International Journal of Statistics and Probability; Vol. 9, No. 1; January 2020
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

T^2 Type Test Statistic and Simultaneous Confidence Intervals for Two Sub-mean Vectors

Tamae Kawasaki[1], Toshiki Naito[1] & Takashi Seo[1]

[1] Department of Applied Mathematics, Tokyo University of Science, Tokyo, Japan

Correspondence: Tamae Kawasaki, Department of Applied Mathematics, Tokyo University of Science, Tokyo, Japan.

Received: October 27, 2019 Accepted: December 4, 2019 Online Published: December 16, 2019

doi:10.5539/ijsp.v9n1p1 URL: https://doi.org/10.5539/ijsp.v9n1p1

Abstract

In this paper, we consider tests for sub-mean vectors and its simultaneous confidence intervals in two-sample problems. We give the T^2 type test statistic and the simultaneous confidence intervals by using two approximate upper percentiles of T^2 type test statistic. One of the approximate percentiles is obtained by normal approximation for a part of the T^2 type statistic, and the other is an approximation obtained by correcting the degrees of freedom of the F distribution. Finally, we investigate the asymptotic behavior of the approximate upper percentiles of T^2 type statistic by Monte Carlo simulation, and we give an example to illustrate the simultaneous confidence intervals.

Keywords: approximate degrees of freedom, F approximation, Monte Carlo simulation, simultaneous confidence interval, two-sample problem, type I error

1. Introduction

Let $\boldsymbol{x}_1^{(i)}, \boldsymbol{x}_2^{(i)}, \ldots, \boldsymbol{x}_{N^{(i)}}^{(i)}$ be p-dimensional random vectors from $N_p(\boldsymbol{\mu}^{(i)}, \Sigma)$, $i = 1, 2$. we consider the following hypothesis

$$H_0 : \boldsymbol{\mu}_1^{(1)} = \boldsymbol{\mu}_1^{(2)} \text{ given } \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)} \text{ vs. } H_1 : \boldsymbol{\mu}_1^{(1)} \neq \boldsymbol{\mu}_1^{(2)} \text{ given } \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)}, \tag{1}$$

where

$$\underset{p\times 1}{\boldsymbol{\mu}^{(i)}} = \begin{pmatrix} \boldsymbol{\mu}_1^{(i)} \\ \boldsymbol{\mu}_2^{(i)} \end{pmatrix}, \ \underset{r\times 1}{\boldsymbol{\mu}_1^{(i)}} = \begin{pmatrix} \mu_1^{(i)} \\ \mu_2^{(i)} \\ \vdots \\ \mu_r^{(i)} \end{pmatrix}, \ \underset{s\times 1}{\boldsymbol{\mu}_2^{(i)}} = \begin{pmatrix} \mu_{r+1}^{(i)} \\ \mu_{r+2}^{(i)} \\ \vdots \\ \mu_p^{(i)} \end{pmatrix}, p = r + s. \tag{2}$$

For the problem of sub-mean vectors, Eaton and Kariya (1983) derived tests for the independence of two normally distributed sub-mean vectors for the case that an additional random sample is available. Provost (1990) obtained explicit expressions for the case that the maximum likelihood estimators (MLEs) of all the parameters of the multi-normal random vector are given, and the likelihood ratio statistic for testing the independence between sub-mean vectors has been obtained. For the one-sample problem, Rao (1949) gave Rao's U-statistic and additional information. The null distribution of Rao's U-statistic has been introduced by Siotani et al. (1985). A test for sub-mean vectors with two-step monotone missing data was discussed by Kawasaki and Seo (2016). A test for sub-mean vectors in two-sample problem was introduced by Rencher (2012). For the k-sample problem, Fujikoshi et al. (2010) gave an asymptotic expansion of the distribution of the generalized U-statistic under normality. Gupta et al. (2006) gave an asymptotic expansion of the distribution of the generalized U-statistic under a general condition. However, the problem for sub-mean vectors in terms of simultaneous confidence intervals does not appear to have been discussed.

The aim of this article is to provide simultaneous confidence intervals for sub-mean vectors in two-sample problems. We consider two procedures. The first procedure is to give the T^2 type test statistic of testing two-normal sub-mean vectors and its approximate upper percentile using normal approximation for a part of the test statistic in Section 2. The second procedure is to obtain the asymptotic expansions of the moments of test statistic, and then the approximating the null distribution of the T^2 type test statistic using an F distribution is also given in Section 2. In Sections 3, the approximate simultaneous confidence intervals for all linear compounds of the difference of two-normal sub-mean vectors are outlined. In Section 4, we investigate the asymptotic behavior of the approximate upper percentiles of the T^2 test statistic by Monte Carlo simulation. In Section 5, we give an example to illustrate simultaneous confidence intervals. This paper is a revised version of the Technical Report Naito et al. (2018) and includes some of the content of the paper.

2. T^2 Type Test Statistic for Sub-mean Vectors

In this section, we provide Hotelling's T^2 type test statistic for testing the hypothesis (1) and its simultaneous confidence intervals. We partition $\boldsymbol{x}_j^{(i)}$ into a $r \times 1$ random vector and a $s \times 1$ random vector, as $\boldsymbol{x}_j^{(i)} = (\boldsymbol{x}_{1j}^{(i)'}, \boldsymbol{x}_{2j}^{(i)'})'$, where $j = 1, 2, \ldots, N^{(i)}$, $p = r + s$. The hypothesis (1) is the same as the following hypothesis,

$$H_0' : \boldsymbol{\mu}_{1\cdot 2}^{(1)} = \boldsymbol{\mu}_{1\cdot 2}^{(2)} \text{ vs. } H_1' : \boldsymbol{\mu}_{1\cdot 2}^{(1)} \neq \boldsymbol{\mu}_{1\cdot 2}^{(2)}, \tag{3}$$

where

$$\boldsymbol{\mu}_{1\cdot 2}^{(i)} = \boldsymbol{\mu}_1^{(i)} - \Sigma_{12}\Sigma_{22}^{-1}\boldsymbol{\mu}_2,\ \boldsymbol{\mu}_2 = \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)},\ \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}. \tag{4}$$

First, we derive the maximum likelihood estimators (MLEs) of $\boldsymbol{\mu}$ and Σ.

