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New Families of Generalized Lomax Distributions: Properties and Applications

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Abstract

In this paper, we propose new families of generalized Lomax distributions named T-Lomax$\{Y\}$. Using the methodology of the Transformed-Transformer, known as T-X framework, the T-Lomax families introduced are arising from the quantile functions of exponential, Weibull, log-logistic, logistic, Cauchy and extreme value distributions. Various structural properties of the new families are derived including moments, modes and Shannon entropies. Several new generalized Lomax distributions are studied. The shapes of these T-Lomax$\{Y\}$ distributions are very flexible and can be symmetric, skewed to the right, skewed to the left, or bimodal. The method of maximum likelihood is proposed for estimating the distributions parameters and a simulation study is carried out to assess its performance. Four applications of real data sets are used to demonstrate the flexibility of T-Lomax$\{Y\}$ family of distributions in fitting unimodal and bimodal data sets from different disciplines.

Keywords: Lomax distribution, Shannon entropy, quantile function, moment, *T-X* family

1. Introduction

The Lomax distribution, also known as Pareto type-II distribution, is one of the important continuous distributions with a heavy tail defined by one shape and one scale parameters. The Lomax distribution was first used in Lomax (1954) to analyze business failure data. After that, researchers turned to using the Lomax distribution extensively in applications in the different fields of sciences; this included but was not limited to modeling business records by Atkinson and Harrison (1978), reliability and life testing studies by Hassan and Al-Ghamdi (2009). Bryson (1974) suggested the use of the Lomax distribution as an alternative to exponential, gamma and Weibull distributions. Other applications of the Lomax distribution can be found in modeling heavy tailed data in wealth, income, business and biological sciences.

Over the past decade, many researchers have studied Lomax distribution properties and applications in depth to have a better understanding of its modelling capabilities. Other researchers used several methodologies for generating generalized families of distributions to introduce extensions and generalizations of the two parameters Lomax distribution. The different generalizations vary between adding one, two, or three extra parameters to increase Lomax flexibility to model the different shapes of real-world data.

The development of several methodologies for generating generalized families of distributions initiated the interest in extending the Lomax distribution. Some of the known techniques to generalize distributions in the recent decades are introduced by Marshall and Olkin (1997), Eugene, Lee and Famoye (2002), Shaw and Buckley (2009), Alzaatreh, Lee, and Famoye (2013), and Alzaghal, Famoye, and Lee (2013). Generalization examples of the Lomax distribution include the Multivariate Lomax distribution of mixing exponential variables, which was introduced by Nayak (1987), and the beta-Lomax as defined by Rajab, Aleem, Nawaz, and Daniyal (2013). Using Zografos and Balakrishnans technique (2009), gamma Lomax distribution was proposed by Cordeiro, Ortega, and Popović (2015). Additionally, the McDonald Lomax distribution was proposed by Lemonte and Cordeiro (2013), the Weibull-Lomax was introduced by Alzaghal, Ghosh and Alzaatreh (2016), and another Lomax extension was defined by Mead (2016) who investigated the beta exponentiated Lomax. More recently, the Gompertz Lomax distribution with increasing, decreasing and constant failure rates was developed by Oguntunde, Khaleel, Ahmed, Adejumo, and Odetunmibi (2017).

The probability density function (PDF) of the Lomax distribution is given by

$$f(x;\alpha,\lambda) = (\alpha/\lambda)(1 + x/\lambda)^{-\alpha-1}, \ x > 0. \tag{1}$$

where $\alpha > 0$ is a shape parameter and $\lambda > 0$ is a scale parameter. The cumulative distribution function (CDF) corresponding to Equation (1) is

$$F(x; \alpha, \lambda) = 1 - (1 + x/\lambda)^{-\alpha}, \ x > 0. \tag{2}$$

Alzaatreh et al. (2013) developed a new class, named the *T-X* family, of distributions as an extension to the beta-generated family of distributions proposed by Eugene et al. (2002). Let T be a continuous random variable such that $T \in [a, b]$, $-\infty \leq a < b \leq \infty$. The CDF of the *T-X* family of distributions is given as

$$G(x) = \int_a^{W(F(x))} r(t)dt = R\{W(F(x))\},$$

where $R(t)$ is the CDF of the random variable T. The $W(F(x))$ is a monotonic and absolutely continuous function of the CDF $F(x)$ of any random variable X. Several $W(F(x))$ functions were defined in Alzaatreh et al. (2013). Later, Alzaatreh, Lee, and Famoye (2014) gave a unified notation for the definition of the *T-X* family and named it the *T-R*{*Y*} family. The definition of the *T-R*{*Y*} family is as following: Let T, R, and Y be random variables with the CDFs $F_T(x) = P(T \leq x)$, $F_R(x) = P(R \leq x)$, and $F_Y(x) = P(Y \leq x)$ where the PDFs of the T, R, and Y are $f_T(x), f_R(x)$, and $f_Y(x)$, respectively. Also, define the quantile function as $Q_Z(p) = \inf\{z : F_Z(z) \geq p\}$, $0 < p < 1$. Then, the corresponding quantile function for the random variable Y is $Q_Y(p)$. If X is a random variable that follows the *T-R*{*Y*} family of distribution, then the CDF and the PDF of the random variable X are respectively defined as

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = F_T\left(Q_Y\left(F_R(x)\right)\right), \text{and} \tag{3}$$

$$f_X(x) = f_R(x) \times \frac{f_T\left(Q_Y\left(F_R(x)\right)\right)}{f_Y\left(Q_Y\left(F_R(x)\right)\right)}. \tag{4}$$

The *T-R*{*Y*} framework is a broad methodology for developing flexible generalized distributions. For example, the *T*-normal{*Y*} family of distributions studied by Alzaatreh et al. (2014) is a family of generalized normal distributions. The *T*-Pareto{*Y*} family of distributions studied by Hamed, Famoye and Lee (2018) defined new families of generalized Pareto distributions.

This article proposes new families of generalized Lomax distributions by using the *T-R*{*Y*} methodology. The motivation of this initiative is to improve the flexibility of the known right tailed Lomax distribution to fit a variety of shapes of data including unimodal (left skewed or symmetric), as well as bimodal from different disciplines. The strength of this generalization is shown consistently by providing better fits than other generated distributions having the same or higher number of parameters.

The rest of the article is outlined as follows: Section 2 introduces new families of Lomax distribution named the *T*-Lomax{*Y*}. In Section 3, some structural properties of the proposed families are discussed. Some new members of these new families are studied in Section 4. A simulation analysis to study the performance of the Maximum Likelihood Estimation (MLE) method in estimating the parameters for the Normal-Lomax{Cauchy} distribution is presented in Section 5. The usefulness of the new proposed families is illustrated through applications to real data sets in Section 6. Lastly, Section 7 provides a summary for this article.

2. Some *T*-Lomax{*Y*} Families of Distributions

In this section, six different closed form quantile functions are used to define six generalized *T*-Lomax{*Y*} families of distributions. Table 1 lists the different quantiles used in this paper and the domain of the corresponding T random variable that can be combined with each one of them.

Table 1. Some quantile functions of Y and the domains of T

	Random variable Y	The quantile function $Q_Y(p)$	Domain of T
(i)	Exponential	$-\log(1-p)$	$(0, \infty)$
(ii)	Weibull	$\gamma(-\log(1-p))^{1/k}, \gamma, k > 0$	$(0, \infty)$
(iii)	Log-logistic	$[p/(1-p)]^{1/\beta}, \beta > 0$	$(0, \infty)$
(iv)	Logistic	$\gamma \log[p/(1-p)], \gamma > 0$	$(-\infty, \infty)$
(v)	Cauchy	$\tan(\pi(p-0.5))$	$(-\infty, \infty)$
(vi)	Extreme value	$\log(-\log(1-p))$	$(-\infty, \infty)$

The CDF and PDF of each of the *T*-Lomax{*Y*} families of distributions resulting of these quantiles are derived using Equations (3) and (4), respectively, and are defined as following:

(i) The T-Lomax{exponential} family of distributions: By using the quantile function of exponential distribution given in Table 1, $Q_Y(p) = -\log(1-p)$, in equations (3) and (4). The CDF and PDE of the T-Lomax{exponential} family are respectively given by

$$F_X(x) = F_T\{-\log(1-F_R(x))\} = F_T\{\alpha\log(1+x/\lambda)\}, \text{and} \tag{5}$$

$$f_X(x) = \frac{f_R(x)}{1-F_R(x)} f_T\{-\log(1-F_R(x))\} = (\alpha/\lambda)(1+x/\lambda)^{-1} f_T\{\alpha\log(1+x/\lambda)\},$$

where $F_R(x)$ and $f_R(x)$ are the CDF and PDF of Lomax random variable given in Equations (1) and (2). The CDF and PDF of the T-Lomax{exponential} family can be written as $F_X(x) = F_T(H_R(x))$ and $f_X(x) = h_R(x)f_T(H_R(x))$, where $h_R(x)$ and $H_R(x)$ are the hazard and cumulative hazard functions of the Lomax distribution, respectively. This shows that the T-Lomax{exponential} family of distributions arises from the hazard function of the Lomax distribution. A member of this family, the Weibull-Lomax{exponential} distribution, was defined and studied by Alzaghal et al. (2016) by taking T to be a Weibull random variable.

