International Journal of Statistics and Probability; Vol. 8, No. 2; March 2019
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

M/M/1 Model With Unreliable Service and a Working Vacation

Joshua Patterson[1] & Andrzej Korzeniowski[1]

[1] Department of Mathematics, University of Texas at Arlington, Arlington TX, USA

Correspondence: Joshua Patterson, Department of Mathematics, University of Texas at Arlington, Arlington TX, 76019, USA. E-mail: Joshua.Patterson@uta.edu

Received: December 3, 2018 Accepted: December 28, 2018 Online Published: January 14, 2019
doi:10.5539/ijsp.v8n2p1 URL: https://doi.org/10.5539/ijsp.v8n2p1

Abstract

We derive an explicit closed form of the stationary distribution of an M/M/1 queue with unreliable service and a working vacation. We also show that the work in (Patterson & Korzeniowski, 2018) can be obtained as a special case of this model. Future work remains to be done; specifically, it may be possible to use the explicit stationary distribution given here to decompose the queue length into the sum of independent random variables. Consequently, it may then be possible to utilize Little's Law (Little, 1961) to decompose the customer waiting time as well.

Keywords: unreliable service, quasi birth death process, matrix-geometric, working vacation

1. Introduction

Within the literature, a vacation queue is typically defined as a queue where the server is considered to be in a different state, generally defined to be a 'vacation' when the number of customers in the queue is below a certain threshold. For an example of a vacation queue within the M/G/1 framework, see (Levy & Yechiali, 1975). Working vacation queues typically refer to queues where the server utilizes its time on vacation for other purposes. For an example of a working vacation queue where the server uses its vacation to service the main queue, but at a reduced rate, see (Xu & Tian, 2009).

A queue with unreliable service is a queue where service may be unsuccessful any number of times before it is successful. This type of queue is important to study because it occurs naturally within a lot of systems. For example, imagine trying to have a conversation with someone in a quiet environment, such as a library–words spoken are generally heard and understood (i.e. service is rendered successfully every time). Now, imagine trying to have the same conversation in a noisy environment, such as a busy restaurant–it can be done, but you may need to repeat yourself (i.e. service may fail). Within the literature, this has been achieved two ways. First, using the M/G/1 framework, a M/PH/1 queue allows for a random number of Poisson distributed 'stages' within the service time. However, this method necessitates the restriction that: $\mu < \beta_1 + \beta_2$, where μ is the service rate, β_1 is the success rate, and β_2 is the failure rate (Latouche & Ramaswami, 1999). Second, one may use the framework of (Nuets, 1981) to construct a queue with identical stationary distribution to the M/PH/1 but does not inherit any restrictions on μ, β_1 or β_2 besides those necessary for positive recurrence, as seen in (Patterson & Korzeniowski, 2018). We extend the M/M/1 model with unreliable service defined in (Patterson & Korzeniowski, 2018) by including two service rates. The use of multiple service rates is important since the customer service time depends not only on the customer, as as it would be in the M/PH/1 queue, but on the state of the server at the time of service as well. This not only has the benefit of generalizing the results of (Patterson & Korzeniowski, 2018) by recovering this stationary distribution as a special case, but also cannot be recovered by previous results related to the M/PH/1.

We adopt assumptions and terminology from (Patterson & Korzeniowski, 2018). Namely, service failure is not due to the server as it would be in breakdown models, nor due to the customer as it would be in some interruption models. Customer's do not leave the queue–that is we preserve the FCFS (First Come First Served) service protocol. We consider service failures to be due to external, random forces and repeat a customer's service until it has been completed successfully. Furthermore, neither the server nor customer know whether the service was successful until the service time has been completed, at which time we envision a 'quality check' to take place which determines if the service was a success or failure.

2. Definitions

We define our process, state space, and parameters as follows:

Definition 2.1. Let $\{N(t) \mid t \geqslant 0\}$ be the number of customers in the queue at time t,

$$J(t) = \begin{cases} 0 & \text{the server is on working vacation} \\ 1 & \text{the server is in a busy state} \end{cases}$$

and

$$S(t) = \begin{cases} 1 & \text{immediately after service is rendered} \\ 0 & \text{otherwise} \end{cases}$$

Then $\{(N(t), J(t), S(t)) \mid t \geqslant 0\}$ is a Markov process on the state space:
$\Omega = \{(0, 0, 0)\} \cup \{(k, j, s) \mid k \in \mathbb{N},\ \ s, t \in \{0, 1\}\}$

Define the following parameters:

- λ : the rate of the Poisson arrivals process.
- μ_b : the rate of service when the server is 'busy,' successful or not.
- μ_v : the rate of service when the server is on 'vacation,' successful or not.
- β_1 : the rate of a successful service.
- β_2 : the rate of a failed service.
- θ : vacation duration is exponentially distributed with rate α.