We use the following transformed parameters $(\boldsymbol{\eta}^{(i)}, \Psi)$,

$$\boldsymbol{\eta}^{(i)} = \begin{pmatrix} \boldsymbol{\eta}_1^{(i)} \\ \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_1^{(i)} - \Sigma_{12}\Sigma_{22}^{-1}\boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_2 \end{pmatrix},\ \Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} = \begin{pmatrix} \Sigma_{11\cdot 2} & \Sigma_{12}\Sigma_{22}^{-1} \\ \Sigma_{22}^{-1}\Sigma_{21} & \Sigma_{22} \end{pmatrix}, \tag{5}$$

where $\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, $i = 1, 2$. We note that $(\boldsymbol{\eta}^{(i)}, \Psi)$ is in one-to-one correspondence with $(\boldsymbol{\mu}^{(i)}, \Sigma)$. Using the transformed parameters $(\boldsymbol{\eta}^{(i)}, \Psi)$, the likelihood function is given by

$$L(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \Psi) = (2\pi)^{-\frac{Np}{2}} |\Psi_{11}|^{-\frac{N}{2}} |\Psi_{22}|^{-\frac{N}{2}} \tag{6}$$

$$\times \prod_{i=1}^{2} \left[\exp\left\{ -\frac{1}{2} \sum_{j=1}^{N^{(i)}} (\boldsymbol{x}_{1j}^{(i)} - \Psi_{12}\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_1^{(i)})' \Psi_{11}^{-1} (\boldsymbol{x}_{1j}^{(i)} - \Psi_{12}\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_1^{(i)}) \right\} \right] \tag{7}$$

$$\times \prod_{i=1}^{2} \left[\exp\left\{ -\frac{1}{2} \sum_{j=1}^{N^{(i)}} (\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_2)' \Psi_{22}^{-1} (\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_2) \right\} \right], N = N^{(1)} + N^{(2)}. \tag{8}$$

Then, we will derive the MLEs as follows:

$$\widehat{\boldsymbol{\eta}}_1^{(i)} = \overline{\boldsymbol{x}}_1^{(i)} - \widehat{\Psi}_{12}\overline{\boldsymbol{x}}_2^{(i)}, i = 1, 2,\ \widehat{\boldsymbol{\eta}}_2 = \overline{\boldsymbol{x}}_2,\ \widehat{\Psi}_{11} = \frac{1}{N} V_{11\cdot 2},\ \widehat{\Psi}_{12} = V_{12}V_{22}^{-1}\widehat{\Psi}_{22}, \tag{9}$$

$$\widehat{\Psi}_{22} = \frac{1}{N}\left\{ V_{22} + \sum_{j=1}^{2} (\overline{\boldsymbol{x}}_2^{(j)} - \overline{\boldsymbol{x}}_2)(\overline{\boldsymbol{x}}_2^{(j)} - \overline{\boldsymbol{x}}_2)' \right\}, \tag{10}$$

where

$$\overline{\boldsymbol{x}}^{(i)} = \frac{1}{N^{(i)}} \sum_{j=1}^{N^{(i)}} \boldsymbol{x}_j^{(i)} = \begin{pmatrix} \overline{\boldsymbol{x}}_1^{(i)} \\ \overline{\boldsymbol{x}}_2^{(i)} \end{pmatrix}, \overline{\boldsymbol{x}}_2 = \frac{1}{N} \sum_{i=1}^{2} N^{(i)} \overline{\boldsymbol{x}}_2^{(i)}, \tag{11}$$

$$V^{(i)} = \sum_{j=1}^{N^{(i)}} (\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})(\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})' = \begin{pmatrix} V_{11}^{(i)} & V_{12}^{(i)} \\ V_{21}^{(i)} & V_{22}^{(i)} \end{pmatrix}, \tag{12}$$

$$V = \sum_{i=1}^{2} V^{(i)} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},\ V_{11\cdot 2} = V_{11} - V_{12}V_{22}^{-1}V_{21}. \tag{13}$$

Therefore, using the relation that $(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \Psi)$ is in one-to-one correspondence with $(\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \Sigma)$, the MLEs of $\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}$, and Σ are given by

$$\widehat{\boldsymbol{\mu}}^{(i)} = \begin{pmatrix} \widehat{\boldsymbol{\mu}}_1^{(i)} \\ \widehat{\boldsymbol{\mu}}_2 \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{x}}_1^{(i)} - \widehat{\Sigma}_{12}\widehat{\Sigma}_{22}^{-1}(\overline{\boldsymbol{x}}_2^{(i)} - \overline{\boldsymbol{x}}_2) \\ \overline{\boldsymbol{x}}_2 \end{pmatrix}, i = 1, 2, \tag{14}$$

$$\widehat{\Sigma} = \begin{pmatrix} \widehat{\Sigma}_{11} & \widehat{\Sigma}_{12} \\ \widehat{\Sigma}_{21} & \widehat{\Sigma}_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{N} V_{11\cdot 2} + \widehat{\Sigma}_{12}\widehat{\Sigma}_{22}^{-1}\widehat{\Sigma}_{21} & V_{12}V_{22}^{-1}\widehat{\Sigma}_{22} \\ \widehat{\Sigma}_{22}V_{22}^{-1}V_{21} & \frac{1}{N}\left\{ V_{22} + \sum_{i=1}^{2} (\overline{\boldsymbol{x}}_2^{(i)} - \overline{\boldsymbol{x}}_2)(\overline{\boldsymbol{x}}_2^{(i)} - \overline{\boldsymbol{x}}_2)' \right\} \end{pmatrix}, \tag{15}$$