(ii) The T-Lomax{Weibull} family of distributions: By using the quantile function of the Weibull distribution given in Table 1, $Q_Y(p) = \gamma(-\log(1-p))^{1/k}$, in Equations (3) and (4).The CDF and PDE of the T-Lomax{Weibull} family are respectively given by

$$\begin{aligned}F_X(x) &= F_T\left\{\gamma[-\log(1-F_R(x))]^{1/k}\right\} = F_T\left\{\gamma[\alpha\log(1+x/\lambda)]^{1/k}\right\}, \text{and}\\ f_X(x) &= \frac{\lambda f_R(x)[-\log(1-F_R(x))]^{(1-k)/k}}{k(1-F_R(x))} f_T\left\{\lambda[-\log(1-F_R(x))]^{1/k}\right\}\\ &= (\alpha\gamma/\lambda k)(1+x/\lambda)^{-1}[\alpha\log(x/\lambda)]^{(1-k)/k} f_T\left\{\gamma[\alpha\log(x/\lambda)]^{1/k}\right\}.\end{aligned}$$

Note that when $k = 1$, the T-Lomax{Weibull} family of distributions is reduced to the T-Lomax{exponential} family of distributions.

(iii) The T-Lomax{log-logistic} family of distributions: By using the quantile function of log-logistic distribution given in Table 1, $Q_Y(p) = [p/(1-p)]^{1/\beta}$, in (3) and (4). The CDF and PDE of the T-Lomax{log-logistic} family are respectively given by

$$F_X(x) = F_T\left\{[F_R(x)/(1-F_R(x))]^{1/\beta}\right\} = F_T\left\{[(1+x/\lambda)^{\alpha}-1]^{1/\beta}\right\}, \text{and} \tag{6}$$

$$\begin{aligned}f_X(x) &= \frac{f_R(x)}{\beta F_R{}^{(\beta-1)/\beta}(x)(1-F_R(x))^{(\beta+1)/\beta}} f_T\left\{[F_R(x)/(1-F_R(x))]^{1/\beta}\right\}\\ &= \alpha(\lambda\beta)^{-1}(1-(1+x/\lambda)^{-\alpha})^{(1-\beta)/\beta}(x/\lambda)^{(\alpha-\beta)/\beta} f_T\left\{[(x/\lambda)^{\alpha}-1]^{1/\beta}\right\}.\end{aligned} \tag{7}$$

If $\beta = 1$, then the T-Lomax{log-logistic} is consider a family arising from the odds of Lomax distribution with the following PDF:

$$f_X(x) = (\alpha/\lambda)(x/\lambda)^{(\alpha-1)} f_T\{[(x/\lambda)^{\alpha}-1]\}.$$

(iv) The T-Lomax{logistic} family of distributions: By using the quantile function of logistic distribution given in Table 1, $Q_Y(p) = \gamma\log[p/(1-p)]$, in Equations (3) and (4). The CDF and PDE of the T-Lomax{logistic} family are respectively given by

$$\begin{aligned}F_X(x) &= F_T\{\gamma\log[F_R(x)/(1-F_R(x))]\} = F_T\{\gamma\log[(1+x/\lambda)^{\alpha}-1]\}, \text{and}\\ f_X(x) &= \frac{\lambda f_R(x)}{F_R(x)(1-F_R(x))} f_T\{\gamma\log[F_R(x)/(1-F_R(x))]\}\\ &= \alpha\gamma\lambda^{-1}(1+x/\lambda)^{-1}(1-(1+x/\lambda)^{-\alpha})^{-1} f_T\{\gamma\log[(1+x/\lambda)^{\alpha}-1]\}.\end{aligned}$$

When $\gamma = 1$, the T-Lomax{logistic} family of distributions is considered a family arising from the logit function of the Lomax distribution.

(v) The T-Lomax{Cauchy} family of distributions: By using the quantile function of Cauchy distribution given in Table 1, $Q_Y(p) = \tan(\pi(p - 0.5))$, in Equations (3) and (4). The corresponding CDF and PDE of the T-Lomax{Cauchy} family are respectively given by

$$F_X(x) = F_T\{\tan[\pi(F_R(x) - 0.5)]\} = F_T\{\tan[\pi(0.5 - (1 + x/\lambda)^{-\alpha})]\}, \text{ and}$$
$$f_X(x) = \pi f_R(x)\sec^2[\pi(F_R(x) - 0.5)] f_T\{\tan[\pi(F_R(x) - 0.5)]\}$$
$$= \alpha\gamma\lambda^{-1}(1 + x/\lambda)^{-1}(1 - (1 + x/\lambda)^{-\alpha})^{-1} f_T\{\gamma\log[(1 + x/\lambda)^{\alpha} - 1]\}. \quad (8)$$

(vi) The T-Lomax{extreme value} family of distributions: By using the quantile function of extreme value distribution in Table 1, $Q_Y(p) = \log(-\log(1 - p))$, in Equations (3) and (4). The CDF and PDE of the T-Lomax{extreme value} family are respectively given by

$$F_X(x) = F_T\{\log[-\log(1 - F_R(x))]\} = F_T\{\log[\alpha\log(1 + x/\lambda)]\}, \text{ and}$$
$$f_X(x) = \frac{f_R(x)}{-[1 - F_R(x)]\log[1 - F_R(x)]} f_T\{\log[-\log(1 - F_R(x))]\}$$
$$= \lambda^{-1}(1 + x/\lambda)^{-1}(\log(1 + x/\lambda))^{-1} f_T\{\log[\alpha\log(1 + x/\lambda)]\}.$$

3. Some Properties of the T-Lomax{Y} Family of Distributions

In this part of the paper, various statistical properties of the proposed T-Lomax{Y} families of distributions are investigated.

Lemma 1 (Transformation) *Consider any random variable T with PDF $f_T(x)$, then the random variable*

$$X = Q_R(F_Y(T)) = \lambda\left\{(1 - F_Y(T))^{-1/\alpha} - 1\right\},$$

where $Q_R(\cdot)$ is the quantile function of the Lomax distribution, follows the T-Lomax{Y} distribution.

Corollary 1 *Based on Lemma 1, we have*

(i) $X = \lambda\left\{e^{T/\alpha} - 1\right\}$ *follows the distribution of T-Lomax{exponential} family.*

(ii) $X = \lambda\left\{e^{(T/\gamma)^k/\alpha} - 1\right\}$ *follows the distribution of T-Lomax{Weibull} family.*

(iii) $X = \lambda\left\{\left(1 + T^{\beta}\right)^{1/\alpha} - 1\right\}$ *follows the distribution of T-Lomax{log-logistic} family.*

(iv) $X = \lambda\left\{e^{T/\alpha} - 1\right\}$ *follows the distribution of T-Lomax{logistic} family.*

(v) $X = \lambda\left\{e^{T/\alpha} - 1\right\}$ *follows the distribution of T-Lomax{Cauchy} family.*

(vi) $X = \lambda\left\{e^{T/\alpha} - 1\right\}$ *follows the distribution of T-Lomax{extreme value} family.*

If X follows the T-Lomax{Y} distribution, then Corollary 1 can be used to generate a random sample of the random variable X using a specific random variable T and a specific quantile function $Q_Y(p)$.

Lemma 2 (Quantiles) *Let $Q_X(p)$, $0 < p < 1$, denote a quantile function of the random variable X. Then, the quantile function for a specific T-Lomax{Y} family of distributions is given by*

$$Q_X(p) = Q_R\{F_Y(Q_T(p)\} = \lambda\left\{(1 - F_Y(Q_T(p))^{-1/\alpha} - 1\right\}.$$

Corollary 2 *Based on Lemma 2, the quantile functions for the (i) T-Lomax{exponential}, (ii) T-Lomax{Weibull}, (iii) T-Lomax{log-logistic}, (iv) T-Lomax{logistic}, (v) T-Lomax{Cauchy}, and (vi) T-Lomax{extreme value} distributions, respectively, are given by*

(i) $Q_X(p) = \lambda\left\{e^{(Q_T(p)/\alpha)} - 1\right\}$,

(ii) $Q_X(p) = \lambda\left\{e^{([Q_T(p)/\gamma]^k/\alpha)} - 1\right\}$,

(iii) $Q_X(p) = \lambda\left\{\left((Q_T(p))^{\beta} + 1\right)^{1/\alpha} - 1\right\}$,

(iv) $Q_X(p) = \lambda\left\{\left(e^{(Q_T(p)/\gamma)} + 1\right)^{1/\alpha} - 1\right\}$,

(v) $Q_X(p) = \lambda\left\{(0.5 - \arctan(Q_T(p))/\pi)^{-1/\alpha} - 1\right\}$, *and*

(vi) $Q_X(p) = \lambda\left\{e^{(e^{Q_T(p)}/\alpha)} - 1\right\}$.

Theorem 1 *The mode(s) of the T-Lomax{Y} family are the solutions of*

$$(x+\lambda)\Psi\left\{f_T\left(Q_y(F_R(x))\right)\right\} = \alpha + 1 + (x+\lambda)\Psi\left\{f_y\left(Q_y(F_R(x))\right)\right\},$$

where $\Psi(f)=f'/f$.

Proof. The derivative of $f_X(x)$ with respect to x can be written as $f'_X(x) = f_X(x)R(x)$, where $R(x) = -(\alpha+1)/(x+\lambda) + \Psi\left\{f_T\left(Q_y(F_R(x))\right)\right\} - \Psi\left\{f_y\left(Q_y(F_R(x))\right)\right\}$. The mode(s) of $f_X(x)$ can be obtained by setting $R(x) = 0$, and then solving for x.

