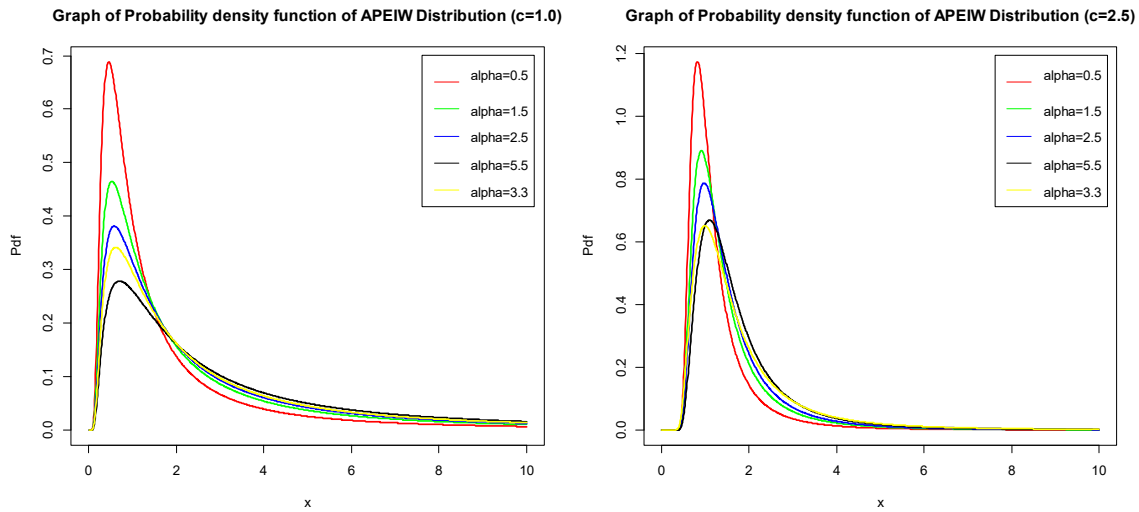


Figure 1. Graph of the pdf of APEIW distribution

The survival and the hazard rate function of the APEIW distribution are respectively given by:

$$S(x; \mathcal{Z}) = \begin{cases} \frac{\alpha}{\alpha-1} (1 - \alpha e^{-x^{-c}}), & \text{if } \alpha > 0, \alpha \neq 1, x > 0 \\ 1 - e^{-x^{-c}}, & \text{if } \alpha = 1, x > 0 \end{cases} \tag{7}$$

And

$$h(x; \mathcal{Z}) = \begin{cases} \frac{\log \alpha}{(1 - \alpha e^{-x^{-c}})} c x^{-c} e^{-x^{-c}} \alpha e^{-x^{-c}} - 1, & \text{if } \alpha > 0, \alpha \neq 1, x > 0 \\ c x^{-c} e^{-x^{-c}}, & \text{if } \alpha = 1, x > 0 \end{cases} \tag{8}$$

Figure 2 and figure 3 respectively are the graph of the survival function and hazard function of APEIW distribution for various values of the parameters. Figure 3 indicates that the hazard function of the APEIW distribution exhibits the upside down bathtub failure rate for the values of the parameters considered.

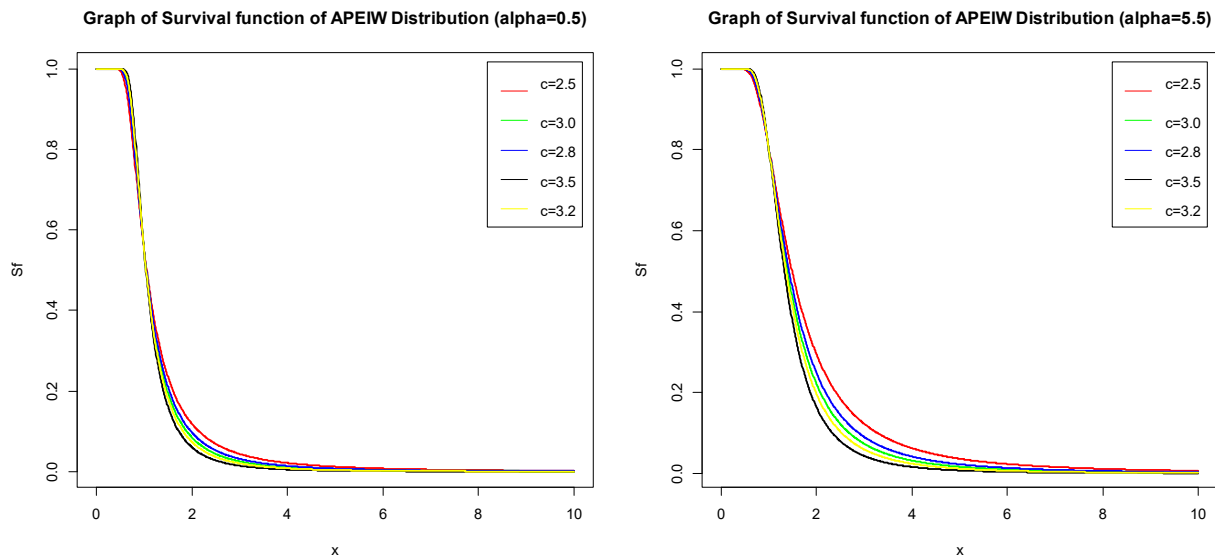


Figure 2. The Graph of the survival function of APEIW distribution

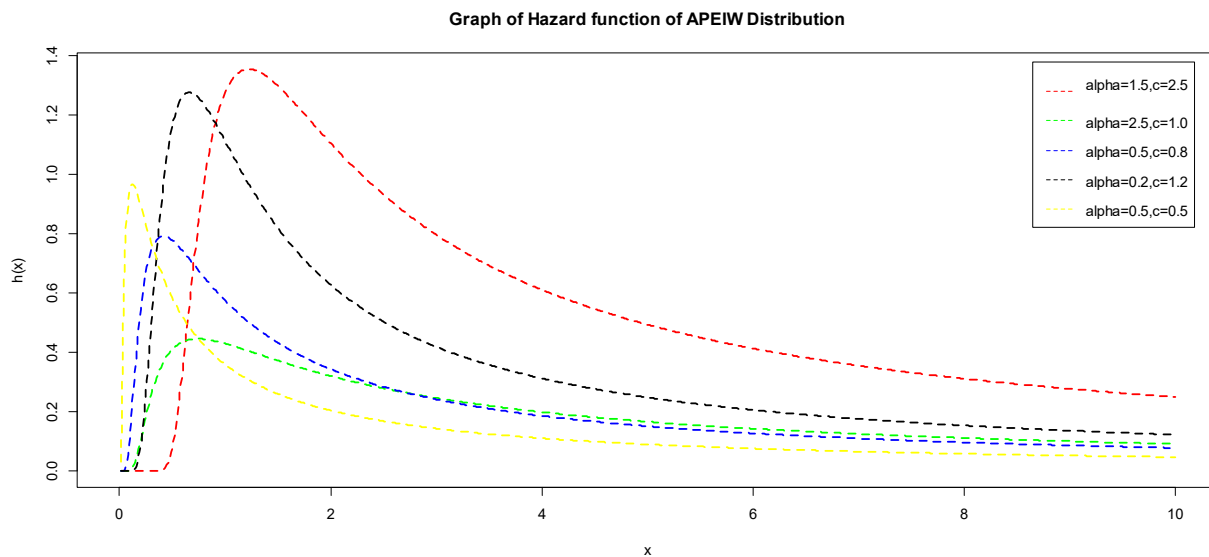


Figure 3. The Graph of the hazard function of APEIW distribution

2.1 Mixture Representation

The mixture representation of the density function plays an important role in deriving the statistical properties of generalised distribution. In this section the mixture representation of the APEIW density function is derived. Using the series representation

$$\alpha^v = \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} v^i, \tag{9}$$