2.1 T^2 Type Test Statistic and Approximation Upper Percentile

For sub-mean vectors, we construct a test statistic based on Hotelling's T^2 statistic structure:

$$T^2 = (\widehat{\boldsymbol{\mu}}^{(1)}_{1\cdot 2} - \widehat{\boldsymbol{\mu}}^{(2)}_{1\cdot 2})'\{\widehat{\mathrm{Cov}}\left(\widehat{\boldsymbol{\mu}}^{(1)}_{1\cdot 2} - \widehat{\boldsymbol{\mu}}^{(2)}_{1\cdot 2}\right)\}^{-1}(\widehat{\boldsymbol{\mu}}^{(1)}_{1\cdot 2} - \widehat{\boldsymbol{\mu}}^{(2)}_{1\cdot 2}), \tag{16}$$

where

$$\widehat{\mathrm{Cov}}\left(\widehat{\boldsymbol{\mu}}^{(1)}_{1\cdot 2} - \widehat{\boldsymbol{\mu}}^{(2)}_{1\cdot 2}\right) = \frac{N(N-3)}{N^{(1)}N^{(2)}(N-2)(N-s-3)}V_{11\cdot 2}, \tag{17}$$

$\widehat{\boldsymbol{\mu}}^{(i)}_{1\cdot 2}$ and $\widehat{\mathrm{Cov}}\left(\widehat{\boldsymbol{\mu}}^{(1)}_{1\cdot 2} - \widehat{\boldsymbol{\mu}}^{(2)}_{1\cdot 2}\right)$ are the estimators of $\boldsymbol{\mu}^{(i)}_{1\cdot 2}$ and $\mathrm{Cov}\left(\widehat{\boldsymbol{\mu}}^{(1)}_{1\cdot 2} - \widehat{\boldsymbol{\mu}}^{(2)}_{1\cdot 2}\right)$, respectively. We call this test statistic the Hotelling's T^2 type statistic. We note that under H_0' in (3), T^2 is asymptotically distributed as a χ^2 distribution with r degrees of freedom. However, when the sample is not large, the χ^2 distribution is not a good approximation of the upper percentile of T^2.

Let

$$\boldsymbol{u} = \overline{\boldsymbol{x}}^{(1)}_1 - \overline{\boldsymbol{x}}^{(2)}_1 - V_{12}V_{22}^{-1}(\overline{\boldsymbol{x}}^{(1)}_2 - \overline{\boldsymbol{x}}^{(2)}_2),\ c = \frac{N(N-3)}{N^{(1)}N^{(2)}(N-s-3)}. \tag{18}$$

We can then rewrite T^2 as

$$T^2 = (N-2)c^{-1}\boldsymbol{u}'V_{11\cdot 2}^{-1}\boldsymbol{u} = \boldsymbol{z}'W^{-1}\boldsymbol{z} = \boldsymbol{z}'\boldsymbol{z}\frac{\boldsymbol{z}'W^{-1}\boldsymbol{z}}{\boldsymbol{z}'\boldsymbol{z}}, \tag{19}$$

where $\boldsymbol{z} = c^{-\frac{1}{2}}\Sigma_{11\cdot 2}^{-\frac{1}{2}}\boldsymbol{u}$, $W = \Sigma_{11\cdot 2}^{-\frac{1}{2}}S_{11\cdot 2}\Sigma_{11\cdot 2}^{-\frac{1}{2}}$,

$$S = \frac{1}{N-2}V = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},\ S_{11\cdot 2} = S_{11} - S_{12}S_{22}^{-1}S_{21}. \tag{20}$$

We note that $\boldsymbol{u}$ is distributed on $N_r(\boldsymbol{\mu}^{(1)}_1 - \boldsymbol{\mu}^{(2)}_1, c\Sigma_{11\cdot 2})$ when $N^{(1)}, N^{(2)} \to \infty$. Therefore, the distribution of $\boldsymbol{z}'\boldsymbol{z}$ is a χ^2 distribution with r degrees of freedom. We note that under H_0', $(N-p-s-1)T^2/\{(N-2)r\}$ is approximately distributed as a F distribution with r and $N-p-s-1$ degrees of freedom. Using this result, the approximate upper 100α percentile of the T^2 statistic is given by

$$t^2_{\mathrm{app}}(\alpha) = \frac{(N-2)r}{N-p-s-1}F_{r,N-p-s-1}(\alpha), \tag{21}$$

where $F_{r,N-p-s-1}(\alpha)$ is the upper 100α percentiles of the F distribution with r and $N-p-s-1$ degrees of freedom. The details of result follow from Naito et al. (2018).

