International Journal of Statistics and Probability; Vol. 9, No. 3; May 2020
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Characterizations of Exponentiated Generalized Power Lindley and Geometric-Zero Truncated Poisson Distributions

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Received: February 24, 2020 Accepted: April 8, 2020 Online Published: April 12, 2020

doi:10.5539/ijsp.v9n3p36 URL: https://doi.org/10.5539/ijsp.v9n3p36

Abstract

MirMostafaee et al. (2019) proposed a continuous univariate distribution called Exponentiated Generalized Power Lindley (EGPL) distribution and studied certain properties and applications of their distribution. Akdogan et al. (2019) introduced a discrete distribution called Geometric-Zero Truncated Poisson (GZTP) distribution and provided its properties and applications. The present short note is intended to complete, in some way, the works cited above via establishing certain characterizations of the EGPL and GZTP distributions in different directions.

Keywords: discrete distributions, power Lindley distribution, zero truncated distribution, hazard function, characterizations

1. Introduction

Characterizations of distributions is an important research area which has recently attracted the attention of many researchers. This short note deals with various characterizations of EGPL and GZTP distributions to complete, in some way, the works of MirMostafaee et al. (2019) and Akdogan et al. (2019). These characterizations are based on: (*i*) a simple relationship between two truncated moments; (*ii*) the hazard function; (*iii*) the reverse hazard function and (*iv*) the conditional expectation of a function of the random variable. It should be mentioned that for characterization (*i*) the cdf (cumulative distribution function) is not required to have a closed form.

MirMostafaee et al. (2019) proposed the EGPL distribution with cdf and pdf (probability density function) given, respectively, by

$$F(x;\beta,\lambda,a,b)=\left\{1-\left[\left(1+\frac{\lambda}{1+\lambda}x^{\beta}\right)e^{-\lambda x^{\beta}}\right]^{a}\right\}^{b},\quad x\geq 0, \tag{1}$$

and

$$f(x;\beta,\lambda,a,b)=\frac{ab\beta\lambda^{2}}{1+\lambda}x^{\beta-1}\left(1+x^{\beta}\right)e^{-\lambda x^{\beta}}\left[\left(1+\frac{\lambda}{1+\lambda}x^{\beta}\right)e^{-\lambda x^{\beta}}\right]^{a-1}$$
$$\times\left\{1-\left[\left(1+\frac{\lambda}{1+\lambda}x^{\beta}\right)e^{-\lambda x^{\beta}}\right]^{a}\right\}^{b-1},\quad x>0, \tag{2}$$

where β,λ,a,b are all positive parameters.

Akdigan et al. (2019) introduced the GZTP distribution with cdf and pmf (probability mass function) given, respectively, by

$$F(x;\theta,q)=\frac{\theta^{q^{x}}-\theta}{1-\theta},\quad x=1,2,\ldots, \tag{3}$$

and

$$f(x;\theta,q)=\frac{\theta^{q^{x}}-\theta^{q^{x-1}}}{1-\theta},\quad x=1,2,\ldots, \tag{4}$$

where $\theta \in (0,1)$ and $q \in (0,1)$ are parameters.

Remark 1.1. Clearly, one can write down the formulas for the hazard and reverse hazard functions corresponding to these distributions as needed.

2. Characterizations of EGPL Distribution

We present our characterizations of EGPL in four subsections.

2.1 Characterizations Based on Two Truncated Moments

In this subsection we present characterizations of EGPL distribution in terms of a simple relationship between two truncated moments. The first characterization result employs a theorem due to Glänzel (1987), see Theorem 2.1.1 below. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Theorem 2.1.1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty,\ e = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that

$$\mathbf{E}\left[g(X) \mid X \geq x\right] = \mathbf{E}\left[h(X) \mid X \geq x\right]\xi(x), \quad x \in H,$$

is defined with some real function ξ. Assume that $g,\ h \in C^1(H)$, $\xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi h = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and ξ , particularly

$$F(x) = \int_a^x C \left| \frac{\xi'(u)}{\xi(u)h(u) - g(u)} \right| \exp(-s(u))\ du\ ,$$

where the function s is a solution of the differential equation $s' = \frac{\xi' h}{\xi h - g}$ and C is the normalization constant, such that $\int_H dF = 1$.