Corollary 3 *Based on Theorem 1, the mode(s) of the (i) T-Lomax{exponential}, (ii) T-Lomax{Weibull}, (iii) T-Lomax{log-logistic}, (iv) T-Lomax{logistic}, (v) T-Lomax{Cauchy}, and (vi) T-Lomax{extreme value} distributions, respectively, are the solutions of the equations*

(i) $(x+\lambda)\Psi\{f_T(\alpha\log(1+x/\lambda))\} = 1$,

(ii) $(x+\lambda)\Psi\left\{f_T\left(\gamma\{\alpha\log(1+x/\lambda)\}^{1/k}\right)\right\} = 1 + 2\alpha - (1-1/k)\{1/\log(1+x/\lambda)\}$,

(iii) $(x+\lambda)\Psi\left\{f_T((1+x/\lambda)^{\alpha} - 1)^{1/\beta}\right\} = \alpha + 1 - \frac{\alpha(\beta+1)(1+x/\lambda)^{\alpha} - 2\alpha\beta}{\beta((1+x/\lambda)^{\alpha} - 1)}$,

(iv) $(x+\lambda)\Psi\left\{f_T\left(\tan\left\{\pi\left(0.5 - (1+x/\lambda)^{-\alpha}\right)\right\}\right)^{1/\beta}\right\} = \alpha + 1 - 2\pi\alpha(1+x/\lambda)^{-\alpha}\cot\left(\pi(1+x/\lambda)^{-\alpha}\right)$,

(v) $(x+\lambda)\Psi\{f_T(\gamma\log\{(1+x/\lambda)^{\alpha} - 1\})\} = \alpha + 1 + \frac{\alpha\{-2+(1+x/\lambda)^{\alpha}\}}{-1+(1+x/\lambda)^{\alpha}}$, *and*

(vi) $(x+\lambda)\Psi\{f_T(\log\{-\log((1+x/\lambda)^{-\alpha})\})\} = 1 + 1/\log(1+x/\lambda)$.

The Normal-Lomax{Cauchy} distribution provided in Section 4 is an example of a bimodal distribution, which means that Corollary 3 (v) could have more than one solution to represent a bimodal distribution.

The entropy of a random variable X is a measure of variation of uncertainty. Entropy has several applications in engineering, chemistry, physics, and information theory. The Shannons entropy (Shannon, 1948) for a continuous random variable X with PDF $f(x)$ is defined as $\eta_X = E[-\log f(x)]$.

Theorem 2 *The Shannon entropy for the T-Lomax{Y} family of distributions in Equation* (4) *is given by*

$$\eta_X = \eta_T + E(\log f_Y(T)) + \log(\lambda/\alpha) + (\alpha+1)E(\log(1+X/\lambda)),$$

where η_T *is the Shannon entropy for the random variable T.*

Proof. By the definition of the Shannon entropy,

$$\eta_X = E(-\log[f_X(X)]) = E\left\{-\log f_T\left(Q_y\{F_R(X)\}\right)\right\} + E\left\{\log f_Y\left(Q_y\{F_R(X)\}\right)\right\} - E\{\log f_R(X)\}.$$

Since the random variable $T = Q_y\{F_R(X)\}$ for the T-Lomax{Y} family, we have

$E\left\{-\log f_T\left(Q_y\{F_R(X)\}\right)\right\} = E\{-\log f_T(T)\} = \eta_T$, and $E\left\{\log f_Y\left(Q_y\{F_R(X)\}\right)\right\} = E\{\log f_Y(T)\}$.

Now, $\log(f_R(x)) = \log(\alpha/\lambda) - (\alpha+1)\log(1+x/\lambda)$, which gives

$E\{\log f_R(X)\} = \log(\alpha/\lambda) - (\alpha+1)E(\log(1+X/\lambda))$,

Hence, $\eta_X = \eta_T + E(\log f_Y(T)) + \log(\lambda/\alpha) + (\alpha+1)E(\log(1+X/\lambda))$.

Corollary 4. *Based on Theorem 2, The Shannon entropies for the (i) T-Lomax{exponential}, (ii) T-Lomax{Weibull}, (i-ii) T-Lomax{log-logistic}, (iv) T-Lomax{logistic}, (v) T-Lomax{Cauchy}, and (vi) T-Lomax{extreme value} distributions, respectively, are given by*

(i) $\eta_X = \eta_T + \log(\lambda/\alpha) + (\mu_T/\alpha)$,

(ii) $\eta_X = \eta_T + \log(\lambda k/\alpha\gamma^k) + \alpha E\left(T^k\right)/\gamma^k + (k-1)E(\log T)$,

(iii) $\eta_X = \eta_T + \log(\beta\lambda/\alpha) + ((1-\alpha)/\alpha)\,E\left(\log\left(1+T^{\beta}\right)\right) + (\beta-1)E(\log T)$,

(iv) $\eta_X = \eta_T + \log(\lambda/\alpha\gamma) + ((1-\alpha)/\alpha)E\left(\log\left(e^{T/\gamma}+1\right)\right) + \mu_T/\gamma$,

(v) $\eta_X = \eta_T + \log(\lambda/\pi\alpha) - (1+1/\alpha)\,E\left(\log\left(0.5 - (\arctan(T)/\pi)\right)\right) - E\left(\log(T^2+1)\right)$, *and*

(vi) $\eta_X = \eta_T + \log(\lambda/\alpha) + E(e^T)/\alpha + \mu_T$.

Here, μ_T and η_T are the mean and the Shannon entropy for the random variable T.

Theorem 3 *Let X be a random variable that follows the T-Lomax{Y} family. Assume that $E(X^r) < \infty$ for all r, then*

$$E\left(X^r\right) \le \left[\lambda^r\,\gamma\left(\alpha - r\right)\gamma\left(1+r\right)/\gamma(\alpha)\right]E\left(\{\bar{F}_Y(T)\}^{-1}\right),$$

whenever $\alpha > r$.

Proof. If $f_R(x)$ is the PDF of a non-negative random variable R, then the r^{th} non-central moment of the random variable T-$R\{Y\}$ satisfies $E\left(X^r\right) \le E(R^r)E\left(\{\bar{F}_Y(T)\}^{-1}\right)$. (see Theorem 1, Aljarrah, Lee, and Famoye (2014)). The result follows using the fact that the r^{th} non-central moment of the Lomax distribution with parameters λ and α is

$$E\left(R^r\right) = \lambda^r\,\gamma\left(\alpha - r\right)\gamma\left(1+r\right)/\gamma(\alpha).$$

Using the upper bound provided in Theorem 3, Theorem 4 provides the r^{th} non-central moment for T-Lomax$\{Y\}$ family of distributions.

Theorem 4 *The r^{th} non-central moments for the T-Lomax{Y} family of distributions is given by*

$$E(X^r) = \lambda^r \sum_{n=0}^{r} a_n E\left[\left(1 - F_Y\left(T\right)\right)^{-n/\alpha}\right],$$

where $a_n = \binom{r}{n}(-1)^{r-n}$.

Proof. Using Lemma 1, $E\left(X^r\right) = E(Q_R(F_Y(T)))^r$ where $Q_R(p)$ is the quantile function of the Lomax distribution. By applying the generalized binomial expansion, $(Q_R(p))^r$ can be written as $(Q_R(p))^r = \lambda^r \sum_{n=0}^{r} a_n\, p^{-n/\alpha}$, where a_n defined in the statement of Theorem 4, which in turn implies the result.

Corollary 5. *Based on Theorem 4, the r^{th} non-central moments for the (i) T-Lomax{exponential}, (ii) T-Lomax{Weibull}, (iii) T-Lomax{log-logistic}, (iv) T-Lomax{logistic}, (v) T-Lomax{Cauchy}, and (vi) T-Lomax{extreme value} distributions, respectively, are given by*

(i) $E\left(X^r\right) = \lambda^r \sum_{n=0}^{r} a_n M_T(n/\alpha)$, exists if $M_T(n/\alpha) < \infty$,

(ii) $E\left(X^r\right) = \lambda^r \sum_{n=0}^{r} a_n M_{T^k}(n/\gamma^k\alpha)$, exists if $M_{T^k}(n/\gamma^k\alpha) < \infty$,

(iii) $E\left(X^r\right) = \lambda^r \sum_{n=0}^{r}\sum_{j=0}^{\infty} a_n \binom{n/\alpha}{j} E(T)^{\beta j}$, exists if $E(T)^{\beta j}$ exist.

(iv) $E\left(X^r\right) = \lambda^r \sum_{n=0}^{r}\sum_{j=0}^{\infty} a_n \binom{n/\alpha}{j} M_T(n/\gamma)$, exists if $M_T(n/\gamma) < \infty$.

(v) $E\left(X^r\right) = \lambda^r \sum_{n=0}^{r} (2)^{n/\alpha} \sum_{j=0}^{\infty} a_n \binom{-n/\alpha}{j} (-2/\pi)^j E(\arctan T)^j$, exists if $E(\arctan T)^j$ exist.

(vi) $E\left(X^r\right) = \lambda^r \sum_{n=0}^{r} a_n M_{e^T}(n/\alpha)$, exists if $M_{e^T}(n/\alpha) < \infty$, where $M_X(t) = E(e^{tX})$.

Theorem 5 *The mean deviation from the mean, D_μ, and the mean deviation from the median, D_M, for the T-Lomax{Y} family of distributions, respectively, are given by*

$$D_\mu = 2\mu\, F_T(Q_Y(F_R(\mu))) - 2I_\mu, and\ D_M = \mu - 2I_M,$$

where μ and M are the mean and median of the random variable X, and $I_q = \lambda \sum_{n=0}^{\infty} a_n \int_{-\infty}^{Q_Y(F_R(q))} f_T(u) \left(F_y(u)\right)^n du.$

Proof. For a nonnegative random variable X, we have

$D_\mu = E(|X-\mu|) = 2\mu\, F_X(\mu) - 2I_\mu$, and $D_M = E(|X-M|) = \mu - 2I_M$, where $I_q = \int_0^q x\, f_X(x)\, dx$.