2.2 Approximate Degrees of Freedom

In this session, we consider the approximate distribution of T^2. By approximating the distribution of $\boldsymbol{z}'\boldsymbol{z}$ and $\boldsymbol{z}'\boldsymbol{z}/\boldsymbol{z}'W^{-1}\boldsymbol{z}$ as

$$\boldsymbol{z}'\boldsymbol{z} \approx \chi^2_\xi,\ \frac{\boldsymbol{z}'\boldsymbol{z}}{\boldsymbol{z}'W^{-1}\boldsymbol{z}} \approx \frac{\chi^2_\nu}{\phi}, \tag{22}$$

we have

$$T^2 \approx \frac{\phi\xi}{\nu}F_{\xi,\nu}, \tag{23}$$

where ξ, ν, and ϕ are unknown constants. It follows from (22) that

$$\mathrm{E}[\boldsymbol{z}'\boldsymbol{z}] \approx \xi,\ \mathrm{E}[\frac{\boldsymbol{z}'\boldsymbol{z}}{\boldsymbol{z}'W^{-1}\boldsymbol{z}}] \approx \frac{\nu}{\phi},\ \mathrm{E}[\left(\frac{\boldsymbol{z}'\boldsymbol{z}}{\boldsymbol{z}'W^{-1}\boldsymbol{z}}\right)^2] \approx \frac{\nu(\nu+2)}{\phi^2}. \tag{24}$$

We consider the asymptotic expansion of the first and second order moment of T^2 in a situation when

$$\gamma_i = \frac{N^{(i)}}{N} \to \text{positive constants} \tag{25}$$

as $N^{(i)}$'s tend to infinity. Therefore, without loss of generality, we assume that $\boldsymbol{\mu}^{(i)} = \mathbf{0}, \Sigma = I_p$, hereafter. In our derivations, we consider the asymptotic expansions of $z'z$ and $z'z/z'W^{-1}z$ in terms of

$$\begin{pmatrix} \overline{\boldsymbol{x}}_1^{(i)} \\ \overline{\boldsymbol{x}}_2^{(i)} \end{pmatrix} = \frac{1}{\sqrt{N^{(i)}}} \begin{pmatrix} z_1^{(i)} \\ z_2^{(i)} \end{pmatrix},\ S = I_p + \frac{1}{\sqrt{N-2}} U = \begin{pmatrix} I_{11} + \frac{1}{\sqrt{N-2}} U_{11} & \frac{1}{\sqrt{N-2}} U_{12} \\ \frac{1}{\sqrt{N-2}} U_{21} & I_{22} + \frac{1}{\sqrt{N-2}} U_{22} \end{pmatrix}. \tag{26}$$

Then, $\overline{\boldsymbol{x}}_1^{(i)} - V_{12} V_{22}^{-1} \overline{\boldsymbol{x}}_2^{(i)}$ and z can be expanded as

$$\overline{\boldsymbol{x}}_1^{(i)} - V_{12} V_{22}^{-1} \overline{\boldsymbol{x}}_2^{(i)} = \frac{1}{\sqrt{N\gamma_i}} \left(z_1^{(i)} - \frac{1}{\sqrt{N}} U_{12} z_2^{(i)} + \frac{1}{N} U_{12} U_{22} z_2^{(i)} \right) + O_p(N^{-2}), \tag{27}$$

$$z = (\sqrt{\gamma_2} z_1^{(1)} - \sqrt{\gamma_1} z_1^{(2)}) - \frac{1}{\sqrt{N}} (\sqrt{\gamma_2} U_{12} z_2^{(1)} - \sqrt{\gamma_1} U_{12} z_2^{(2)}) + \frac{1}{N} (\sqrt{\gamma_2} U_{12} U_{22} z_2^{(1)} - \sqrt{\gamma_1} U_{12} U_{22} z_2^{(2)}) + O_p(N^{-3/2}), \tag{28}$$

respectively. We have

$$z'z = (\sqrt{\gamma_2} z_1^{(1)} - \sqrt{\gamma_1} z_1^{(2)})'(\sqrt{\gamma_2} z_1^{(1)} - \sqrt{\gamma_1} z_1^{(2)}) - \frac{2}{\sqrt{N}} (\sqrt{\gamma_2} z_1^{(1)} - \sqrt{\gamma_1} z_1^{(2)})'(\sqrt{\gamma_2} U_{12} z_2^{(1)} - \sqrt{\gamma_1} U_{12} z_2^{(2)}) \tag{29}$$

$$+ \frac{1}{N} \{ (\sqrt{\gamma_2} z_1^{(1)} - \sqrt{\gamma_1} z_1^{(2)})'(\sqrt{\gamma_2} U_{12} U_{22} z_2^{(1)} - \sqrt{\gamma_1} U_{12} U_{22} z_1^{(2)}) \tag{30}$$

$$+ (\sqrt{\gamma_2} U_{12} z_2^{(1)} - \sqrt{\gamma_1} U_{12} z_2^{(2)})'(\sqrt{\gamma_2} U_{12} z_2^{(1)} - \sqrt{\gamma_1} U_{12} z_2^{(2)}) \tag{31}$$

$$+ (\sqrt{\gamma_2} U_{12} U_{22} z_2^{(1)} - \sqrt{\gamma_1} U_{12} U_{22} z_1^{(2)})'(\sqrt{\gamma_2} z_1^{(1)} - \sqrt{\gamma_1} z_1^{(2)}) \} + O_p(N^{-3/2}). \tag{32}$$