Definition 2.2. We define the vacation policy:

- When the server becomes idle (i.e. $N(t) = 0$), the server goes on a working vacation; by this we mean that customers arriving while the server is on vacation get served at a reduced rate $\mu_v < \mu_b$.
- When the server is not idle (i.e. $N(t) \neq 0$), a vacationing server begins a working vacation duration that is exponentially distributed with rate θ, after which it begins a busy period and operates at rate μ_b until the server becomes idle again, renewing the process.
- If a customer is served successfully while the server is on a working vacation and there are additional customers waiting in the queue, the server then immediately ends its vacation and enters into a busy state until the queue is emptied.

To help visualize this 3-dimensional Markovian process in 2-dimensions, we informally construct the state transition rate diagram in 2D.

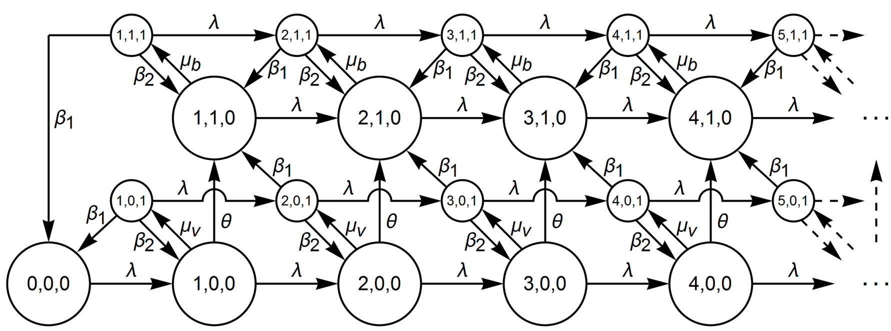

Figure 1. 3D Markovian state transition rate diagram in 2D

We define a 'successful service' similarly to that done in (Patterson & Korzeniowski, 2018) to be a transition from $(n, j, 1) \longrightarrow (n-1, 0|1, 0)$, which is represented in the state transition diagram as having rate β_1. Accordingly, we will define a 'failed service' to be a transition from $(n, j, 1) \longrightarrow (n, j, 0)$ with transition rate β_2. We will compute the probabilities of a 'successful' or 'failed' service in an explicit manner by considering the transition probabilities of the embedded Markov Chain and will note a similarity with the results from (Patterson & Korzeniowski, 2018).

Let $E_s = \{\text{a customer was served successfully}\}$
$S_v = \{\text{the server is on a working vacation}\}$

$S_b = \{\text{the server is busy}\}$

$$p_s = P(E_s \cap S_v) + P(E_s \cap S_b) = P(E_s|S_v)P(S_v) + P(E_s|S_b)P(S_b)$$

$$p_s = \frac{\beta_1}{\beta_1+\beta_2+\lambda}\sum_{i=0}^{\infty}\left(\frac{\lambda}{\beta_1+\beta_2+\lambda}\right)^i P(S_v) + \frac{\beta_1}{\beta_1+\beta_2+\lambda}\sum_{i=0}^{\infty}\left(\frac{\lambda}{\beta_1+\beta_2+\lambda}\right)^i P(S_b)$$

$$p_s = \frac{\beta_1}{\beta_1+\beta_2+\lambda}\sum_{i=0}^{\infty}\left(\frac{\lambda}{\beta_1+\beta_2+\lambda}\right)^i \big(P(S_v) + P(S_b)\big)$$

$$p_s = \frac{\beta_1}{\beta_1+\beta_2+\lambda}\left(\frac{1}{1-\frac{\lambda}{\beta_1+\beta_2+\lambda}}\right) = \frac{\beta_1}{\beta_1+\beta_2+\lambda}\left(\frac{1}{\frac{\beta_1+\beta_2}{\beta_1+\beta_2+\lambda}}\right) = \frac{\beta_1}{\beta_1+\beta_2+\lambda}\left(\frac{\beta_1+\beta_2+\lambda}{\beta_1+\beta_2}\right) = \frac{\beta_1}{\beta_1+\beta_2}$$

We list the countable state space in lexicographical order; formally defined for triplets below.