From Equation (4) and Lemma 1, we have $I_q = \int_{-\infty}^{Q_Y(F_R(q))} f_T(u)\, Q_R(F_Y(u))\, du$. By using the series expansion of $Q_R(\cdot)$ we obtain the result in Theorem 5.

Corollary 6. *Based on Theorem 5, the I_qs for (i) T-Lomax{exponential}, (ii) T-Lomax{Weibull}, (iii) T-Lomax{log-logistic}, (iv) T-Lomax{logistic}, (v) T-Lomax{Cauchy}, and (vi) T-Lomax{extreme value} distributions, respectively, are*

(i) $I_q = \lambda \sum_{n=0}^{\infty}\sum_{j=0}^{n} a_n \binom{n}{j} (-1)^j S_{e^u}(q, 0, -j)$,
where $S_\xi(q, a, b) = \int_a^{Q_Y(F_R(q))} \xi^b f_T(u) du$, and $Q_Y(F_R(q)) = -\log(1 - F_R(q))$ for exponential distribution.

(ii) $I_q = \lambda \sum_{n=0}^{\infty}\sum_{j=0}^{n} a_n \binom{n}{j} (-1)^j S_{e^{(u/\gamma)^k}}(q, 0, -j)$,
where $Q_Y(F_R(q)) = \lambda(-\log(1 - F_R(q)))^{1/k}$ for Weibull distribution.

(iii) $I_q = \lambda \sum_{n=0}^{\infty}\sum_{j=0}^{n} a_n \binom{n}{j} (-1)^j S_{1+u^\beta}(q, 0, -j)$,
where $Q_Y(F_R(q)) = (F_R(q)/(1 - F_R(q)))^{1/\beta}$ for log-logistic distribution.

(iv) $I_q = \lambda \sum_{n=0}^{\infty} a_n S_{1+e^{-u/\gamma}}(q, -\infty, -j)$,
where $Q_Y(F_R(q)) = \gamma \log(F_R(q)/(1 - F_R(q)))$ for logistic distribution.

(v) $I_q = \lambda \sum_{n=0}^{\infty}\sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{2}\right)^{n-j} \frac{a_n}{\pi^j} S_{\arctan(u)}(q, -\infty, j)$,
where $Q_Y(F_R(c)) = \tan(\pi(F_R(c) - 0.5))$ for Cauchy distribution.

(vi) $I_q = \lambda \sum_{n=0}^{\infty} a_n S_{1+e^{-u/\gamma}}(q, -\infty, -j)$,
where $Q_Y(F_R(c)) = \log(-\log(1 - F_R(c)))$ for extreme value distribution.

Theorem 5 and Corollary 6 can be used to obtain the mean deviations for T-Lomax{exponential}, T-Lomax{Weibull}, T-Lomax{log-logistic}, T-Lomax{logistic}, T-Lomax{Cauchy}, and T-Lomax{extreme value} distributions.

4. Some New Generalized Lomax Distributions

Based on the general format of the different T-Lomax{Y} families of distributions provided in Section 2, the Weibull-Lomax{log-logistic}, the gamma-Lomax{log-logistic}, the Exponentiated Weibull-Lomax{exponential}, and the Normal-Lomax{Cauchy} distributions are investigated in this section.

4.1 The Weibull-Lomax{Log-logistic} Distribution

If the random variable T follows the Weibull distribution with the CDF $F_T(x) = 1 - e^{-x^k}$, where $x \geq 0$ and $k > 0$. Using Equation (6), the CDF of the Weibull-Lomax{log-logistic} (*W-L*{*LL*}) distribution can be defined as

$$F_X(x) = 1 - e^{-\{-1+(1+x/\lambda)^\alpha\}^{k/\beta}}.$$

Setting $c = k/\beta$ and using Equation (7), the PDF of the *W-L*{*LL*} distribution is given by

$$f_X(x) = \frac{\alpha c}{\lambda}(1 + x/\lambda)^{\alpha-1} \left\{-1 + (1 + x/\lambda)^\alpha\right\}^{c-1} e^{-\{-1+(1+x/\lambda)^\alpha\}^c},\ x > 0, \alpha, \lambda, c > 0.$$

When $\alpha = 1$, the *W-L*{*LL*} reduces to the Weibull distribution with the parameters c and λ, and when $\alpha = c = 1$, the *W-L*{*LL*} reduces to the exponential distribution with the parameter $1/\lambda$. In Figure 1, various plots of the *W-L*{*LL*} are provided for different values of the parameters α, λ and c. The graphs show that the *W-L*{*LL*} with the two shape parameters α and c can be unimodal with monotonically decreasing (reversed J-shape), skewed to the right, symmetric, or skewed to the left.

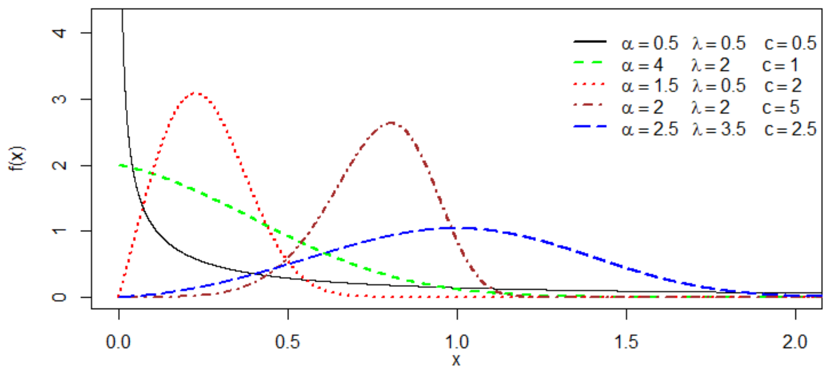

Figure 1. PDFs of *W-L*{*LL*} for various values of α, λ and c

Based on Section 2, some properties of *W-L*{*LL*} distribution are derived from the general properties of the T-Lomax{log-logistic} family as follows:

(i) The quantile function: By using Corollary 1 part (iii), the quantile function of the *W-L*{*LL*} distribution is given by

$$Q_X(p) = \lambda\left\{\left(\{-\log(1-p)\}^{1/c} + 1\right)^{1/\alpha} - 1\right\}.$$

(ii) The mode: By using Corollary 3 part (iii), the unique mode of *W-L*{*LL*} distribution is the solution of the following equation

$$(1 + x/\lambda)^{\alpha}\left\{\alpha c\left(-1 + (-1 + (1 + x/\lambda)^{\alpha})^{c}\right) + 1\right\} = (3\alpha + 1),$$

which can be evaluated numerically.

(iii) The moments: By using Corollary 5 part (iii), the r^{th} non-central moments of *W-L*{*LL*} distribution are given by

$$E\left(X^r\right) = \lambda^r \sum_{n=0}^{r} \sum_{j=0}^{\infty} a_n \begin{pmatrix} n/\alpha \\ j \end{pmatrix} \gamma\left(j/c + 1\right).$$

(iv) The mean deviations: By using Corollary 6 part (iii), the D_μ and the D_M of *W-L*{*LL*} distribution are given by

$$D_\mu = 2\mu\, F_T(Q_Y(F_R(\mu))) - 2I_\mu, \text{and } D_M = \mu - 2I_M,$$

where I_q is given by $I_q = \lambda \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{i=0}^{\infty} \frac{(-1)^j j^i \gamma^i}{i!} \begin{pmatrix} n \\ j \end{pmatrix} a_n\, \gamma\left[1 + i/k, \left(\frac{\alpha}{\gamma}\log(1 + q/\lambda)\right)^k\right]$, and $\gamma(\alpha, x) = \int_0^x u^{\alpha-1} e^{-u} du$ is the incomplete gamma function.

4.2 The Gamma-Lomax{Log-logistic} Distribution

Let the random variable T follow the gamma distribution with parameters c and θ with the PDF $f_T(x) = \frac{1}{\gamma(c)\theta^c} x^{c-1} e^{-x/\theta}$, $x > 0$. Using Equation (7), the PDF of the gamma-Lomax{log-logistic}(*G-L*{*LL*}) distribution, when setting the scale parameter $\theta = 1$ for the gamma distribution, is given by

$$f_X(x) = \frac{\alpha}{\beta\,\gamma(c)\lambda}(1 + x/\lambda)^{\alpha-1}(-1 + (1 + x/\lambda)^{\alpha})^{-1+c/\beta} e^{-(-1+(1+x/\lambda)^{\alpha})^{1/\beta}}, x > 0, \alpha, \lambda, \beta, c > 0.$$

When $\beta = \alpha = 1$, the *G-L*{*LL*} reduces to the gamma distribution with the parameters c and λ. Figure 2 provides graphs of the *G-L*{*LL*} PDF for various values of α, λ, c and β. These plots indicate that the *G-L*{*LL*} can be monotonically decreasing (reversed J-shape), skewed to left, symmetric, and also skewed to the right.