Further, since

$$W^{-1} = I_{11} - \frac{1}{\sqrt{N}} U_{11} + \frac{1}{N} (U_{11}^2 + U_{12} U_{21}) + O_p(N^{-3/2}), \tag{33}$$

$$\frac{z'z}{z'W^{-1}z} = 1 + \frac{1}{\sqrt{N}} \frac{z'U_{11}z}{z'z} - \frac{1}{N} \left\{ \frac{z'U_{11}^2 z}{z'z} + \frac{z'U_{12}U_{21}z}{z'z} + \left(\frac{z'U_{11}^2 z}{z'z} \right)^2 \right\} O_p(N^{-3/2}), \tag{34}$$

$$\left(\frac{z'z}{z'W^{-1}z} \right)^2 = 1 + \frac{2}{\sqrt{N}} \frac{z'U_{11}z}{z'z} - \frac{1}{N} \left\{ \frac{2z'U_{11}^2 z}{z'z} + \frac{2z'U_{12}U_{21}z}{z'z} + \left(\frac{z'U_{11}^2 z}{z'z} \right)^2 \right\} O_p(N^{-3/2}). \tag{35}$$

By calculating the expectations of $z'z$ and $\frac{z'z}{z'W^{-1}z}$, we obtain

$$\mathrm{E}[z'z] = r + \frac{1}{N} rs + O(N^{-3/2}),\ \mathrm{E}[\frac{z'z}{z'W^{-1}z}] = 1 - \frac{1}{N}(p+3) + O(N^{-3/2}), \tag{36}$$

$$\mathrm{E}[\left(\frac{z'z}{z'W^{-1}z} \right)^2] = 1 - \frac{2}{N}(p+2) + O(N^{-3/2}). \tag{37}$$

By equating (24), (36) and (37) to the N^{-1} terms, the coefficients ξ, ν, and ϕ are determined as

$$\xi = r + \frac{1}{N} rs,\ \nu = \frac{2(N-p-3)^2}{2N-(p+3)^2},\ \phi = \frac{2N(N-p-3)}{2N-(p+3)^2}. \tag{38}$$

Using this results, for $N > (p+3)^2/2$, the approximate upper 100α percentile of the T^2 statistic is given by

$$t_{df}^2(\alpha) = \frac{\xi\phi}{\nu} F_{\xi,\nu}(\alpha). \tag{39}$$

3. Simultaneous Confidence Intervals

In this section, we consider the simultaneous confidence intervals for any and all linear compounds of the sub-mean. Using the upper percentiles of T^2 from Section 2.1, for any nonnull vector $\boldsymbol{a} = (a_1, a_2, \cdots, a_r)'$, the simultaneous confidence intervals for $\boldsymbol{a}'(\boldsymbol{\mu}_{1\cdot2}^{(1)} - \boldsymbol{\mu}_{1\cdot2}^{(2)})$ are given by

$$\boldsymbol{a}'\boldsymbol{u} - \sqrt{\frac{c}{N-2}} M \le \boldsymbol{a}'(\boldsymbol{\mu}_{1\cdot2}^{(1)} - \boldsymbol{\mu}_{1\cdot2}^{(2)}) \le \boldsymbol{a}'\boldsymbol{u} + \sqrt{\frac{c}{N-2}} M,\ \forall \boldsymbol{a} \in \mathbb{R}^r - \{\mathbf{0}\}, \tag{40}$$

where $M = (t^2(\alpha)\boldsymbol{a}'V_{11\cdot 2}\boldsymbol{a})^{\frac{1}{2}}$, and $t^2(\alpha)$ is the upper 100α percentiles of the T^2 test statistic. However, it is not easy to obtain $t^2(\alpha)$.

Therefore, as the first method, using the approximate upper 100α percentiles of the T^2 test statistic, $t^2_{\text{app}}(\alpha)$ by (21), the approximate simultaneous confidence intervals for $\boldsymbol{a}'(\boldsymbol{\mu}^{(1)}_{1\cdot 2} - \boldsymbol{\mu}^{(2)}_{1\cdot 2})$ can obtained by

$$\boldsymbol{a}'\boldsymbol{u} - \sqrt{\frac{c}{N-2}}M_{\text{app}} \le \boldsymbol{a}'(\boldsymbol{\mu}^{(1)}_{1\cdot 2} - \boldsymbol{\mu}^{(2)}_{1\cdot 2}) \le \boldsymbol{a}'\boldsymbol{u} + \sqrt{\frac{c}{N-2}}M_{\text{app}}, \ \forall \boldsymbol{a} \in \mathbb{R}^r - \{\mathbf{0}\}, \tag{41}$$

where $M_{\text{app}} = (t^2_{\text{app}}(\alpha)\boldsymbol{a}'V_{11\cdot 2}\boldsymbol{a})^{\frac{1}{2}}$.

As a second method, using the approximate upper 100α percentiles of the T^2 test statistic, $t^2_{\text{df}}(\alpha)$ by (39), the approximate simultaneous confidence intervals for $\boldsymbol{a}'(\boldsymbol{\mu}^{(1)}_{1\cdot 2} - \boldsymbol{\mu}^{(2)}_{1\cdot 2})$ can obtained by

$$\boldsymbol{a}'\boldsymbol{u} - \sqrt{\frac{c}{N-2}}M_{\text{df}} \le \boldsymbol{a}'(\boldsymbol{\mu}^{(1)}_{1\cdot 2} - \boldsymbol{\mu}^{(2)}_{1\cdot 2}) \le \boldsymbol{a}'\boldsymbol{u} + \sqrt{\frac{c}{N-2}}M_{\text{df}}, \ \forall \boldsymbol{a} \in \mathbb{R}^r - \{\mathbf{0}\}, \tag{42}$$

where $M_{\text{df}} = (t^2_{\text{df}}(\alpha)\boldsymbol{a}'V_{11\cdot 2}\boldsymbol{a})^{\frac{1}{2}}$.