Definition 2.3. Lexicographical Ordering

We say $(k_1, j_1, s_1) < (k_2, j_2, s_2)$ if and only if $k_1 \frown j_1 \frown s_1 < k_2 \frown j_2 \frown s_2$,

where $\frown$ denotes concatenation (Quine, 1946). For example, $7 \frown 0 \frown 1 = 701$.

It should be noted that this definition for lexicographical ordering can easily be extended to n-tuples and is equivalent to that found on pg. 353 of (Ibe, 2013) when applied to the case of twoples. Using this re-ordering convention, we write: $\Omega = \{(0,0,0), (1,0,0), (1,0,1), (1,1,0), (1,1,1), \dots\}$ and define the corresponding infinitesimal matrix Q.

3. Infinitesimal Matrix Q

$$\mathbf{Q} = \begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{C}} & 0 & 0 & 0 & \dots \\ \hat{\mathbf{B}} & \mathbf{A} & \mathbf{C} & 0 & 0 & \dots \\ 0 & \mathbf{B} & \mathbf{A} & \mathbf{C} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix} \tag{1}$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} -\lambda \end{bmatrix} \qquad \hat{\mathbf{B}} = \begin{bmatrix} 0 \\ \beta_1 \\ 0 \\ \beta_1 \end{bmatrix} \qquad \hat{\mathbf{C}} = \begin{bmatrix} \lambda & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -(\lambda+\mu_v+\theta) & \mu_v & \theta & 0 \\ \beta_2 & -(\lambda+\beta_1+\beta_2) & 0 & 0 \\ 0 & 0 & -(\lambda+\mu_b) & \mu_b \\ 0 & 0 & \beta_2 & -(\lambda+\beta_1+\beta_2) \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

4. The Quadratic Matrix Equation

Thanks to Lemma 4.1 from (Patterson & Korzeniowski, 2018), which is based on the framework developed by (Neuts, 1981), we seek the minimal non-negative solution **R** to the quadratic matrix equation:

$$\mathbf{R}^2\mathbf{B} + \mathbf{R}\mathbf{A} + \mathbf{C} = \mathbf{0} \tag{2}$$

We will again employ the direct method whereby we solve the system of equations generated by equating the matrices entry by entry.

Let $\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} \Longrightarrow$ (2) can be restated as the following system:

$$\begin{cases} r_{11}\left(\theta+\beta_1\left(r_{12}+r_{14}\right)\right)+\beta_1 r_{12}\left(r_{22}+r_{24}\right)+r_{14}\left(\beta_2+\beta_1\left(r_{42}+r_{44}\right)\right)=r_{13}\left(\mu_b+\lambda-\beta_1\left(r_{32}+r_{34}\right)\right) \\ r_{21}\left(\theta+\beta_1\left(r_{12}+r_{14}\right)\right)+\beta_1 r_{22}\left(r_{22}+r_{24}\right)+r_{24}\left(\beta_2+\beta_1\left(r_{42}+r_{44}\right)\right)=r_{23}\left(\mu_b+\lambda-\beta_1\left(r_{32}+r_{34}\right)\right) \\ \lambda+r_{31}\left(\theta+\beta_1\left(r_{12}+r_{14}\right)\right)+\beta_1\left(r_{22}+r_{24}\right)r_{32}+r_{34}\left(\beta_2+\beta_1\left(r_{42}+r_{44}\right)\right)=r_{33}\left(\mu_b+\lambda-\beta_1\left(r_{32}+r_{34}\right)\right) \\ r_{41}\left(\theta+\beta_1\left(r_{12}+r_{14}\right)\right)+\beta_1\left(r_{22}+r_{24}\right)r_{42}+r_{44}\left(\beta_2+\beta_1\left(r_{42}+r_{44}\right)\right)=r_{43}\left(\mu_b+\lambda-\beta_1\left(r_{32}+r_{34}\right)\right) \\ r_{11}\left(\theta+\lambda+\mu_v\right)-\beta_2 r_{12}-\lambda=0 \\ r_{12}\left(\beta_1+\beta_2+\lambda\right)-r_{11}\mu_v=0 \\ r_{14}\left(\lambda+\beta_1+\beta_2\right)-r_{13}\mu_b=0 \\ \beta_2 r_{22}-r_{21}\left(\theta+\lambda+\mu_v\right)=0 \\ r_{22}\left(\beta_1+\beta_2+\lambda\right)-\lambda-r_{21}\mu_v=0 \\ r_{24}\left(\lambda+\beta_1+\beta_2\right)-r_{23}\mu_b=0 \\ \beta_2 r_{32}-r_{31}\left(\theta+\lambda+\mu_v\right)=0 \\ r_{32}\left(\lambda+\beta_1+\beta_2\right)-r_{31}\mu_v=0 \\ r_{34}\left(\lambda+\beta_1+\beta_2\right)-r_{33}\mu_b=0 \\ \beta_2 r_{42}-r_{41}\left(\theta+\lambda+\mu_v\right)=0 \\ r_{42}\left(\beta_1+\beta_2+\lambda\right)-r_{41}\mu_v=0 \\ r_{44}\left(\lambda+\beta_1+\beta_2\right)-r_{43}\mu_b-\lambda=0 \end{cases} \tag{3}$$