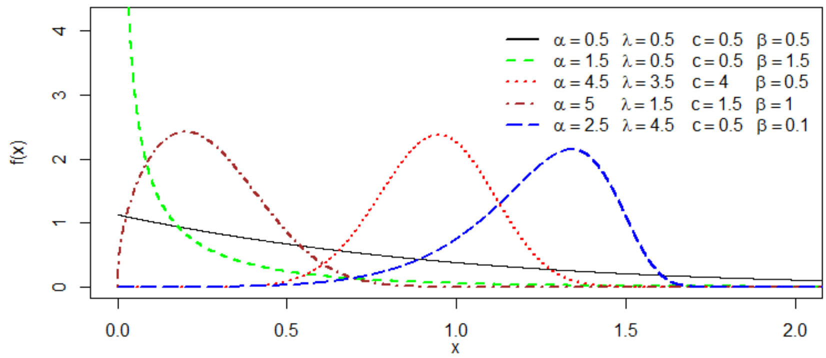

Figure 2. PDFs of G-$L\{LL\}$ for various values of α, λ, c and β

4.3 The Exponentiated Weibull-Lomax{Exponential} Distribution

Let the random variable T follow the Exponentiated Weibull distribution with parameters k, c and γ, the CDF of T is then $F_T(x) = \left(1 - e^{-(x/\gamma)^k}\right)^c$, where $x \geq 0$, $k, c, \gamma > 0$. Using Equation (5), the CDF of the Exponentiated Weibull-Lomax{exponential} (EW-$L\{E\}$) distribution is given by

$$F_X(x) = \left(1 - e^{-\left\{(\alpha/\gamma)\log(1+x/\lambda)\right\}^k}\right)^c.$$

By setting $\beta = \frac{\alpha}{\gamma}$, the PDF of the EW-$L\{E\}$ distribution is given by

$$f_X(x) = \frac{\beta k c}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-1}\left\{\beta \log\left(1 + \frac{x}{\lambda}\right)\right\}^{k-1}\left(1 - e^{-\left\{\beta\log(1+x/\lambda)\right\}^k}\right)^{c-1} e^{-\left\{\beta\log(1+x/\lambda)\right\}^k},\ x > 0, k, \lambda, \beta, c > 0.$$

When $c = 1$, we get the Weibull-Lomax{exponential} distribution as a sub-model of the EW-$L\{E\}$ distribution. When $c = k = 1$, the EW-$L\{E\}$ reduces to the Lomax distribution with β and λ. Figure 3 shows that EW-$L\{E\}$ PDF takes unimodal shapes that can be skewed to the right, skewed to left, symmetric, as well as monotonically decreasing.

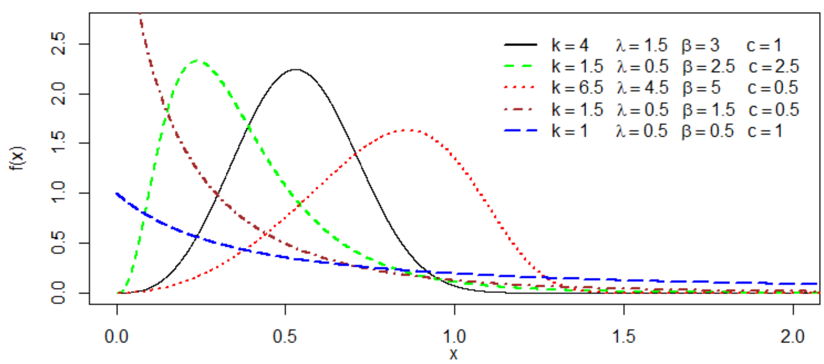

Figure 3. PDFs of EW-$L\{E\}$ for various values of k, λ, β and c

4.4 The Normal-Lomax{Cauchy} Distribution

If the random variable T follows the normal distribution with parameters μ and σ with the PDF $f_T(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}$, then using equation (8) the PDF of the normal-Lomax{Cauchy} (N-$L\{C\}$) distribution is defined as

$$f_X(x) = \frac{\sqrt{\pi}\alpha\sec^2(\pi[0.5 - (1 + x/\lambda)^{-\alpha}])}{\sqrt{2}\lambda\sigma(1 + x/\lambda)^{(\alpha+1)}} \exp\left(-(\tan(\pi[0.5 - (1 + x/\lambda)^{-\alpha}]) - \mu)^2/2\sigma^2\right),$$

where $x > 0$, $\sigma^2, \alpha, \lambda > 0$ and $\mu \in (-\infty, \infty)$.

In Figure 4 and 5, various plots of the N-$L\{C\}$ for various values of α, λ, μ and σ are provided. The plots show that N-$L\{C\}$ is flexible and can be unimodal; right skewed, symmetric, or left skewed. When $\sigma > 1$, the N-$L\{C\}$ distribution can be bimodal.

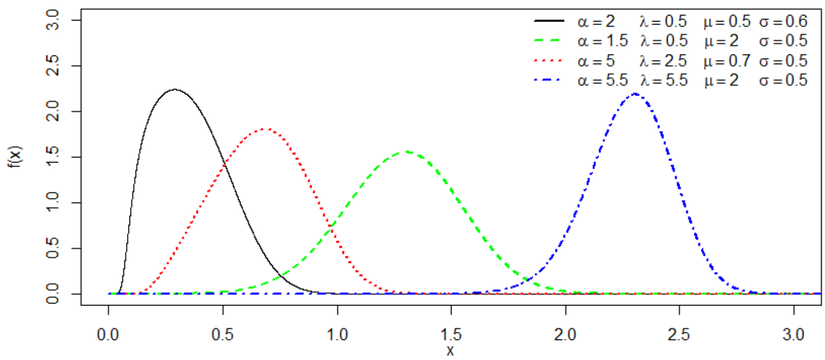

Figure 4. PDFs of N-$L\{C\}$ for various values of α, λ, μ and σ

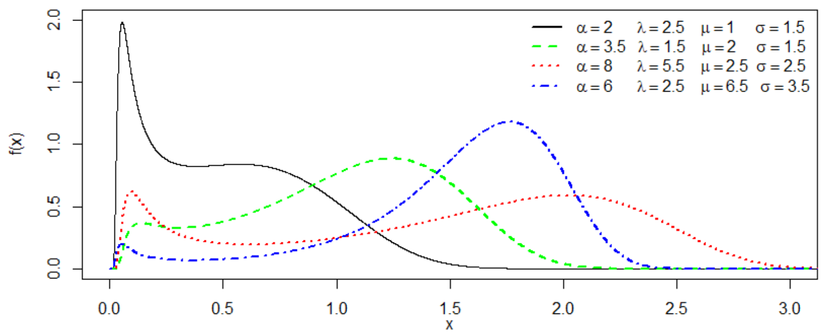

Figure 5. PDFs of N-$L\{C\}$ for various values of α, λ, μ and σ

5. Estimation and Simulation for the Parameters of the *N-L{C}* Distribution

Let X_1, X_2, ..., X_n be a random sample of size n drawn from the N-$L\{C\}$ distribution as defined in subsection 4.4. Let $\Theta = (\alpha, \lambda, \mu, \sigma)^T$ be a vector of parameters of dimension 4. By setting $z_i = (1 + x_i/\lambda)^{-\alpha}$, the corresponding log-likelihood function for Θ is given by

$$\ell(\Theta) = 0.5\, n \log(\pi/2) + n \log\alpha - n\log\lambda - n\log\sigma - n(\alpha+1)\sum\nolimits_{i=1}^{n} \log z_i + 2\sum\nolimits_{i=1}^{n} \log\left(\sec\pi(0.5 - z_i)\right)$$
$$-2^{-1}\sigma^{-2}\sum\nolimits_{i=1}^{n}\left(\tan\{\pi(0.5 - z_i)\} - \mu\right)^2.$$

And by setting $t_i = \log\left(z_i^{-1/\alpha}\right)\sec^2\left(\pi(0.5 - z_i)\right)(\mu - \tan(\pi(0.5 - z_i)))$, $u_i = \log\left(z_i^{-1/\alpha}\right)\tan(\pi(0.5 - z_i))$, $r_i = \tan(\pi(0.5 - z_i))$, $p_i = \sec^2\left(\pi(0.5 - p_i)\right)(\mu - \tan(\pi(0.5 - z_i)))$, and $q_i = \log\left(z_i^{-1/\alpha}\right)\sec^2(\pi\,(0.5 - z_i))$, the score vector

$$U(\Theta) = (U_\alpha = \partial\ell(\Theta)/\partial\alpha, U_\lambda = \partial\ell(\Theta)/\partial\lambda, U_\mu = \partial\ell(\Theta)/\partial\mu, U_\sigma = \partial\ell(\Theta)/\partial\sigma)^T$$

for the parameters α, λ, μ, and σ are derived analytically as

$$U_\alpha(\Theta) = \tfrac{n}{\alpha} - \sum\nolimits_{i=1}^{n}\log\left(z_i^{-1/\alpha}\right) + \tfrac{\pi}{2\sigma^2}\sum\nolimits_{i=1}^{n} z_i\, t_i + 2\pi\sum\nolimits_{i=1}^{n} z_i\, u_i,$$

$$U_\lambda(\Theta) = \tfrac{1}{\lambda^2}\sum\nolimits_{i=1}^{n} x_i z_i^{1/\alpha} - \tfrac{2\pi\alpha}{\lambda^2}\sum\nolimits_{i=1}^{n} z_i^{1+1/\alpha} x_i\left(\pi q_i z_i + r_i - \alpha u_i\right)$$
$$+\tfrac{\pi\alpha}{\lambda^2\sigma^2}\sum\nolimits_{i=1}^{n} x_i z_i^{1+1/\alpha}\left(q_i\, z_i\pi\sec^2\left(\pi\,(0.5 - z_i)\right) - p_i/\alpha + t_i - 2\pi t_i \tan\pi\,(0.5 - z_i)\right),$$

$$U_\mu(\Theta) = \tfrac{-\pi}{\sigma^2}\sum\nolimits_{i=1}^{n} z_i\, q_i, \text{ and}$$

$$U_\sigma(\Theta) = \tfrac{-2\pi}{\sigma^3}\sum\nolimits_{i=1}^{n} z_i\, t_i, \text{ respectively.}$$

The MLEs for the parameters α, λ, μ and σ are $\hat\alpha, \hat\lambda, \hat\mu$ and $\hat\sigma$, respectively, can be obtained by solving the nonlinear likelihood equations $U(\Theta) = 0$ simultaneously.