4. Simulation Studies

In this section, we perform a Monte Carlo simulation (with 10^6 runs) in order to evaluate the asymptotic behavior of the F approximations and the accuracy of the approximate upper 100α percentiles of the T^2 statistic.

Tables 1 and 2 present the simulated upper 100α percentile of the T^2 test statistic, $t^2(\alpha)$, the approximate upper 100α percentile of the T^2 test statistic, $t^2_{\text{app}}(\alpha)$ and $t^2_{\text{df}}(\alpha)$ for the two-sample problem;

$$(p, r, s) = (4, 1, 3), (4, 2, 2), (4, 3, 1), (8, 2, 6), (8, 4, 4), (8, 6, 2); \alpha = 0.05, 0.01; \tag{43}$$

and for the following two cases of $(N^{(1)}, N^{(2)})$:

$$(N^{(1)},\ N^{(2)}) = \begin{cases} (\ell, \ell), \ \ell = 20, 40, 100, 200, 400 \\ (\ell, 2\ell), \ \ell = 20, 40, 100, 200 \end{cases}. \tag{44}$$

Tables 1 and 2 present the type I errors for the upper 100α percentile of the χ^2 distribution with r degrees of freedom and the approximate upper 100α percentile of the T^2 test statistic given by

$$\alpha_1 = \Pr(T^2 > \chi^2_r(\alpha)), \ \alpha_2 = \Pr(T^2 > t^2_{\text{app}}(\alpha)), \ \alpha_3 = \Pr(T^2 > t^2_{\text{df}}(\alpha)), \tag{45}$$

respectively. It may be noted from Tables 1 and 2 that the simulated values approach closer to the upper percentile of the χ^2 distribution when both of the sample sizes $N^{(1)}$ and $N^{(2)}$ become large. In addition, it can be seen from both tables that the proposed approximation values are good for all cases even when the sample size is small. The results for the type I error of the proposed approximation value are closer than those of the χ^2 value for all cases. Since t^2_{df} has restrictions on sample size and dimensions, there are combinations in which values cannot be obtained. However, it is a better approximation than t^2_{app}, especially when the r dimension corresponding to hypothesis "given" is small. Also, comparing α_2 and α_3, it can be seen that in the case of 0.01, α_3 is a better approximation even when the sample size is small. On the other hand, it can be seen that t^2_{app} is a stable good approximation. t^2_{app} can be used in more cases than t^2_{df} because it has less sample size and dimensionality constraints. However, it should be noted that t^2_{app} is a result obtained using the assumption that $\boldsymbol{u}$ from (18) is normally distributed.

5. Numerical Example

In this section, we discuss an example to illustrate the results. In this example, we utilize the data in the iris plant taken from Fisher (1936). The data consists of four different measurements, x_1: petal width, x_2: petal length: , x_3: sepal width, and x_4: sepal length, for three irises, however, we use two irises, virginica and versicolor. The population mean vectors are $\boldsymbol{\mu}^{(i)} = (\boldsymbol{\mu}^{(i)\prime}_1, \boldsymbol{\mu}^{(i)\prime}_2)' = (\mu^{(i)}_1, \mu^{(i)}_2, \mu^{(i)}_3, \mu^{(i)}_4)'$, where $\mu^{(i)}_1$: mean of petal length, $\mu^{(i)}_2$: mean of sepal length, $\mu^{(i)}_3$: mean of petal width, $\mu^{(i)}_4$: mean of sepal width, $\boldsymbol{\mu}^{(i)}_1 = (\mu^{(i)}_1, \mu^{(i)}_2, \mu^{(i)}_3)'$, and $\boldsymbol{\mu}^{(i)}_2 = \mu^{(i)}_4$. We assume that these data are distributed normality, and $\boldsymbol{\mu}_2 = \boldsymbol{\mu}^{(1)}_2 = \boldsymbol{\mu}^{(2)}_2$. Therefore, we have the data of $N^{(1)} = N^{(2)} = 50, p = 4, r = 3, s = 1$. The hypothesis (3) is considered on this example. We computed $T^2 = 320.20$. For this example case, the simulated upper 100α percentiles pf T^2 type statistic is $t^2(0.05) = 8.37$, the null hypothesis is rejected. When we use $t^2_{\text{app}}(0.05) = 8.45$ and $t^2_{\text{df}}(0.05) = 8.72$, the null hypothesis is also rejected. Table 3 gives 95% simultaneous confidence intervals using three different upper percentiles. $t^2_{\text{app}}(0.05)$ is very similar results to the true values that are given by the simulated upper percentiles of T^2, $t^2(0.05)$. $t^2_{\text{df}}(0.05)$ is almost similar, and the result is the same as the other two. Therefore, our approaches are able to give very good approximation to the true results in this example.