Remark 2.1.1. The goal of Theorem 2.1.1 is for $\xi(x)$ to have a simple form.

Proposition 2.1.1. Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let , $h(x) = \frac{(1+x^\beta)^{-1}\left[\left(1+\frac{\lambda}{1+\lambda}x^\beta\right)e^{-\lambda x^\beta}\right]^{1-a}}{\left\{1-\left[\left(1+\frac{\lambda}{1+\lambda}x^\beta\right)e^{-\lambda x^\beta}\right]^a\right\}^{b-1}}$ and $g(x) = h(x)e^{-\lambda x^\beta}$ for $x > 0$. The random variable X has pdf (2) if and only if the function ξ defined in Theorem 2.1.1 has the form

$$\xi(x) = \frac{1}{2}e^{-\lambda x^\beta}, \quad x > 0.$$

Proof. Let X be a random variable with pdf (2), then

$$(1 - F(x))\,E\left[h(X) \mid X \geq x\right] = \frac{ab\lambda}{1+\lambda}e^{-\lambda x^\beta}, \quad x > 0,$$

and

$$(1 - F(x))\,E\left[g(X) \mid X \geq x\right] = \frac{ab\lambda}{2(1+\lambda)}e^{-2\lambda x^\beta}, \quad x > 0,$$

and finally

$$\xi(x)h(x) - g(x) = -\frac{1}{2}h(x)e^{-\lambda x^\beta} < 0 \quad for\ \ x > 0.$$

Conversely, if ξ is given as above, then

$$s'(x) = \frac{\xi'(x)h(x)}{\xi(x)h(x) - g(x)} = \lambda\beta x^{\beta-1}, \quad x > 0,$$

and hence

$$s(x) = \lambda x^{\beta}, \quad x > 0.$$

Now, in view of Theorem 2.1.1, X has density (2).

Corollary 2.1.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x)$ be as in Proposition 2.1.1. The pdf of X is (2) if and only if there exist functions g and ξ defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\xi'(x)\,h(x)}{\xi(x)\,h(x) - g(x)} = \lambda\beta x^{\beta-1}, \quad x > 0.$$

The general solution of the differential equation in Corollary 2.1.1 is

$$\xi(x) = e^{\lambda x^{\beta}}\left[-\int \lambda\beta x^{\beta-1} e^{-\lambda x^{\beta}} (h(x))^{-1} g(x) + D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 2.1.1 with $D = 0$. However, it should be also noted that there are other triplets (h, g, ξ) satisfying the conditions of Theorem 2.1.1.

2.2 Characterization Based on Hazard Function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following proposition establishes a characterization of EGPL distribution, for $b = 1$, in terms of the hazard function, which is not of the above trivial form.

Proposition 2.2.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is (2), for $b = 1$, if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h'_F(x) - (\beta - 1)\,x^{-1} h_F(x) = \frac{a\beta^2\lambda^2 x^{2(\beta-1)}}{\left(1 + \lambda + \lambda x^{\beta}\right)^2}, \quad x > 0,$$

with the initial condition $h_F(0) = 0$ for $\beta > 1$.

Proof. If X has pdf (2), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{x^{-(\beta-1)} h_F(x)\right\} = a\beta\lambda^2 \frac{d}{dx}\left\{\frac{1 + x^{\beta}}{1 + \lambda + \lambda x^{\beta}}\right\},$$

or

$$h_F(x) = a\beta\lambda^2 \left\{\frac{x^{\beta-1}\left(1 + x^{\beta}\right)}{1 + \lambda + \lambda x^{\beta}}\right\}, \quad x > 0,$$

which is the hazard function of the EGPL distribution for $b = 1$.