And $\alpha \neq 1$, the density of the APEIW distribution can be expressed as:

$$f(x; \mathcal{Z}) = \frac{cx^{-c}}{\alpha-1} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} e^{-(i+1)x^{-c}} \tag{10}$$

Several properties of the APEIW distribution are investigated which includes: expressions for quantile function, median, moment generating function, stress-strength parameter, stochastic ordering, order statistics, mean residual life function and entropy. Proposition 3.1, 3.2, 3.3 and 3.4 contains expressions for moments, incomplete moment, Bonferroni curve and Lorenz curve respectively. The next section provides simulation studied and method of maximum likelihood estimation of parameters. Three real data applications are used to check the tractability of the proposed model. Conclusions are given in the last section.

3. Statistical Properties

This section focuses on the statistical properties of APEIW distribution. Here, we derived the statistical properties of APEIW distribution for only when $\alpha \neq 1$. Since for $\alpha = 1$, the properties is shifted to that of Inverted Weibull distribution.

3.1 Quantile Function

In statistical analysis, the quantile function is mostly used in simulating random variates from a statistical distribution. The quantile function of the APEIW distribution, say $x = Q_z$ is given by:

$$Q_z = \left\{ -\log \left[\frac{\log(1+z(\alpha-1))}{\log \alpha} \right] \right\}^{-\frac{1}{c}}, \quad 0 < z < 1 \tag{11}$$

The median can be obtained by putting $z = \frac{1}{2}$, into equation (11). Hence we have

$$Q_{\frac{1}{2}} = \left\{ -\log \left[\frac{\log\left(1+\frac{1}{2}(\alpha-1)\right)}{\log \alpha} \right] \right\}^{-\frac{1}{c}} \tag{12}$$

Expression for the lower quartile and the upper quartile can be obtained by taking $z = \frac{1}{4}$ and $z = \frac{3}{4}$ respectively.

When a distribution is heavy tailed, the classical measures of kurtosis and skewness may be difficult to obtain due to nonexistence of higher moment. In such situation the Bowley's measures of skewness (Kenney and Keeping 1962) can be considered because it is based on quartiles. It is given by

$$\mathfrak{B} = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}}$$

Consequently, the coefficient of kurtosis can be computed using the Moor's coefficient (\mathcal{M}) of kurtosis (Moors 1998) is based on octiles can be employed. It is given by

$$\mathcal{M} = \frac{Q_{0.875} - Q_{0.625} - Q_{0.375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}}$$

Table 1 drawn below gives various values of Bowley's of skewness and Moor's coefficient of kurtosis for various arbitrary values of the parameters for APEIW distribution..

Table 1. Values of coefficient of \mathfrak{B} and \mathcal{M} for various parameters value

α	c	0.25	0.5	0.75	0.125	0.625	0.875	\mathfrak{B}	\mathcal{M}
1.5	0.2	0.0055	0.0014	0.0003	6.33e-7	0.0029	0.0102	-0.5535	-1.2529
	1.2	0.2501	0.3362	0.4212	0.1997	0.3782	0.4660	-0.0058	-0.0446
2.0	0.2	0.0210	0.2669	2.4272	0.0036	0.8124	7.4197	0.7957	2.7137
	1.2	0.5256	0.8024	1.1593	0.3910	0.9660	1.3966	0.1263	0.2574
2.5	0.2	0.3292	9.7971	357.93	0.0381	53.495	3787.55	0.9470	10.436
	1.2	0.8310	1.4628	2.6646	0.5801	1.9411	3.9481	0.3108	0.8046

It can be observed from Table 1 that as the values of α increase the value of \mathfrak{B} and \mathcal{M} also increase but when the value of c increase keeping the value of α constant value of \mathfrak{B} and \mathcal{M} decreases. Also, we can conclude that the APEIW distribution can be used to model data that is positively or negatively skewed with varying degree of kurtosis (leptokurtic, mesokurtic and kleptokurtic)

3.2 Moments

Moment of a distribution plays a very important role in statistical analysis. They are used for estimating features and characteristics of a distribution such as skewness, kurtosis, measures of central tendency and measures of dispersion.

Proposition 3.1. If $X \sim APEIW(\mathcal{Z})$, where $\mathcal{Z} = \{\alpha, c\}$, then the k^{th} non-central moment of X is given by

$$E(X^k) = \mu'_k = \left(\frac{1}{\alpha - 1}\right) \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{k!} (i + 1)^{\frac{k-c}{c}} \Gamma\left(1 - \frac{k}{c}\right), \quad k < c$$

Proof.: By definition, the k^{th} non-central moment is given by

$$\begin{aligned} \mu'_k &= \int_{-\infty}^{\infty} x^k f(x; \mathcal{Z}) dx \\ &= \int_{-\infty}^{\infty} x^k \frac{cx^{-c}}{\alpha - 1} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} e^{-(i+1)x^{-c}} dx \\ &= \frac{c}{\alpha - 1} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} \int_{-\infty}^{\infty} x^{k-c} e^{-(i+1)x^{-c}} dx \end{aligned}$$

Taking $y = (i + 1)x^{-c}$, then we have

$$= \left(\frac{1}{\alpha - 1}\right) \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{k!} (i + 1)^{\frac{k-c}{c}} \int_0^{\infty} x^{\frac{k}{c}} e^{-y} dy$$

$$E(X^k) = \mu'_k = \left(\frac{1}{\alpha - 1}\right) \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{k!} (i + 1)^{\frac{k-c}{c}} \Gamma\left(1 - \frac{k}{c}\right), \quad k < c \tag{13}$$

For $k = 1, 2, \dots$, $\Gamma(\cdot)$ is the gamma function. Table 2 drawn below gives the first four moments, variance(δ^2), Coefficient of Variation(CV). The values for δ^2 and CV , are respectively given by

$$\delta^2 = (\mu'_2 - \mu^2)^{\frac{1}{2}}$$

$$CV = \frac{\delta}{\mu} = \left(\frac{\mu'_2}{\mu^2} - 1\right)^{\frac{1}{2}}$$