Table 1. $t^2(\alpha)$, $t^2_{app}(\alpha)$ and $t^2_{df}(\alpha)$ when $p = 4$

Sample Size		$\alpha = 0.05$						$\alpha = 0.01$					
$N^{(1)}$	$N^{(2)}$	$t^2(\alpha)$	$t^2_{app}(\alpha)$	$t^2_{df}(\alpha)$	α_1	α_2	α_3	$t^2(\alpha)$	$t^2_{app}(\alpha)$	$t^2_{df}(\alpha)$	α_1	α_2	α_3
$(r, s) = (1, 3)$, $\chi^2_1(0.05) = 3.84$, $\chi^2_1(0.01) = 6.64$													
20	20	4.48	4.93	5.05	0.068	0.040	0.038	8.06	8.91	8.79	0.018	0.007	0.008
40	40	4.13	4.31	4.42	0.058	0.045	0.043	7.27	7.58	6.64	0.014	0.009	0.008
100	100	3.94	4.01	4.07	0.053	0.048	0.047	7.15	7.37	7.05	0.013	0.009	0.009
200	200	3.89	3.92	3.95	0.052	0.049	0.049	6.84	6.98	6.84	0.011	0.009	0.010
400	400	3.88	3.88	3.90	0.051	0.050	0.050	6.76	6.80	6.74	0.011	0.010	0.010
20	40	4.23	4.49	4.63	0.062	0.044	0.041	7.55	7.97	8.07	0.015	0.008	0.008
40	80	4.04	4.14	4.22	0.056	0.047	0.045	7.05	7.23	7.34	0.012	0.009	0.008
100	200	3.92	3.95	3.99	0.052	0.049	0.048	6.80	6.86	6.91	0.011	0.010	0.009
200	400	3.88	3.90	3.92	0.051	0.050	0.049	6.71	6.75	6.77	0.010	0.010	0.010
$(r, s) = (2, 2)$, $\chi^2_2(0.05) = 5.99$, $\chi^2_2(0.01) = 9.21$													
20	20	7.10	7.57	7.83	0.077	0.042	0.038	11.45	12.23	12.23	0.022	0.008	0.008
40	40	6.50	6.67	6.89	0.062	0.046	0.042	10.20	10.49	10.73	0.015	0.009	0.008
100	100	6.19	6.24	6.34	0.055	0.049	0.047	9.60	9.68	9.80	0.012	0.010	0.009
200	200	6.07	6.11	6.16	0.052	0.049	0.048	9.38	9.44	9.50	0.011	0.010	0.010
400	400	6.03	6.05	5.99	0.051	0.049	0.049	9.30	9.32	9.21	0.010	0.010	0.010
20	40	6.68	6.94	5.50	0.067	0.045	0.040	10.63	11.01	11.25	0.018	0.009	0.008
40	80	6.32	6.43	5.99	0.058	0.048	0.044	9.86	10.02	9.21	0.013	0.009	0.008
100	200	6.12	6.16	5.99	0.053	0.049	0.048	9.46	9.52	9.21	0.011	0.010	0.010
200	400	6.05	6.07	6.11	0.051	0.049	0.049	9.32	9.36	9.41	0.011	0.010	0.010
$(r, s) = (3, 1)$, $\chi^2_3(0.05) = 7.82$, $\chi^2_3(0.01) = 11.35$													
20	20	9.37	9.67	10.12	0.085	0.045	0.039	14.38	14.81	15.02	0.026	0.009	0.008
40	40	8.53	8.63	8.95	0.066	0.048	0.042	12.68	12.83	13.22	0.017	0.009	0.008
100	100	8.09	8.12	8.26	0.056	0.049	0.046	11.84	11.89	12.08	0.012	0.010	0.009
200	200	7.94	7.96	7.82	0.053	0.049	0.048	11.59	11.61	11.35	0.011	0.010	0.009
400	400	7.87	7.89	7.92	0.051	0.050	0.049	11.43	11.48	11.53	0.010	0.010	0.010
20	40	8.77	8.94	7.82	0.071	0.047	0.040	13.14	13.43	11.35	0.019	0.009	0.008
40	80	8.27	8.36	8.56	0.060	0.049	0.045	12.18	12.29	12.58	0.014	0.010	0.008
100	200	7.99	8.01	7.82	0.054	0.050	0.047	11.66	11.70	11.35	0.012	0.010	0.009
200	400	7.91	7.91	7.96	0.052	0.050	0.049	11.54	11.52	11.59	0.011	0.010	0.010

Table 2. $t^2(\alpha)$, $t^2_{\text{app}}(\alpha)$ and $t^2_{\text{df}}(\alpha)$ when $p = 8$