2.3 Characterization Based on Reverse Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in support\ of\ F.$$

In this subsection we present a characterization of EGPL distribution in terms of the reverse hazard function.

Proposition 2.3.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has pdf (2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r'_F(x) + \lambda x^{\beta-1} r_F(x) = \left(\frac{ab\beta\lambda^2}{1+\lambda}\right) e^{-\lambda x^\beta} \frac{d}{dx}\left\{\frac{x^{\beta-1}\left(1+x^\beta\right)\left[\left(1+\frac{\lambda}{1+\lambda}x^\beta\right)e^{-\lambda x^\beta}\right]^{a-1}}{1-\left[\left(1+\frac{\lambda}{1+\lambda}x^\beta\right)e^{-\lambda x^\beta}\right]^a}\right\}, \quad x>0.$$

Proof. If X has pdf (2), then clearly the above differential equation holds. If the differential equation holds, then

$$\frac{d}{dx}\left\{e^{\lambda x^\beta} r_F(x)\right\} = \left(\frac{ab\beta\lambda^2}{1+\lambda}\right) \frac{d}{dx}\left\{\frac{x^{\beta-1}\left(1+x^\beta\right)\left[\left(1+\frac{\lambda}{1+\lambda}x^\beta\right)e^{-\lambda x^\beta}\right]^{a-1}}{1-\left[\left(1+\frac{\lambda}{1+\lambda}x^\beta\right)e^{-\lambda x^\beta}\right]^a}\right\}, \quad x>0,$$

from which we arrive at the reverse hazard function corresponding to the pdf (2).

2.4 Characterizations Based on Conditional Expectation

The following propositions have already appeared in (Hamedani, 2013), so we will just state them here which can be used to characterize the EGPL distribution.

Proposition 2.4.1. Let $X : \Omega \to (c, d)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on (c, d) with $\lim_{x\to c^+} \psi(x) = 1$. Then for $\delta \neq 1$,

$$E[\psi(X) \mid X \geq x] = \delta\psi(x), \quad x \in (c, d),$$

if and only if

$$\psi(x) = (1 - F(x))^{\frac{1}{\delta}-1}, \quad x \in (c, d).$$

Proposition 2.4.2. Let $X : \Omega \to (c, d)$ be a continuous random variable with cdf F. Let $\psi_1(x)$ be a differentiable function on (c, d) with $\lim_{x\to d^-} \psi_1(x) = 1$. Then for $\delta_1 \neq 1$,

$$E[\psi_1(X) \mid X \leq x] = \delta_1\psi_1(x), \quad x \in (c, d),$$

implies that

$$\psi_1(x) = (F_1(x))^{\frac{1}{\delta}-1}, \quad x \in (c, d).$$

Remarks 2.4.1. (A) For $(c, d) = (0, \infty)$, $b = 1$, $\psi(x) = \left(1+\frac{\lambda}{1+\lambda}x^\beta\right)e^{-\lambda x^\beta}$ and $\delta = \frac{a}{a+1}$, Proposition 2.4.1 provides a characterization of EGPL distribution. (B) For $(c, d) = (0, \infty)$, $\psi_1(x) = 1-\left[\left(1+\frac{\lambda}{1+\lambda}x^\beta\right)e^{-\lambda x^\beta}\right]^a$ and $\delta_1 = \frac{b}{b+1}$, Proposition 2.4.2 provides a characterization of EGPL distribution. (C) Of course there are other suitable functions than the ones we mentioned above, which are chosen for the sake of simplicity.

3. Characterizations of GZTP Distribution

We present our characterizations of GZTP via the following two Propositions.