Table 2. First four moments, δ^2 and CV

α	c	μ'_1	μ'_2	μ'_3	μ'_4	δ^2	CV
1.5	4.5	1.2311	1.7215	3.0121	10.0816	0.2059	0.3686
	8.5	1.1022	1.2505	1.4706	1.8118	0.0357	0.1714
	15.5	1.0513	1.1143	1.1914	1.2864	0.0091	0.0907
3.0	4.5	1.3036	1.9417	3.6406	13.1397	0.2423	0.3776
	8.5	1.1353	1.3288	1.6150	2.0597	0.0310	0.1551
	15.5	1.0684	1.1511	1.2518	1.3756	0.0096	0.0917
10.0	4.5	1.4298	2.3450	4.8552	19.3736	0.3007	0.3836
	8.5	1.1915	1.4657	1.8738	2.5163	0.0460	0.1800
	15.5	1.0969	1.2139	1.3564	1.5320	0.0107	0.0943

From Table 2, we observed that as the values of α and c increases, the values of mean, variance and coefficient of correlation increases. whereas, increasing the value c and keeping α constant will decrease their value.

3.3 Moment Generating Functions

Moment Generating Functions (MGF): These are special functions that are used to obtain the moments and its functions such as: mean and variance of a random variable in a simpler way. proposition 3.2. If $X \sim APEIW(\mathcal{Z})$, where $\mathcal{Z} = \{\alpha, c\}$, then the MGF of X is given by

Proof: By definition, the MGF is given by

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} f(x; \mathcal{Z}) dx \tag{14}$$

Using the series expansion e^{tx} gives

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^{\infty} x^k f(x) dx = \sum_{k=0}^{\infty} \frac{t^k \mu'_k}{k!} \tag{15}$$

Substituting μ'_k into equation (13), yields

$$M_X(t) = \left(\frac{1}{\alpha - 1}\right) \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i! k!} (i + 1)^{\frac{k-c}{c}} t^k \Gamma\left(1 - \frac{k}{c}\right), \quad k < c \tag{16}$$

3.4 Incomplete Moment

The incomplete moment is used to estimate the median deviation, mean deviation and measures of inequalities such as the Lorenz and Bonferroni curves.

Proposition 3.2: The k^{th} incomplete moment of the APEIW distribution is given by

$$\gamma_k(t) = \left(\frac{1}{\alpha - 1}\right) \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{k!} (i + 1)^{\frac{k-c}{c}} \left\{ \Gamma\left[\left(1 - \frac{k}{c}\right), t^{-c}(i + 1)\right] \right\}, \quad t > 0, k = 1, 2, \dots \tag{17}$$

Proof. By definition

$$\begin{aligned} \gamma_k(t) &= \int_0^t x^k f(x) dx \\ &= \int_0^t x^k \frac{cx^{-c}}{\alpha-1} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} e^{-(i+1)x^{-c}} dx \\ &= \frac{c}{\alpha-1} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} \int_0^t x^{k-c} e^{-(i+1)x^{-c}} dx \end{aligned} \tag{18}$$

Taking $y = (i + 1)x^{-c}$, then we have

$$= \left(\frac{1}{\alpha-1}\right) \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{k!} (i+1)^{\frac{k-c}{c}} \int_{(i+1)x^{-c}}^t x^{\frac{k}{c}} e^{-y} dy$$

Using the complementary incomplete gamma function, we have

$$\gamma_k(t) = \left(\frac{1}{\alpha-1}\right) \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{k!} (i+1)^{\frac{k-c}{c}} \left\{ \Gamma \left[\left(1 - \frac{k}{c}\right), t^{-c}(i+1) \right] \right\}, t > 0, k < c$$

Where $\Gamma(q, w) \int_w^{\infty} v^{q-1} e^{-v} dv$ is the complementary incomplete gamma function.

The mean deviation, $\uparrow_1(x)$ and median deviation, $\uparrow_2(x)$, can be calculated using the relationships $\uparrow_1(x) = 2\mu F(\mu) - 2\gamma_1(\mu)$ and $\uparrow_2(x) = \mu - 2\gamma_1(M)$. Where $\mu = E(X)$ and M is the median of the APEIW random variable. $\gamma_1(\mu)$ and $\gamma_1(M)$ are calculated using the first incomplete moment.

3.5 Inequality Measures

The Bonferroni and Lorenz curves are the most commonly used measures of income inequality of a given population and have various applications in, reliability, insurance, economics and medicine.

Proposition 3.3: The Bonferroni curve for the APEIW distribution is given by

$$\mathbb{B}_F(t) = \frac{1}{\mu(\alpha^{e^{-t^{-\beta}}}-1)} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{k!} (i+1)^{\frac{k-c}{c}} \left\{ \Gamma \left[\left(1 - \frac{k}{c}\right), t^{-c}(i+1) \right] \right\} \tag{19}$$

Proof: By definition

$$\mathbb{B}_F(t) = \frac{1}{\mu F(t)} \int_0^t x^k f(x; \mathcal{Z}) dx \tag{20}$$

$$\mathbb{B}_F(t) = \frac{1}{\mu(\alpha^{e^{-t^{-c}}}-1)} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{k!} (i+1)^{\frac{k-c}{c}} \left\{ \Gamma \left[\left(1 - \frac{k}{c}\right), t^{-c}(i+1) \right] \right\}$$

Proposition 3.4: The Lorenz curve for the APEIW distribution is given by

$$L_F(t) = \left(\frac{1}{\alpha-1}\right) \left(\frac{1}{\mu}\right) \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{k!} (i+1)^{\frac{k-c}{c}} \left\{ \Gamma \left[\left(1 - \frac{k}{c}\right), t^{-c}(i+1) \right] \right\}, t > 0, k < c \tag{21}$$

Proof: By definition

$$L_F(t) = \frac{1}{\mu} \int_0^t x^k f(x; \mathcal{Z}) dx \tag{22}$$

$$L_F(t) = \left(\frac{1}{\alpha-1}\right) \left(\frac{1}{\mu}\right) \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{k!} (i+1)^{\frac{k-c}{c}} \left\{ \Gamma \left[\left(1 - \frac{k}{c}\right), t^{-c}(i+1) \right] \right\}, t > 0, k < \beta$$

It should be noted that $\mu = \mu'_1$ equal the first about the origin and is obtained by taking $k = 1$ in equation (13).