The SAS® software was used to run the simulation analysis, to study the performance of the MLE for different sample sizes (n =25, 50, 100, 200, and 500) and varying combinations of true values of the N-$L\{C\}$ parameters ($\alpha = 0.5, 1.5, 2, 3.5$, $\lambda = 0.5, 1.5, 2.5, 3$, $\mu = 0.5, 1, 2, 3$, $\sigma = 0.5, 1, 1.5$). For each sample size and each parameter combinations, 500 simulations were performed. Table 2 presents the bias (actual estimate) and the standard deviations of the parameter estimates of the N-$L\{C\}$ distribution using the MLE method.

Table 2 shows that the MLE method is an appropriate technique for estimating the parameters of the *N-L*{*C*} distribution. In general, the biases and standard deviations of the parameters are reasonably small. Also, it can be deduced from Table 2 that the standard deviation reduces for all the selected parameter values as the sample size increases. Similar estimation analysis was conducted for other members of the T-Lomax{Y} family of distributions that are defined in Section 4. This shows that the MLE method is an appropriate method for estimating the parameters of T-Lomax{Y} families of distributions.

Table 2. Bias and standard deviations of the parameter estimates of *N-L*{*C*} distribution using MLE method

Actual Values					Bias				Standard deviation			
α	λ	μ	σ	n	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\sigma}$
0.5^R	0.5	0.5	0.5	25	−0.0919	−0.3258	0.0185	−0.0689	0.2832	0.6976	1.7729	0.2726
				50	−0.0749	−0.3177	−0.0278	−0.0362	0.2271	0.6884	0.3493	0.1685
				100	−0.0791	−0.3672	0.0120	−0.0358	0.2108	0.6695	0.2566	0.1483
				200	−0.0602	−0.2924	0.0216	−0.0246	0.1797	0.5914	0.2003	0.1246
				500	−0.0455	−0.2157	0.0252	−0.0210	0.1381	0.4853	0.1420	0.0930
1.5^S	0.5	2	0.5	25	0.0254	−0.1366	0.0201	0.0054	0.8588	0.6879	1.1583	0.3216
				50	−0.2847	−0.4396	0.3245	0.0510	0.8291	0.6849	0.9608	0.2628
				100	−0.2601	−0.2939	0.0065	−0.0066	0.7189	0.5680	0.8943	0.2267
				200	−0.3887	−0.4153	0.1693	−0.0073	0.5767	0.5387	0.8713	0.2218
				500	−0.3660	−0.4105	0.2811	0.0365	0.5205	0.5361	0.5895	0.1299
3^L	3	3	1	25	0.3471	0.1618	0.1993	0.0552	0.9494	1.3838	1.2703	0.4941
				50	0.2224	0.1314	−0.0317	−0.0182	0.9112	1.2945	1.2158	0.4705
				100	0.1708	0.2188	−0.3269	−0.0941	0.8242	1.2626	1.1255	0.4033
				200	0.1234	0.2226	−0.4276	−0.1251	0.7416	1.2168	1.0000	0.3440
				500	0.0037	−0.0063	−0.2976	−0.0919	0.6427	1.1991	0.9199	0.2982
2^B	2.5	1	1.5	25	0.1871	0.3666	−0.2083	0.0857	0.7074	1.3100	0.5922	0.3895
				50	0.1270	0.2141	−0.1129	0.0633	0.6403	1.2617	0.4160	0.2741
				100	−0.0013	−0.0446	−0.0244	0.0317	0.5364	1.1085	0.2821	0.1910
				200	−0.0465	−0.1236	−0.0168	0.0134	0.4888	1.0123	0.2149	0.1435
				500	−0.0763	−0.1752	−0.0028	−0.0037	0.3609	0.7556	0.1261	0.0958
3.5^B	1.5	2	1.5	25	0.5740	0.2954	−0.2042	0.0380	1.2225	0.7736	0.8573	0.4651
				50	0.3169	0.1567	−0.2056	−0.0037	1.2203	0.8030	0.6935	0.3692
				100	0.0929	0.0124	−0.0790	0.0190	1.2110	0.7989	0.5260	0.2748
				200	−0.0531	−0.0562	−0.0490	0.0061	1.0315	0.6703	0.3918	0.1972
				500	−0.1681	−0.1172	−0.0055	0.0011	0.8670	0.5561	0.2290	0.1202

^{R}Skewed to the right, ^{S}symmetric, ^{L}skewed to the left, ^{B} bimodal distribution.

6. Some Applications of *T*-Lomax{*Y*} Family of Distributions

In this section, different data sets from different disciplines with different shapes are used to illustrate the flexibility of the Weibull-Lomax{log-logistic}, gamma-Lomax{log-logistic}, Exponentiated Weibull-Lomax {exponential}, Normal-Lomax{Cauchy}, and Weibull-Lomax {exponential}(*W-L*{*E*}) distributions as representatives of the T-Lomax{Y} family of distributions in modeling and better fitting variety of shapes of data. This include unimodal and bimodal data sets and comparing their results to other known distributions. For each of the applications, the maximized log-likelihood estimates (MLEs), the value of two times the minus log-likelihood function $-2\log l$, the Akaike Information Criterion (AIC) value, the Bayesian Information Criterion (BIC) value, and the Kolmogorov-Smirnov (K-S) test statistic and its corresponding p-value for each of the fitted distributions are reported. Also, the Cramér-von Mises (W^*) and the Anderson-Darling (A^*) statistics, are described in detail in (Chen and Balakrishnan, 1995), are provided to compare the T-Lomax{Y} members ability to fit real world data with other distributions. In general, the smaller the values of the $-2\log l$, AIC, BIC, K-S, W^*, and A^* test statistics, the better the fit to the data.

6.1 Applications With Unimodal Data Sets

Three data sets from different fields of sciences, medical, entomology and material science are used in this section to reveal the power of members of the T-Lomax$\{Y\}$ family of distributions in modeling the three different shapes of unimodal data sets; right skewed, symmetric, and left skewed, compared with other strong extended distributions of the Lomax distribution. The three and four parameters T-Lomax$\{Y\}$ members: *W-L*$\{LL\}$, *G-L*$\{LL\}$, *EW-L*$\{E\}$, *N-L*$\{C\}$, and the *W-L*$\{E\}$ are used to fit these unimodal data sets. Their results are compared to the fitting results of the McDonald Lomax (McLomax), the Beta Lomax (BLomax), and the Kumaraswamy Lomax (KwLomax) distributions which are known extensions of the Lomax distribution defined by Lemonte and Cardeiro (2013) with four and five parameters. The goodness of fit tests results for fitting the different distributions to the different data sets are reported in Tables 3-8.

6.1.1 Data 1: Remission Times of Bladder Cancer Patients

This first data set represents the remission times (in months) of a random sample of 128 bladder cancer patients, the distribution of this data is heavily skewed to the right (skewness = 3.286). This data was analyzed by Lemonte and Cordeiro (2013). The results in Tables 3 and 4 for the McLomx, BLomax, and KwLomax distributions are obtained from Lemonte and Cardeiro (2013). The three parameters distribution *W-L*$\{E\}$ is providing the best fit with the lowest AIC and BIC test statistics. Comparing the fit of the four and five parameters distributions reveals that the *EW-L*$\{E\}$ (four parameters), KwLomax (four parameters) and McLomax (five parameters) distributions provide the best fit in modeling this data set with some slight differences in the different tests values.

Table 3. MLEs and the measures $-2\log l$, AIC, BIC results for the remission times of bladder cancer data

	Estimates								Statistics	
Distributions	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{c}$	$\hat{\beta}$	$\hat{k}$	$\hat{\mu}$	$\hat{\sigma}$	$-2logl$	AIC	BIC
W-L$\{LL\}$	0.3926 (0.15)*	1.8489 (1.5689)	1.6121 (0.3916)					821.8	**827.8**	**836.3**
G-L$\{LL\}$	0.5187 (0.40)	2.1028 (1.53)	1.3848 (1.07)	0.8314 (0.59)				821.6	829.6	841.0
EW-L$\{E\}$		10.8715 (18.70)	1.1090 (0.83)	1.8021 (3.07)	1.3723 (0.97)			819.9	827.9	839.4
N-L$\{C\}$	0.1862 (0.04)	0.000049 (0.00001)				2.728 (1.46)	0.5655 (0.38)	821.4	829.4	840.8
W-L$\{E\}$		8.8627 (7.4999)		1.4606 (0.868)	1.5114 (0.2764)			820.0	**826.0**	**834.5**
Mclomax	0.8085 (3.36)	11.2929 (15.82)	2.1046 (3.08)	1.506 (0.24)	5.1886 (25.03)			**819.8**	829.8	844.1
BLomax	3.9191 (18.19)	23.9281 (27.34)		1.5853 (0.280)	1.1572 (5.02)			820.1	828.1	839.6
KwLomax	0.39191 (2.386)	12.2973 (17.32)	1.5162 (0.23)		12.0323 (87.14)			819.9	827.9	839.4

*Standard error.