Sample Size		$\alpha = 0.05$						$\alpha = 0.01$					
$N^{(1)}$	$N^{(2)}$	$t^2(\alpha)$	$t^2_{\text{app}}(\alpha)$	$t^2_{\text{df}}(\alpha)$	α_1	α_2	α_3	$t^2(\alpha)$	$t^2_{\text{app}}(\alpha)$	$t^2_{\text{df}}(\alpha)$	α_1	α_2	α_3
$(r, s) = (2, 6)$, $\chi^2_2(0.05) = 5.99$, $\chi^2_2(0.01) = 9.21$													
20	20	8.13	10.29	-	0.103	0.025	-	13.36	16.93	-	0.035	0.004	-
40	40	6.87	7.53	7.37	0.072	0.038	0.041	10.82	11.87	11.29	0.019	0.007	0.009
100	100	6.31	6.52	6.54	0.058	0.045	0.045	9.75	10.11	10.07	0.013	0.009	0.009
200	200	6.13	6.24	6.26	0.054	0.047	0.048	9.49	9.64	9.64	0.011	0.009	0.009
400	400	6.06	6.11	6.13	0.052	0.049	0.049	9.35	9.42	9.43	0.011	0.010	0.010
20	40	7.24	8.26	-	0.082	0.034	-	11.55	13.17	-	0.024	0.006	-
40	80	6.53	6.93	6.91	0.064	0.042	0.042	10.17	10.82	10.64	0.015	0.008	0.008
100	200	6.19	6.33	6.36	0.055	0.047	0.047	9.60	9.79	9.79	0.012	0.009	0.009
200	400	6.08	6.16	6.17	0.052	0.048	0.048	9.38	9.49	9.50	0.011	0.009	0.009
$(r, s) = (4, 4)$, $\chi^2_4(0.05) = 9.49$, $\chi^2_4(0.01) = 13.28$													
20	20	13.16	15.36	-	0.129	0.028	-	19.69	23.11	-	0.048	0.005	-
40	40	11.00	11.68	11.55	0.082	0.040	0.042	15.80	16.82	16.20	0.023	0.007	0.009
100	100	10.02	10.25	10.33	0.061	0.046	0.045	14.15	14.49	14.53	0.014	0.009	0.009
200	200	9.75	9.85	9.91	0.550	0.048	0.047	13.74	13.85	13.90	0.012	0.010	0.009
400	400	9.61	9.67	9.70	0.052	0.049	0.048	13.52	13.56	13.59	0.011	0.010	0.009
20	40	11.62	12.68	-	0.096	0.036	-	16.92	18.50	-	0.030	0.006	-
40	80	10.43	10.84	10.89	0.067	0.043	0.043	14.89	15.44	15.32	0.018	0.008	0.009
100	200	9.84	9.98	8.23	0.057	0.047	0.046	13.91	14.06	14.11	0.013	0.009	0.009
200	400	9.65	9.73	9.77	0.530	0.048	0.048	13.56	13.66	13.70	0.011	0.010	0.009
$(r, s) = (6, 2)$, $\chi^2_6(0.05) = 12.59$, $\chi^2_6(0.01) = 16.81$													
20	20	17.79	19.12	-	0.152	0.038	-	25.49	27.51	-	0.062	0.007	-
40	40	14.71	15.15	15.12	0.091	0.044	0.044	20.27	20.86	20.32	0.027	0.008	0.010
100	100	13.35	13.50	13.65	0.064	0.048	0.046	17.98	18.22	18.35	0.015	0.009	0.009
200	200	12.95	13.03	13.12	0.057	0.049	0.047	17.38	17.49	17.59	0.011	0.010	0.009
400	400	12.80	12.80	12.94	0.054	0.050	0.048	17.07	17.14	17.33	0.010	0.010	0.010
20	40	15.59	16.27	-	0.109	0.042	-	21.74	22.69	-	0.036	0.008	-
40	80	13.93	14.18	14.33	0.076	0.046	0.044	18.95	19.30	19.31	0.020	0.009	0.009
100	200	13.10	13.18	13.30	0.059	0.049	0.047	17.57	17.72	17.85	0.013	0.009	0.009
200	400	12.80	12.80	12.86	0.054	0.050	0.049	17.20	17.26	17.20	0.012	0.010	0.010

Table 3. 95% Simultaneous confidence intervals

$\boldsymbol{a}$	$(1,0,0)'$	$(0,1,0)'$	$(0,0,1)'$
$\boldsymbol{a}'(\mu_{1\cdot2}^{(1)} - \mu_{1\cdot2}^{(2)})$	$\mu_1^{(1)} - \mu_1^{(2)}$	$\mu_2^{(1)} - \mu_2^{(2)}$	$\mu_3^{(1)} - \mu_3^{(2)}$
mean	0.700	1.292	0.144
$t^2(0.05)$	$(0.469, 0.727)$	$(0.648, 0.991)$	$(-0.261, 0.241)$
$t^2_{\text{app}}(0.05)$	$(0.469, 0.726)$	$(0.648, 0.991)$	$(-0.262, 0.243)$
$t^2_{\text{df}}(0.05)$	$(0.467, 0.730)$	$(0.645, 0.994)$	$(-0.266, 0.247)$

Acknowledgements

The authors would like to thank the editorial team and referees for useful comments and suggestions. The first and third author's research was in part supported by Grant-in-Aid for Young Scientists (18K18014) and Scientific Research (C) (17K00058).

References

Eaton, M. L., & Kariya, T. (1983). Multivariate tests with incomplete data. *The Anal. Stat., 11*(2), 654-665. https://doi.org/10.1037/0033-2909.126.6.910

Fisher, R. A. (1936). The use of multiple measurements in taxonomic problems. *Ann. Eugen., 7*, 179-188. https://doi.org/10.1111/j.1469-1809.1936.tb02137.x

Fujikoshi, Y., Ulyanov, V. V., & Shimizu, R. (2010). *Multivariate Statistics: High-Dimensional and Large-Sample Approximations*. Hoboken, NJ: Wiley.

Gupta, A. K., Xu, J., & Fujikoshi, Y. (2006), An asymptotic expansion of the distribution of Rao's U-statistic under a general condition. *J. Multivariate Anal., 97*, 492-513. https://doi.org/10.1016/j.jmva.2005.03.012

Kawasaki, T., & Seo, T. (2016). A test for subvector of mean vector with two-step monotone missing data. *SUT J. Math., 52*, 21-39.

Naito, T., Kawasaki, T., & Seo, T. (2018). T^2 type test statistic and simultaneous confidence intervals for sub-mean vectors in k-sample problem. *Technical Report, No.18-03, Hiroshima Statistical Research Group, Hiroshima University*.

Provost, S. B. (1990). Estimators for the parameters of a multivariate normal random vector with incomplete data on two subvectors and test of independence. *Comput. Statist. Data Anal., 9*, 37-46. https://doi.org/10.1016/0167-9473(90)90069-T

Rao, C. R. (1949). On some problems arising out of discrimination with multiple characters. *Sankhyā, 9*, 343-364.

Rencher, A. C. (2012). *Methods of Multivariate Analysis*. Hoboken, NJ: Wiley. https://doi.org/10.1002/9781118391686

Siotani, M., Hayakawa, T., & Fujikoshi, Y. (1985). *Modern Multivariate Statistical Analysis: A Graduate Course and Handbook*. American Science Press Inc., Ohio.