3.6 Stress-Strength Parameter

If we let X_1 and X_2 be two continuous and independent random variables, where $X_1 \sim APEIW(\alpha_1, \beta)$ and $X_2 \sim APEIW(\alpha_2, \beta)$, then the stress strength parameter, say $\$,$ is defined as

$$\$ = \int_{-\infty}^{\infty} f_1(x)F_2(x)dx \tag{23}$$

Using the pdf and the cdf of APEIW in the expression above, the strength stress parameter $\$,$ can be obtained as

$$\$ = \frac{c}{(\alpha_1-1)(\alpha_2-1)} \sum_{i=0}^{\infty} \frac{(\log \alpha_1)^{i+1}}{i!} x^{-\beta} \int_1^{\infty} e^{-(i+1)x^{-\beta}} \left(\alpha_2 e^{-x^{-\beta}} - 1 \right) \tag{24}$$

Applying equation (9) in (24), we have

$$\$ = \frac{1}{(\alpha_1-1)(\alpha_2-1)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\log \alpha_1)^{i+1} (\log \alpha_2)^j}{i! j!} \left\{ \frac{j}{i+j+1} \right\} \tag{25}$$

3.7 Stochastic Ordering

Stochastic ordering is an important characteristic for assessing the comparative behaviour of continuous random variable. Suppose a distribution has likelihood ration (*lr*) ordering, then it possesses the same ordering in hazard rate (*hr*) and distribution (*st*). Also, it is established that if a family of distribution has likelihood ordering, then there exist a uniformly most powerful test, Shaked and Shanthikumar (2007).

Proposition (3.5): let $X_1 \sim APEIW(\alpha_1, c)$ and $X_2 \sim APEIW(\alpha_2, c)$ be two independent random variables. If $\alpha_1 < \alpha_2$, then

$$X_1 \leq_{lr} X_2 \quad \forall X$$

Proof: Likelihood ratio is given by

$$\frac{f_{X_1}(x)}{f_{X_2}(x)} = \frac{\log(\alpha_1)}{\log(\alpha_2)} \left(\frac{\alpha_2 - 1}{\alpha_1 - 1} \right) \left(\frac{\alpha_1}{\alpha_2} \right)^{e^{-x^{-c}}}$$

$$\frac{d}{dx} \left(\log \frac{f_{X_1}(x)}{f_{X_2}(x)} \right) = \frac{\log(\alpha_1)}{\log(\alpha_2)} c x^{-c-1} e^{-x^{-c}}, \text{ if } \alpha_1 < \alpha_2, \forall x > 0 \tag{26}$$

Hence, for

$$\alpha_1 < \alpha_2, X_1 \leq_{lr} X_2, \quad \forall x$$

It follows that

$$X_1 \leq_{lr} X_2 \implies X_1 \leq_{st} X_2$$

3.8 Order Statistics

Let X_1, X_2, \dots, X_n be a random sample of size n from APEIW distribution and let $Z_{j:n}$ denote the j^{th} order statistics, then the pdf of $X_{j:n}$ is given by

$$f_{j:n}(x) = \frac{n!}{(i-1)!(n-i)!} f_x(z)[F_x(z)]^{j-1}[1 - F_x(z)]^{n-j} \tag{27}$$

Substitute the pdf and the cdf of APEIW distribution in (27), we obtain the pdf of j^{th} order statistics for $z > 1$ as

$$f_{j:n}(z) = W \frac{cz^{-c}}{(\alpha-1)^n} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} e^{-(i+1)z^{-c}} [\alpha e^{-z^{-c}} - 1]^{j-1} [\alpha^{1-e^{-z^{-c}}} - 1]^{n-j} \tag{28}$$

By taking $j = 1$ in equation (28), we obtain first order statistics as

$$f(z_1) = W \frac{cz^{-c}}{(\alpha-1)^n} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\log \alpha)^{i+k+1}}{i!k!} (-1)^l \binom{k(n-1)}{l} e^{-(i+l+1)z^{-c}} \tag{29}$$

By putting $j = n$ in equation (28), we obtain the n^{th} order statistics as

$$f(z_n) = W \frac{cz^{-c}}{(\alpha-1)^n} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\log \alpha)^{i+k+1}}{i!k!} e^{-(i+1)z^{-c}} [\alpha e^{-z^{-c}} - 1]^{n-1} \tag{30}$$

Where

$$W = \frac{n!}{(j-1)!(n-j)!}$$

3.9 Mean Residual Life Function

Suppose that X is a continuous random variable with survival function given in equation (7), the mean residual life function is define as the additional useful life time that a component has until time t or additional useful lifetime until failure occur. The mean residual life function, say, $\gamma(t)$ is given by

$$\gamma(t) = \frac{1}{P(X > t)} \int_t^\infty P(X > x) dx, t \geq 0$$

This can further be simplified as

$$\gamma(t) = \frac{1}{S(t)} \left(\mu - \int_0^t xf(x) dx \right) - t, t \geq 0 \tag{31}$$

Where

$$\int_0^t xf(x) dx = \left(\frac{1}{\alpha-1} \right) \sum_{i=0}^\infty \frac{(\log \alpha)^{i+1}}{k!} (i+1)^{\frac{k-c}{c}} \left\{ \Gamma \left[\left(1 - \frac{k}{c} \right), t^{-c}(i+1) \right] \right\}, \tag{32}$$

Substituting equations (7) and (32) in equation (31), $\gamma(t)$ can be written as

$$\gamma(t) = \frac{\alpha-1}{\alpha(1-\alpha e^{-x^{-c}-1})} \left(\mu - \left(\frac{1}{\alpha-1} \right) \sum_{i=0}^\infty \frac{(\log \alpha)^{i+1}}{k!} (i+1)^{\frac{k-c}{c}} \left\{ \Gamma \left[\left(1 - \frac{k}{c} \right), t^{-c}(i+1) \right] \right\} \right) - t \tag{33}$$

It should be noted that $\mu = \mu'_1$ equal the first about the origin and is obtained by taking $k = 1$ in equation (13).

3.10 Entropy

Entropies is a measure of randomness of a system and has been used extensively in information theory. Two popular entropy measures are Renyi entropy (Neyman, 1961) and Shannon entropy (Shannon, 1948). A large value of the entropy indicates a greater uncertainty in the data. The Shannon entropy is a special case of the Renyi entropy when $z \rightarrow 1$ and is given by

$$E[-\log(f(x; \mathcal{Z}))].$$

Proposition 3.6: If the random variable X has a APEIW distribution, then the Renyi entropy of X is given by

$$E_R(z) = \frac{1}{1-z} \log \left[\frac{c^{z-1}}{(\alpha-1)^z} \left\{ \sum_{i=0}^\infty \frac{(\log \alpha)^i}{i!} \right\}^z \left\{ \frac{1}{((i+1)z)} \right\}^{\frac{z(c+1)+1}{c}} \Gamma \left\{ 1 + \frac{z(c+1)}{c}, (i+1)zx^{-c} \right\} \right] \tag{34}$$

Proof: By definition

$$E_R(z) = \frac{1}{1-z} \log \left\{ \int_0^\infty f^z(x) dx \right\}, z > 0, z \neq 0.$$

From equation (9),

$$f^z(x) = \frac{c^z}{(\alpha-1)^z} \left\{ \sum_{i=0}^\infty \frac{(\log \alpha)^i}{i!} \right\}^z x^{-zc} e^{-(i+1)zx^{-c}}$$

So

$$\int_0^\infty f^z(x) dx = \frac{c^z}{(\alpha-1)^z} \left\{ \sum_{i=0}^\infty \frac{(\log \alpha)^i}{i!} \right\}^z \int_0^\infty x^{-zc} e^{-(i+1)zx^{-c}} dx$$