Proposition 3.1. Let $X : \Omega \to \mathbb{N}$ be a random variable. The pmf of X is (4) if and only if

$$E\left\{\left[\left(\theta^{q^X} + \theta^{q^{X-1}}\right)\right] \mid X \leq k\right\} = \theta^{q^k} + \theta, \quad k \in \mathbb{N}. \tag{5}$$

Proof. If X has pmf (4), then the left-hand side of (5) will be

$$\frac{1}{1-\theta}(F(k))^{-1}\sum_{x=1}^{k}\left\{\left(\theta^{2q^X} - \theta^{2q^{X-1}}\right)\right\} = \frac{\theta^{2q^k} - \theta^2}{\theta^{q^k} - \theta} = \theta^{q^k} + \theta, \quad k \in \mathbb{N}.$$

Conversely, if (5) holds, then

$$\sum_{x=1}^{k}\left\{\left(\theta^{q^{x}}+\theta^{q^{x-1}}\right)\ f(x)\right\}=F(k)\left(\theta^{q^{k}}+\theta\right). \tag{6}$$

From (6), we also have

$$\begin{aligned}\sum_{x=1}^{k-1}\left\{\left(\theta^{q^{x}}+\theta^{q^{x-1}}\right)\ f(x)\right\}&=F(k-1)\left(\theta^{q^{k-1}}+\theta\right)\\&=\{F(k)-f(k)\}\left(\theta^{q^{k-1}}+\theta\right).\end{aligned} \tag{7}$$

Now, subtracting (7) from (6), we arrive at

$$\begin{aligned}F(k)\left[\left(\left(\theta^{q^{k}}+\theta\right)-\left(\theta^{q^{k-1}}+\theta\right)\right)\right]&=F(k)\left[\left(\theta^{q^{k}}-\theta^{q^{k-1}}\right)\right]\\&=\left[\left(\theta^{q^{k}}+\theta^{q^{k-1}}\right)-\left(\theta^{q^{k-1}}+\theta\right)\right]\ f(k)\\&=\left(\theta^{q^{k}}-\theta\right)\ f(k).\end{aligned}$$

From the last equality, we have

$$r_F(k)=\frac{f(k)}{F(k)}=\frac{\left(\theta^{q^{k}}-\theta^{q^{k-1}}\right)}{\left(\theta^{q^{k}}-\theta\right)},$$

which, is the hazard function corresponding to the pmf (4).

Proposition 3.2. Let $X:\Omega\rightarrow\mathbb{N}$ be a random variable. The pmf of X is (4) if and only if its reverse hazard function, r_F, satisfies the following difference equation

$$r_F(k+1)-r_F(k)=\frac{\left(\theta^{q^{k-1}}-\theta\right)}{\left(\theta^{q^{k}}-\theta\right)}-\frac{\left(\theta^{q^{k}}-\theta\right)}{\left(\theta^{q^{k+1}}-\theta\right)},\quad k\in\mathbb{N}, \tag{8}$$

with the initial condition $r_F(0)=1$.

Proof. Clearly, if X has pmf (4), then (8) holds. Now, if (8) holds, then

$$\sum_{k=1}^{x-1}\{r_F(k+1)-r_F(k)\}=\sum_{k=1}^{x-1}\left\{\left(\frac{\left(\theta^{q^{k+1}}-\theta^{q^{k}}\right)}{\left(\theta^{q^{k+1}}-\theta\right)}-\frac{\left(\theta^{q^{k}}-\theta^{q^{k-1}}\right)}{\left(\theta^{q^{k}}-\theta\right)}\right)\right\},$$

or

$$r_F(x)-r_F(0)=\frac{\theta^{q^{x}}-\theta^{q^{x-1}}}{\theta^{q^{x}}-\theta}-1,$$

or, in view of the initial condition $r_F(0)=1$, we have

$$r_F(k)=\frac{\theta^{q^{x}}-\theta^{q^{x-1}}}{\theta^{q^{x}}-\theta},\quad k\in\mathbb{N},$$

which is the reverse hazard function corresponding to the pmf (4)

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