The values of the AIC, BIC, and K-S test statistics are the same for both the KwLomax and *EW-L*$\{E\}$ distributions. But, the *EW-L*$\{E\}$ distribution provides the best fit based on the W^* statistic, and the McLomax distribution has the smallest A^* statistic.

Table 4. Goodness-of-fit tests for data 1

	Statistics		
Distributions	K-S (P-value)	A^*	W^*
W-L{LL}	0.0491 (0.9167)	0.3412	0.0532
G-L{LL}	0.0465 (0.9451)	0.3023	0.0463
EW-L{E}	0.0395 (0.9883)	0.1693	**0.0247**
N-L{C}	0.0464 (0.9452)	0.2770	0.0411
W-L{E}	0.0409 (0.9828)	0.1761	0.0257
McLomax	0.0391 (0.9896)	**0.1685**	0.0254
BLomax	0.0405 (0.9846)	0.1900	0.0283
KwLomax	**0.0389 (0.9902)**	0.1727	0.0259

This application suggests that the four parameters *EW-L{E}* distribution is better than the KwLomax and McLomax distributions in fitting this data set. The values in Table 3 and 4 indicate that the three and four parameters members of the T-Lomax$\{Y\}$ family are having a better or similar fit to this data compared to the other mentioned distributions.

Recently, the Cauchy-Weibull{Logistic} (*C-W{L}*) distribution was used to fit this data set by Almheidat, Famoye, and Lee (2015). The *C-W{L}* distribution provided an adequate fit to this data compared to other distributions used in their comparison (see Almheidat et al. (2015)). But when compared to the T-Lomax$\{Y\}$ members, the *C-W{L}* distribution is considerd the least effective in fitting this data set; with the highest AIC, BIC and K-S, A^*, and W^* statistics and the smallest p-value for the K-S test.

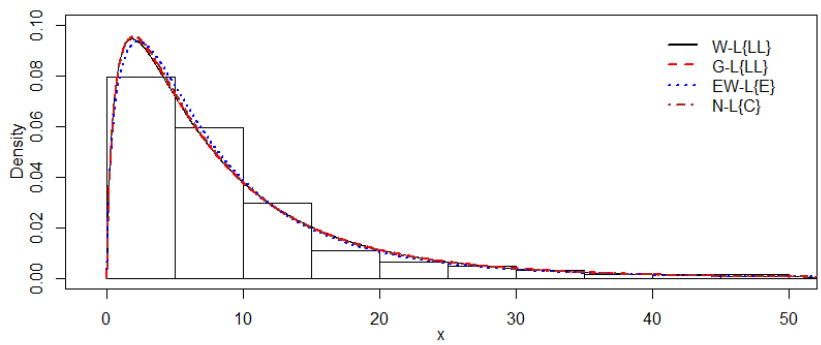

Figure 6. Fitted PDFs of the T-Lomax$\{Y\}$ distributions for for or data 1

This application shows the power of the T-Lomax$\{Y\}$ family of distributions in fitting heavily right-skewed data set. Figure 6 illustrates the fitted density functions of the *W-L{LL}*, *G-L{LL}*, *EW-L{E}* and *N-L{C}*, along with the histogram of the remission times of bladder cancer patients. The plots show the abilities of these members in fitting right skewed data set.

6.1.2 Data 2: Tribolium Castaneum Cultured at 24^oC

This application uses an entomology data set about a sample of size 368 Tribolium Castaneum cultured at 24^oC taken from Park (1954). Alzaghal et al. (2016) used the *W-L{E}* distribtuion, a sub modal of the T-Lomax{exponential} family, to fit this data set. The values of the *W-L{E}* distribtuion provided in Tables 5 and 6 are obtained from Alzaghal et al. (2016). The MLEs of the parameters as well as the goodness-of-fit tests of the T-Lomax$\{Y\}$ members *W-L{LL}*, *G-L{LL}*, *EW-L{E}* and *N-L{C}* with the other competing distributions McLomax, BLomax, and KwLomax are shown in Tables 5 and 6.

The five parameters McLomax distribution rank first based on $-2\log l$, AIC and K-S test statistics. However, the four parameters *G-L{LL}* distribution rank first with the least A^* and W^* test values. With one less parameter and very similar goodness of fit tests values, the *G-L{LL}* distribution is considered a strong replacement to the five parameters McLomax distribution in fitting this data set. The other T-Lomax $\{Y\}$ members of distributions have an adequate fit to the same data set. This application illustrates the flexibility of the T-Lomax$\{Y\}$ family of distributions in modeling symmetric data sets. Figure 7 displays the histogram and the fitted T-Lomax$\{Y\}$ family density functions to the Tribolium Castaneum data set.

Table 5. MLEs and the measures $-2\log l$, AIC, BIC results for the Tribolium Castaneum cultured at 24^oC

Distributions	Estimates								Statistics	
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{c}$	$\hat{\beta}$	$\hat{k}$	$\hat{\mu}$	$\hat{\sigma}$	$-2logl$	AIC	BIC
W-L{LL}	0.2622 (0.14)	5.7287 (0.99)	11.2763 (4.98)					2979.1	2985.1	2996.9
G-L{LL}	0.2242 (0.66)	2.1803 (20.82)	3.5591 (1.94)	0.1453 (0.35)				2968.8	2976.8	2992.4
EW-L{E}		4.2652 (13.25)	2.1655 (0.74)	0.3595 (0.37)	11.4468 (9.14)			2969.1	2977.1	2992.7
N-L{C}	0.6950 (0.21)	0.2325 (0.44)				16.6637 (20.68)	2.3014 (3.20)	2969.3	2977.3	2992.9
W-L{E}		12.63 (28.78)		0.5171 (0.52)	12.3971 (8.42)			2976.2	2982.1	2993.8
Mclomax	278.58 (37.15)	4946.91 (2.18)	381.60 (358.97)	45.293 (16.130)	6.238 (2.67)			**2966.6**	**2976.6**	2996.2
BLomax	1.5003 (0.82)	546.82 (282.52)		27.998 (1.39)	142.30 (70.79)			2973.2	2981.2	2996.8
KwLomax	1.1169 (0.78)	7.9334 (36.32)	66.9955 (123.27)		169.29 (348.49)			2968.7	2976.7	**2992.3**

Table 6. Goodness-of-fit tests for data 2

Distributions	Statistics		
	K-S (P-value)	A^*	W^*
W-L{LL}	0.0902 (0.0049)	2.7506	0.4664
G-L{LL}	0.0838 (0.0113)	**2.2153**	**0.3958**
EW-L{E}	0.0834 (0.0119)	2.2381	0.3993
N-L{C}	0.0829 (0.0125)	2.2507	0.4014
W-L{E}	0.0904 (0.0048)	2.5902	0.4481
McLomax	**0.0808 (0.0164)**	2.2889	0.4042
BLomax	0.0842 (0.0108)	2.4418	0.4053
KwLomax	0.0849 (0.010)	2.2167	0.3961

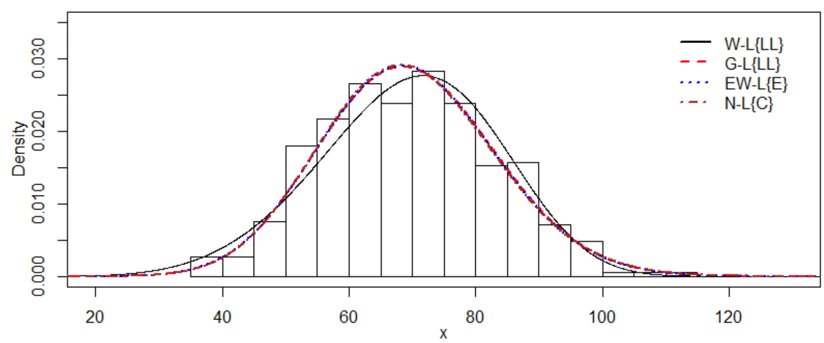

Figure 7. Fitted PDFs of the T-Lomax$\{Y\}$ distributions for data 2

6.1.3 Data 3: Breaking Stress of 50mm Carbon Fibers

The last shape a unimodal data can take is presented in this example. This application shows the ability of the T-Lomax$\{Y\}$ distributions in fitting real life left skewed data set. The data set is taken from Nichols and Padgett (2006) and it is about the breaking stress of carbon fibers of 50 mm in length. The same T-Lomax$\{Y\}$ members *W-L{LL}*, *G-L{LL}*, *EW-L{E}* and *N-L{C}* with the other competing distributions McLomax, BLomax, and KwLomax are used to fit this data set. The MLEs of the parameters as well as the goodness-of-fit tests results are provided in Tables 7 and 8.

The T-Lomax$\{Y\}$ distributions, the three parameters *W-L*{*LL*} and the four parameters *N-L*{*C*} distributions provide the best fit to this data set. The lowest value of AIC and BIC provided by the *W-L*{*LL*} and *N-L*{*C*} distributions provides the lowest $-2\log l$, K-S, A^* and W^* values and the highest p-value for the K-S.