Let

$$B_z = \int_0^\infty x^{-zc} e^{-(i+1)zx^{-c}} dx \text{ and } p = (i+1)zx^{-c}$$

Then $dx = -\frac{1}{c}\{(i+1)z\}^{\frac{1}{c}}p^{-\frac{1}{c}-1}dp$

Thus

$$B_z = \left\{ \frac{1}{(i+1)z} \right\}^{\frac{z(c+1)+1}{c}} \Gamma \left\{ 1 + \frac{z(c+1)}{c}, (i+1)zx^{-c} \right\}$$

Finally,

$$E_R(z) = \frac{1}{1-z} \log \left[\frac{c^{z-1}}{(\alpha-1)^z} \left\{ \sum_{i=0}^\infty \frac{(\log \alpha)^i}{i!} \right\}^z \right] \left\{ \frac{1}{(i+1)z} \right\}^{\frac{z(c+1)+1}{c}} \Gamma \left\{ 1 + \frac{z(c+1)}{c}, (i+1)zx^{-c} \right\}$$

4. Simulation Study

Simulation study was carried out for average MLEs, Absolute biase (AB) and Mean Square Error. We generate 1000 random sample X_1, X_2, \dots, X_n of samples of size $n = 10, 20, 50, 100, 150, 200, 250, \text{ and } 300$ were obtained from APEIW distribution. Random number generation was carried out using an expression for the quantile function of APEIW distribution as given in equation (11). The choice of parameters was taken to be $\alpha = 0.2$ and $c = 1.5$. The Table 3 drawn below summaries the results of simulation for the estimates of the poarameters of APEIW distribution. It can be observed that as the sample size increases the MSE approaches zero

Table 3. MSEs, AB and SE of APEIW distribution

	Parameter	Mean	AB	SE	MSE
n = 10	α	0.1997	0.0003	0.23356	0.0546
	c	0.7459	0.7541	0.1369	0.5870
n = 20	α	0.0963	0.1037	0.0847	0.0179
	c	0.8763	0.6238	0.1083	0.4008
n = 50	α	0.1684	0.0316	0.0891	0.0089
	c	0.9058	0.5942	0.0716	0.3582
n = 100	α	0.1400	0.0600	0.0532	0.0064
	c	0.8937	0.6063	0.0511	0.3702
n = 150	α	0.1350	0.0650	0.0420	0.0060
	c	0.9559	0.5440	0.0445	0.2980
n = 200	α	0.1515	0.0485	0.0404	0.0040
	c	0.9319	0.5681	0.0378	0.3242
n = 250	α	0.1563	0.0437	0.0372	0.0033
	c	0.9690	0.5310	0.0351	0.2852
n = 300	α	0.1644	0.0356	0.0355	0.0025
	c	0.9828	0.5172	0.0326	0.2685

5. Parameters Estimation

5.1 Maximum Likelihood Estimation

Let X_1, X_2, \dots, X_3 be a random sample from APEIW (α, c) then the likelihood function is given by

$$L(\underline{x}; \alpha, c) = \prod_{i=0}^n \left\{ \frac{\log \alpha}{\alpha-1} c x^{-c} e^{-x^{-c}} \alpha^{e^{-x^{-c}}} \right\} \tag{35}$$

Taking logarithm, equation (35) becomes

$$\text{Log}L(\underline{x}; \alpha, c) = n \log c + n \log \left(\frac{\log \alpha}{\alpha-1} \right) - \sum_{i=1}^n x_i^{-c} - c \sum_{i=1}^n \log(x_i) + \log(\alpha) \sum_{i=1}^n e^{-x_i^{-c}} \tag{36}$$

Differentiating equation (36) with respect to α and β and equating it to zero, we obtain the following normal equations:

$$\frac{\partial \text{LogL}(\underline{x}; \alpha, c)}{\partial \alpha} = \frac{n(\alpha - 1 - \alpha \log \alpha)}{\alpha(\alpha - 1) \log \alpha} + \frac{1}{\alpha} \sum_{i=1}^n e^{-x_i^{-c}} = 0 \tag{37}$$

$$\frac{\partial \text{LogL}(\underline{x}; \alpha, c)}{\partial c} = \frac{n}{c} + \sum_{i=1}^n x_i^{-c} \log(x_i) - \log(\alpha) \sum_{i=1}^n x_i^{-c} \log(x_i) e^{-x_i^{-c}} \tag{38}$$

And solving equation (37) and (38) simultaneously, MLE of α and c can be obtained. Standard algorithm such as Bisection method or Newton Rapheson method can be employed in solving the non-linear equations. It is confirm that the MLEs are asyptotically normally distributed i.e., $\sqrt{n}(\hat{\alpha} - \alpha, \hat{c} - c) \sim N_*(0, \Sigma)$ where Σ is variance covariance matrix and can be computed by inverting the Fisheer infpormation matrix F_m as given below

$$F_m = \begin{bmatrix} \frac{\partial^2 \text{LogL}(\underline{x}; \alpha, c)}{\partial \alpha^2} & \frac{\partial^2 \text{LogL}(\underline{x}; \alpha, c)}{\partial \alpha \partial c} \\ \frac{\partial^2 \text{LogL}(\underline{x}; \alpha, c)}{\partial c \partial \alpha} & \frac{\partial^2 \text{LogL}(\underline{x}; \alpha, c)}{\partial c^2} \end{bmatrix}$$

Taking deroivative of equation (37) and (38) w.r.t α and c , then we have

$$\frac{\partial^2 \text{LogL}(\underline{x}; \alpha, c)}{\partial \alpha^2} = \frac{n}{(\alpha - 1)^2} - \frac{n \log \alpha + n}{\alpha^2 \log^2 \alpha} - \frac{1}{\alpha^2} \sum_{i=1}^n e^{-x_i^{-c}} \tag{39}$$

$$\frac{\partial^2 \text{LogL}(\underline{x}; \alpha, c)}{\partial \alpha \partial c} = \frac{1}{\alpha} \sum_{i=1}^n x_i^{-c} \log x_i e^{-x_i^{-c}} \tag{40}$$

$$\frac{\partial^2 \text{LogL}(\underline{x}; \alpha, c)}{\partial c^2} = -\frac{n}{c} - \sum_{i=1}^n (\log x_i)^2 x_i^{-c} - \log \alpha \sum_{i=1}^n e^{-x_i^{-c}} (\log x_i)^2 x_i^{-c} (x_i^{-c} - 1) \tag{41}$$

Asymptotic $(1 - \omega)100\%$ confidence intervals for parameters can be obtained as

$$\hat{\alpha} \pm Z_{\omega/2} \sqrt{\Sigma_{11}}$$

$$\hat{c} \pm Z_{\omega/2} \sqrt{\Sigma_{22}}$$

Where Z_{ω} is the upper ω^{th} percentile of the normal distribution.

5.2 The Crammer-Von Mises Minimum Distance Estimators

Let x_1, x_2, \dots, x_n be a random sample from APEIW distribution and let $x_{(1)} < x_{(2)} < \dots < x_{(n)}$ be the corresponding order statistics. The CV estimator of a set of parameters $\mathcal{Z} = \{\alpha, c\}^T$ has been discribed as a minimum distance estimators that is based on the difference between the estimate of the cdf and the emprical cdf , D'Agostino and Stephens (1986) and Luceno (2006). The CV estimators are obtained by minimizing

$$C(\mathcal{Z}) = \frac{1}{12n} + \sum_{i=1}^n \left(\frac{\alpha e^{-x_i^{-c}} - 1}{\alpha - 1} - \frac{2i-1}{2n} \right)^2 \tag{42}$$

With respect to \mathcal{Z} .

MacDonald (1971) observed that the choice of CV method type with mnumum distance estimator that provides emprical evidence is that bestiamtor which the bias is smaller than the other minimum distance estimators.

5.3 Percentile Estimator (PE)

The PR method of estimation canb be obtained by muinimizing the set of parameters $\mathcal{Z} = \{\alpha, c\}^T$, are obtained by minimizing the following

$$P(\mathcal{Z}) = \sum_{i=1}^n \left\{ \ln(p_i) - \ln \left(\frac{\alpha e^{-x_i^{-c}} - 1}{\alpha - 1} \right) \right\}^2 \tag{43}$$

With respect \mathcal{Z} , where p_i denotes some estiamtes of APEIW($x_{(i)}; \mathcal{Z}$), and $p_i = \frac{1}{n+1}$

5.4 Applications

Three sets of life time data have been used to demonstrate the performance of the proposed model. The first data set is made up of failure time in hours of Kevlar 49/epoxy strands with pressure at 90% and was already studied by Andrews and Herzberg (2012). The data consists of 101 observations and the numbers are. While second data is obtained from Smith and Naylor (2017), and consists of the strength of 1.5cm glass fibers measured at the National Physical Laboratory, England. In the third data set we consider the data presented by Murthy et al. (2004) on the failure times (in weeks) of 50 components. The three data sets are recorded in Table (4) as given below.

Table 4. The failure times in hours of Kevlar 49/epoxy strands, strength of 1.5cm glass fibers and the failure time of components

Data 1	Data 2	Data 3
0.01, 0.01, 0.02, 0.02,0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09,0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20,0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42,0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68,0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85,0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10,1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40,1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58,1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17,2.33, 3.03, 3.03, 3.34, 4.20,4.69, 7.89	0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13,1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48,1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55,1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62,1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70,1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84,1.89, 2.00, 2.01, 2.24	0.013, 0.065,0.111, 0.111,0.163, 0.309, 0.426, 0.535, 0.684, 0.747, 0.997, 1.284, 1.304, 1.647, 1.829, 2.336, 2.838, 3.269, 3.977, 3.981, 4.520,4.789, 4.849, 5.202, 5.291, 5.349, 5.911, 6.018, 6.427, 6.456, 6.572, 7.023, 7.087, 7.291, 7.787, 8.596, 9.388, 10.261, 10.713, 11.658, 13.006, 13.388, 13.842, 17.152, 17.283, 19.418, 23.471, 24.777, 32.795, 48.105

The descriptive statistics of the two data sets are given in Table 5. From this table, the first and the third data sets are over-dispersed and second data sets are under-dispersed. The first and the third data are positively skewed and the data set II is negatively skewed. Moreover, the first and the third data set is platykurtic while the second data set is leptokurtic. This evidence is clearly supported in Figure 4. From the Total Test on Time (TTT) plot drawn in figure 5, the first data set exhibits the inverted Bathtub failure rate, the second data sets exhibits an increasing failure rate and the third data set exhibits a decreasing failure rate.

Table 5. Exploratory data Analysis of Failure data

	Data 1	Data 2	Data 3
Sample size	101	63	50
mean	1.025	1.507	7.821
Lower quartile	0.240	1.375	1.390
Upper quartile	1.45	1.685	10.040
Median	0.80	1.590	5.320
Variance	1.253	0.105	84.756
Skewness	3.047	-0.922	2.378
Kurtosis	14.475	1.103	7.229
Range	7.880	1.690	48.087