Table 7. MLEs and the measures $-2\log l$, AIC, BIC results for the carbon fibers data set

	Estimates								Statistics	
Distributions	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{c}$	$\hat{\beta}$	$\hat{k}$	$\hat{\mu}$	$\hat{\sigma}$	$-2logl$	AIC	BIC
W-L{*LL*}	3.2031 (9.24)	12.77 (41.15)	2.787 (0.85)					171.7	**177.7**	**184.3**
G-L{*LL*}	3.61 (27.28)	14.428 (113.58)	1.0298 (1.45)	0.37 (0.59)				171.7	179.7	188.5
EW-L{*E*}		1758.54 (16.47)	0.8004 (0.35)	544.85 (53.18)	3.9129 (1.07)			171.9	179.9	188.7
N-L{*C*}	1.364 (0.60)	0.6039 (0.99)				3.2494 (3.32)	1.235 (1.07)	**170.5**	178.5	187.2
W-L{*E*}		2772.99 (10.04)		906.18 (30.73)	3.4417 (0.33)			172.2	178.2	184.7
Mclomax	14.277 (62.31)	80.50 (353.93)	6.7221 (4.81)	3.6284 (0.86)	170.76 (444.47)			172.2	182.2	193.1
BLomax	9.823 (10.57)	601.05 (2306.9)		7.512 (128)	163.79 (459.78)			182.4	190.4	199.2
KwLomax	15.016 (51.86)	138.40 (477.25)	4.019 (0.64)		170.30 (293.59)			173.1	181.1	189.8

Recently, Fatima, Jan, and Ahmad (2018) fitted this data using the three parameters Rayleigh Lomax distribution. A comparison between the Rayleigh Lomax distribution's performance and the *W-L*{*LL*} distribution in fitting this data set reveals that both distributions provide an adequate fit for this data set. But, the *W-L*{*LL*} distribution provides a better fit with lower $-2\log l$, AIC, BIC, A^* and W^* values. This application shows the ability of the T-Lomax$\{Y\}$ family of distributions in providing a good fit for a left skewed data set. Figure 8 presents the plots of the T-Lomax$\{Y\}$ members in fitting the histogram of the carbon fibers data set.

Table 8. Goodness-of-fit tests for data 3

	Statistics		
Distributions	K-S (P-value)	A^*	W^*
W-L{*LL*}	0.0821 (0.7653)	0.4741	0.0778
G-L{*LL*}	0.0821 (0.7659)	0.4736	0.0778
EW-L{*E*}	0.0808 (0.7816)	0.4855	0.0778
N-L{*C*}	**0.0770 (0.8289)**	**0.3751**	**0.0591**
W-L{*E*}	0.0825 (0.7595)	0.4866	0.0832
McLomax	0.0843 (0.7367)	0.5026	0.0859
BLomax	0.1272 (0.2360)	1.2552	0.2257
KwLomax	0.0884 (0.6803)	0.5498	0.0973

6.2 Application With Bimodal Data Set

Some members of the T-Lomax$\{Y\}$ family of distributions has the ability of modeling real lifetime data set with bimodal histogram in addition to unimodal data sets. In this subsection, one of the bimodal members of the T-Lomax$\{Y\}$ family of distributions is used to fit a bimodal medical data set and the goodness of fits tests results compared with other known bimodal distributions used for fitting this data set are recorded in Tables 9 and 10.

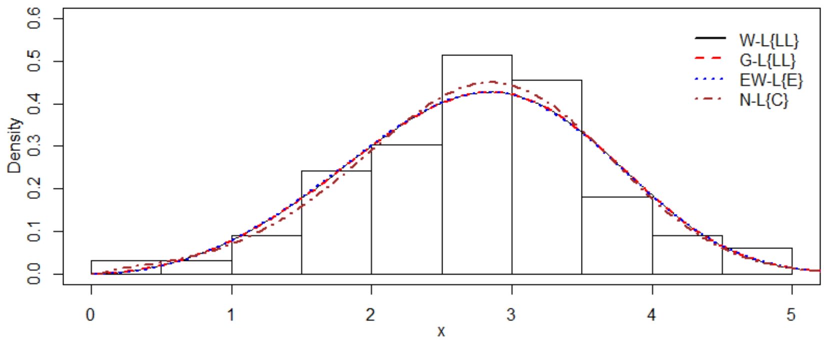

Figure 8. Fitted PDFs of the T-Lomax{Y} distributions for data 3

6.2.1 Data 4: Times to Death of Psychiatric Patients

This data set is taken from the survival data of 26 inpatients admitted to the University of Iowa hospitals during the years 1935-1948. The data consists of the age of each patient at first admission to the hospital, sex, number of years from admission to death and is taken from Klein and Moeschberger (1997). This application considers the times to death of the 26 psychiatric patients. A member of the T-Lomax{Y} family of distributions, namely the Normal-Lomax{Cauchy} distribution, is used in fitting this data set. The *N-L*{*C*} distribution is revealing a superiority in fitting this bimodal data set compared to the following bimodal distributions: the four parameters Beta-Normal (*B-N*) distribution defined by Famoye, Lee and Eugene (2004), the four parameters Weibull-gamma{log-logistic} (*W-G*{*LL*}) distribution introduced by Alzaatreh, Lee and Famoye (2015), and the Weibull Lindley (*W-L*) distribution defined by Asgharzadeh, Nadarajah, and Sharafi (2018). The parameters estimates and the different test results are provided in Tables 9 and 10.

The values in Tables 9 and 10 indicate that the *N-L*{*C*} distribution provides a superior fit to this bimodal data among the five distributions presented. It has the smallest $-2\log l$, AIC, BIC, K-S, A^* and W^* values and the highest p-value of the K-S. the *N-L*{*C*} distribution gives the best fit in comparison with the other distributions performance.

Table 9. MLEs and the measures $-2\log l$, AIC, BIC results for the times to death of psychiatric patients data set

	Estimates						Statistics		
Distributions	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{c}$	$\hat{\beta}$	$\hat{\mu}$	$\hat{\sigma}$	$-2logl$	AIC	BIC
N-L{*C*}	604.69 (61.5094)	6176.50 (6.0013)			6.2362 (2.5175)	5.1188 (1.8651)	**176.6**	**184.6**	**189.6**
B-N	0.1040 (0.1086)			171.20 (318.81)	56.2885 (11.040)	6.2733 (4.1968)	190.0	198.0	203.0
W-G{*LL*}	3.1294 (2.3302)	729.9 (948.98)	0.3938 (0.1394)	2.8238 (1.1677)			186.7	194.7	199.8
W-L	9.901 (2.822)	0.043 (0.0105)		0.0283 (0.0101)			186.8	192.8	196.5

Table 10. Goodness-of-fit tests for data 4

	Statistics		
Distributions	K-S (P-value)	A^*	W^*
N-L{*C*}	**0.0990 (0.9606)**	**0.2068**	**0.0287**
B-N	0.1801 (0.3676)	0.7497	0.1253
W-G{*LL*}	0.1951 (0.2757)	1.0374	0.1561
W-L	0.1158 (0.8765)	1.0035	0.0598

Recently, Asgharzadeh et al. (2018) used the *W-L* distribution to fit this data set and compared the results with seven other distributions. The *W-L* distribution fits the data better in comparison with the seven other distributions. But, in comparison with the *N-L*{*C*} distribution, the *W-L* distribution ranks second in fitting this data set. This application is a good illustration of the power of a T-Lomax{Y} member, *N-L*{*C*}, in modeling a bimodal data set. In Figure 9, the plots support the results of Tables 9 and 10 in showing that the *N-L*{*C*} distribution provides the best fit to this bimodal histogram.

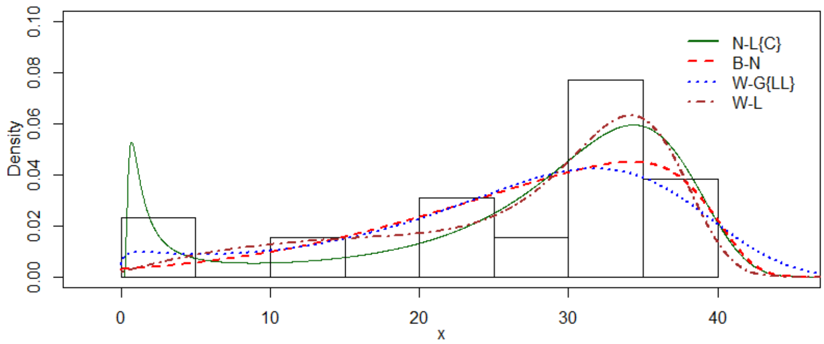

Figure 9. Fitted PDFs for data 4

7. Summary

Using the *T-X* framework, this paper proposed a new generalization of the two-parameter Lomax distribution namely, the *T*-Lomax{*Y*} families of distributions. From which, the *T*-Lomax{exponential}, *T*-Lomax{Weibull}, *T*-Lomax{log-logistic}, *T*-Lomax{logistic}, *T*-Lomax{Cauchy} and *T*-Lomax{extreme value} families of distributions were investigated. Some statistical properties of these defined new families including the moments and Shannon entropies were studied. Four members of the *T*-Lomax{*Y*} family of distributions namely, the Weibull-Lomax {log-logistic} , gamma-Lomax{log-logistic}, Exponentiated Weibull-Lomax{exponential} and Normal-Lomax{Cauchy} distributions were studied in more details. The parameters of the Normal-Lomax{Cauchy} distribution are estimated by the method of maximum likelihood. A simulation study to assess the performance of the parameters using the MLE method of the Normal-Lomax{Cauchy} distribution is provided. Lastly, four different real data sets were used to demonstrate the flexibility of the *T*-Lomax{*Y*} family of distributions in fitting the different shapes of real-world data from different sciences.

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