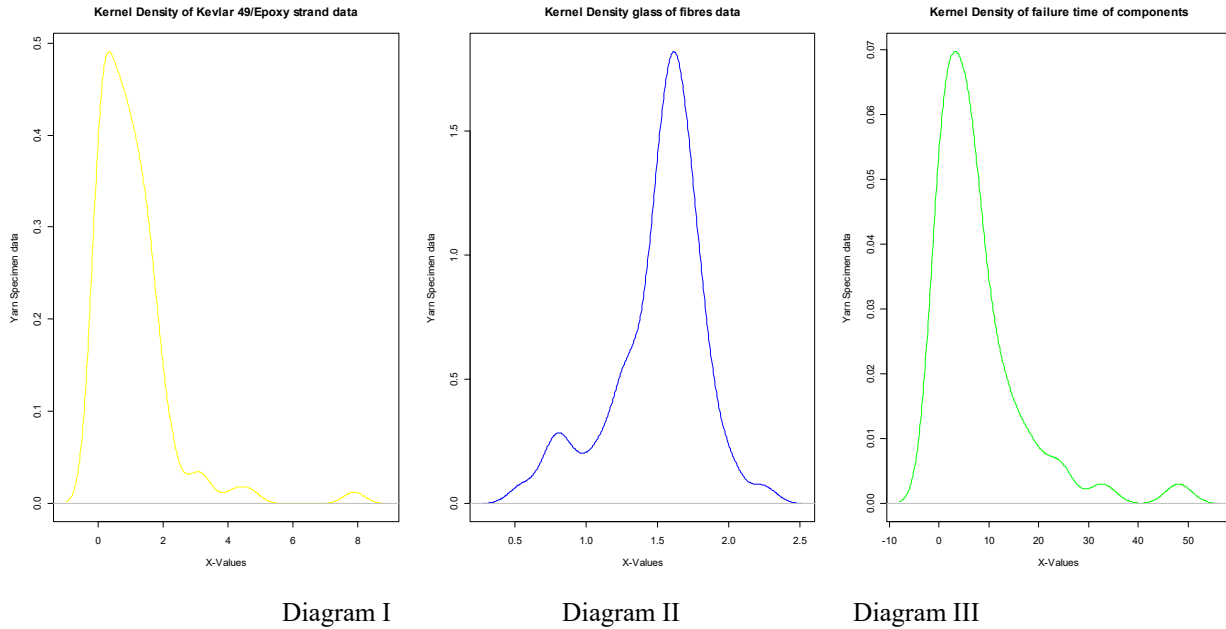


Figure 4. Kernel density Plot for the three failure data

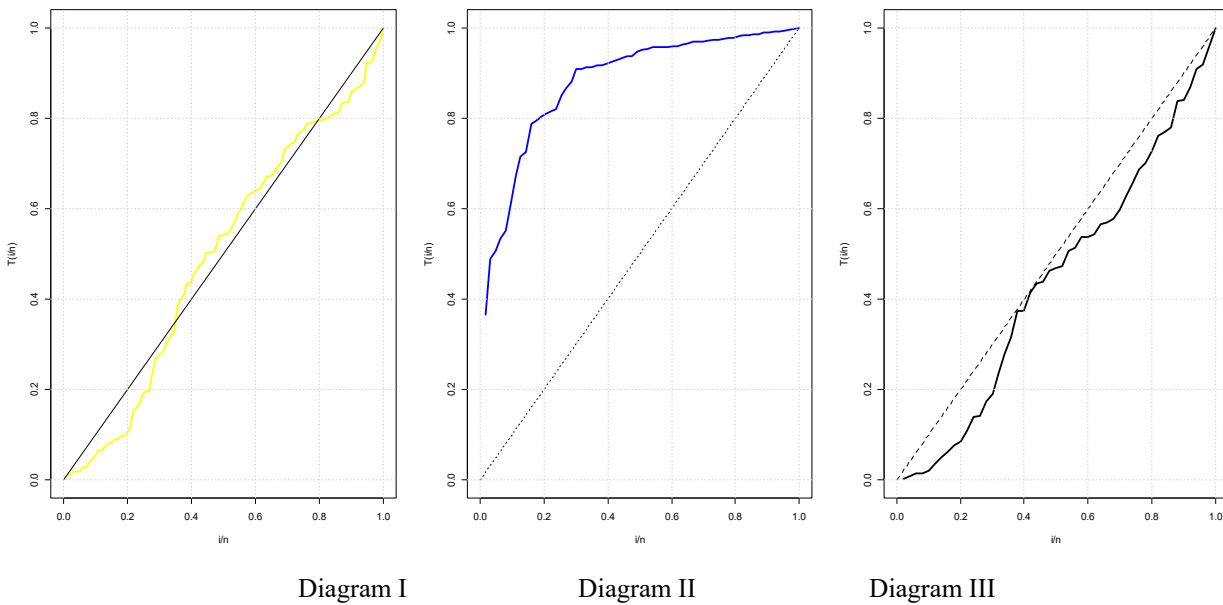


Figure 5. TTT Plot of the three failure data

The ML estimates along with their standard error (SE) of the model parameters are provided in Tables and. In the same tables, the analytical measures including; minus log-likelihood(-log L), Akaike information Criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Kolmogorov Smirnov (KS) test statistic are obtained for the model considered. The fit of the proposed APEIW distribution is compared with three other competitive models namely the conventional Inverted Weibull distribution, Alpha Power Exponential distribution, Alpha Power Inverse Rayleigh distribution and Alpha Power Inverse Exponential distribution, with the following pdfs.

- Alpha Power Exponential distribution

$$f(x) = \frac{\log \alpha}{\alpha - 1} b e^{-cx} \alpha^{e^{-cx}}, \quad x > 0, \alpha, c > 0$$

- Alpha Power Inverse Rayleigh distribution

$$f(x) = \frac{\log \alpha}{\alpha - 1} 2^c c x^{-c-1} e^{-2x^{-c}} \alpha^{e^{-2x^{-c}}}, \quad x > 0, \alpha, b > 0$$

- Alpha Power Inverse Exponential distribution

$$f(x) = \frac{\log \alpha}{\alpha - 1} c x^{-c-1} e^{-cx^{-1}} \alpha^{e^{-cx^{-1}}}, \quad x > 0, \alpha, c > 0$$

The distribution with the best fit to the data will be the one that possesses the smallest AIC, BIC, CAIC, and KS also, with the highest PV values.

Table 6. Analytical results of the APEIW model and other competing models for Kevlar 49/epoxy data

Model	ML Estimates (SE)	-Log L	AIC	BIC	CAIC	KS	PV
APEIW	$\hat{\alpha} = 0.06(0.03)$ $\hat{c} = 0.45 (0.02)$	131.11	266.53	271.76	266.65	0.204	0.0005
APIR	$\hat{\alpha} = 0.07(0.03)$ $\hat{c} = 0.37(0.02)$	151.26	275.39	317.75	312.64	0.015	1.2e-05
APIE	$\hat{\alpha} = 16.08 (5.59)$ $\hat{c} = 0.09 (0.01)$	135.70	371.44	280.62	275.52	0.282	2.07e-7
IW	$\hat{c} = 0.44 (0.07)$	158.03	318.06	320.68	318.01	0.441	2.2e-16

Table 7. Analytical results of the APEIW model and other competing models for glass fibres data.

Model	ML Estimates (SE)	-Log L	AIC	BIC	CAIC	KS	PV
APEIW	$\hat{\alpha} = 25.75(14.20)$ $\hat{c} = 3.75 (0.29)$	39.44	82.88	87.16	83.08	0.217	0.0054
APIR	$\hat{\alpha} = 3.14(1.50)$ $\hat{c} = 1.80(0.13)$	75.41	154.82	159.11	155.02	0.738	2.2e-16
APE	$\hat{\alpha} = 25.34(8.91)$ $\hat{c} = 1.26 (0.11)$	62.07	128.14	132.42	128.34	0.346	5.47e-07
IW	$\hat{c} = 2.81(0.27)$	58.48	118.96	121.11	119.03	0.436	7.98e-11

Table 8. Analytical results of the APEIW model and other competing models for components failure data

Model	ML Estimates (SE)	-Log L	AIC	BIC	CAIC	KS	PV
APEIW	$\hat{\alpha} = 2.18(1.14)$ $\hat{c} = 0.52 (0.05)$	167.83	339.66	343.48	339.92	0.196	0.0433
APIR	$\hat{\alpha} = 1.83(0.95)$ $\hat{c} = 0.47(0.04)$	168.93	341.85	345.68	342.11	0.288	0.0005
APIE	$\hat{\alpha} = 15.28(5.28)$ $\hat{c} = 0.27 (0.04)$	191.63	387.26	391.08	387.51	0.465	8.29e-10
IW	$\hat{c} = 2.81(0.27)$	58.48	118.96	121.11	119.03	0.436	7.98e-11

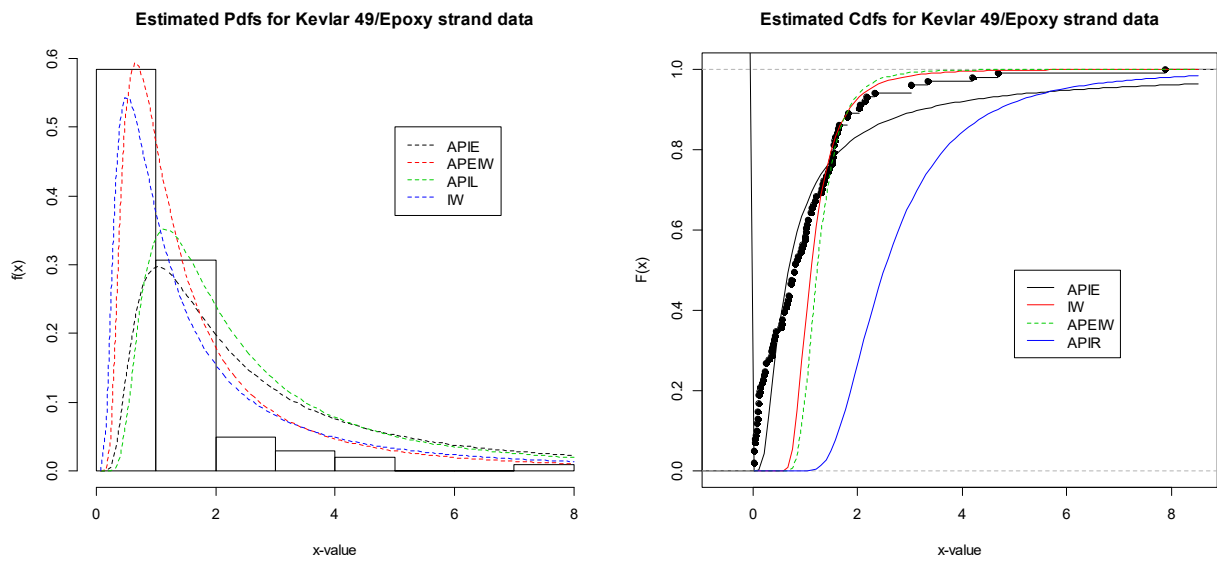


Figure 6. Estimated pdf and cdf function and other competing models for Kevlar 49/Epoxy strand data

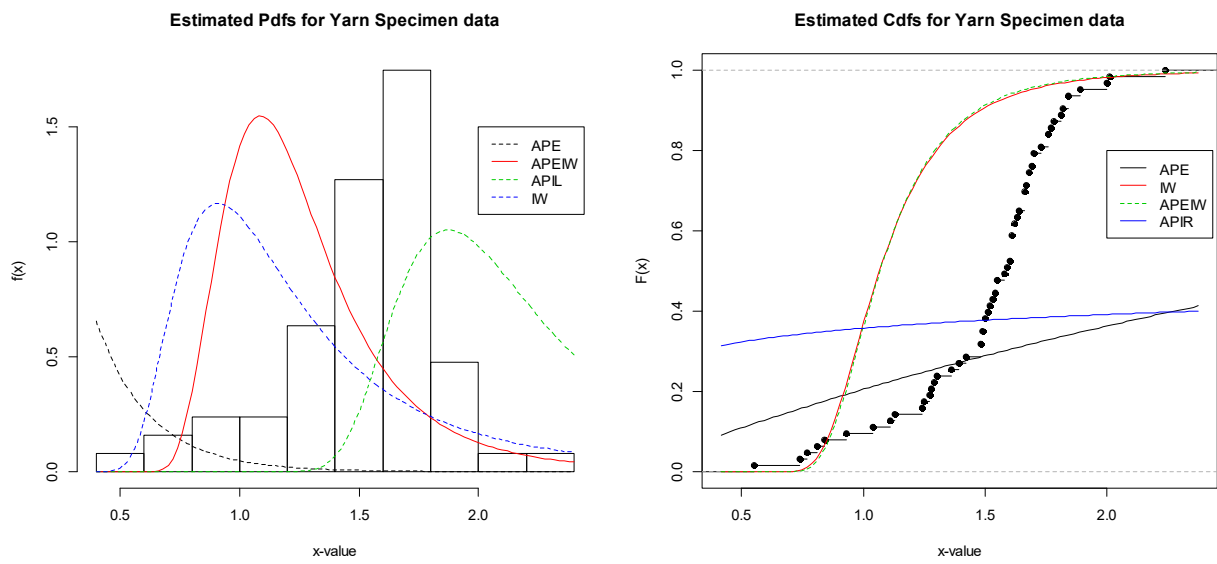


Figure 7. Estimated pdf and cdf function and other competing models for Yarn specimen data

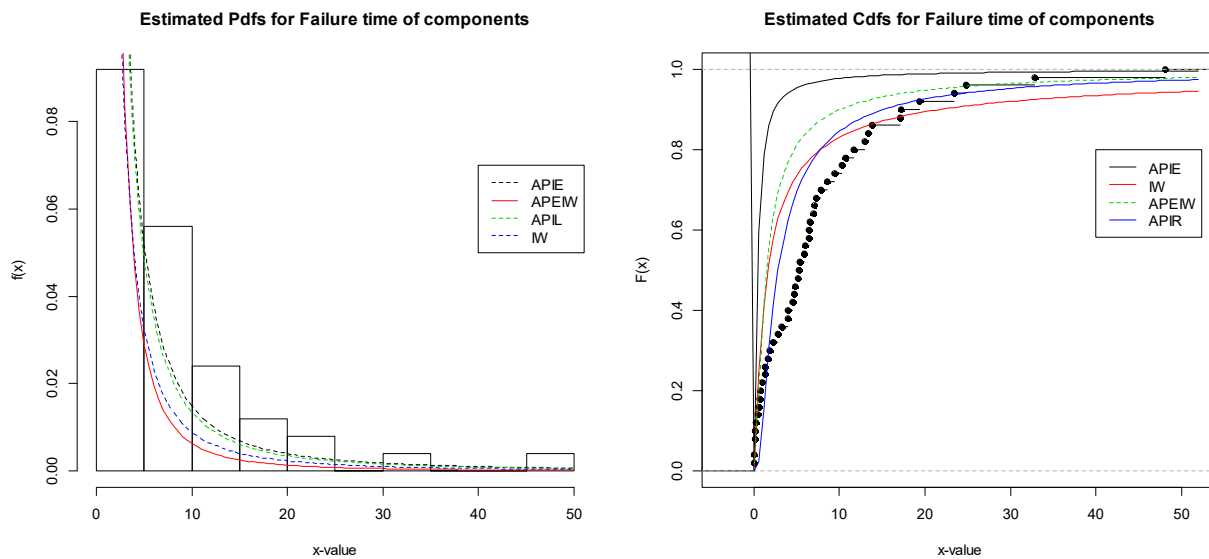


Figure 8. Estimated pdf and cdf function and other competing models for components failure data

Based on Tables 6, 7 and 8, it is evident that APEIW distribution provides the best fit and can therefore be taken as the best model based on the data considered. Figures 6, 7, and 8 provide more information on the flexibility of the APEIW distribution.

6. Conclusion

In this work, we study the Alpha power extended inverted Weibull distribution. Some structural properties of the APEIW distribution are derived. Estimation of the population parameters is achieved by using maximum likelihood estimation method. Simulation study and three life data sets are used to illustrate the tractability of APEIW distribution in modeling real life data. We recommend that further studies should be carried out by using different estimations techniques such as Least Square method, Bayesian method etc. and compare the performance of the estimation techniques.

Data statement

There is no conflict of interest among the authors.

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