The analytical minimal non-negative solution to (3) is given by:

$$\mathbf{R}=\begin{bmatrix} \frac{\lambda(\lambda+\beta_1+\beta_2)}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v} & \frac{\lambda\mu_v}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v} & \frac{\lambda(\beta_1+\beta_2+\lambda)((\lambda+\beta_1+\beta_2)(\theta+\lambda)+\lambda\mu_v)}{\beta_1\mu_b((\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v)} & \frac{\lambda((\lambda+\beta_1+\beta_2)(\theta+\lambda)+\lambda\mu_v)}{\beta_1((\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v)} \\ \frac{\beta_2\lambda}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v} & \frac{\lambda(\theta+\lambda+\mu_v)}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v} & \frac{\lambda(\beta_1+\beta_2+\lambda)((\beta_2+\lambda)(\theta+\lambda)+\lambda\mu_v)}{\beta_1\mu_b((\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v)} & \frac{\lambda((\beta_2+\lambda)(\theta+\lambda)+\lambda\mu_v)}{\beta_1((\lambda+\beta_1+\beta_2)(\theta+\lambda)+(\beta_1+\lambda)\mu_v)} \\ 0 & 0 & \frac{\lambda(\lambda+\beta_1+\beta_2)}{\beta_1\mu_b} & \frac{\lambda}{\beta_1} \\ 0 & 0 & \frac{\lambda(\beta_2+\lambda)}{\beta_1\mu_b} & \frac{\lambda}{\beta_1} \end{bmatrix} \tag{4}$$

5. The Spectral Radius of R

We compute the spectral radius of $\mathbf{R}$ explicitly and show that the sufficient condition under which our model will be positive recurrent has not changed from the case in (Patterson & Korzeniowski, 2018).

Corollary. *By Lemma 4.1 from (Patterson & Korzeniowski, 2018), the infinitesimal matrix* $\mathbf{Q}$ *given in equation* (1) *is positive recurrent if and only if:* $\beta_1(\mu_b-\lambda)-\lambda(\mu_b+\beta_2)>0$.

Proof. The spectral radius of $\mathbf{R}$ is computed by solving the scalar quadratic equation generated by $\det(\mathbf{R}-\rho_i\mathbf{I})=0$, yielding that $\{\rho_i\}_{i=0,1,2,3}$ satisfies the following quadratic equations:

$$\mu_b\beta_1\rho_i^2-\lambda(\lambda+\mu_b+\beta_1+\beta_2)\rho_i+\lambda^2=0 \tag{5}$$

$$\Longrightarrow \rho_i=\frac{\lambda\left(\lambda+\mu_b+\beta_1+\beta_2+(-1)^i\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}\right)}{2\mu_b\beta_1}, \qquad i=0,1$$

$$\left(\left(\lambda+\beta_1+\beta_2\right)\left(\theta+\lambda\right)+\mu_v\left(\beta_1+\lambda\right)\right)\rho_i^2-\lambda\left(\beta_1+\beta_2+\theta+2\lambda+\mu_v\right)\rho_i+\lambda^2=0 \tag{6}$$

$$\Longrightarrow \rho_i=\frac{\lambda\left(\theta+2\lambda+\beta_1+\beta_2+\mu_v+(-1)^i\sqrt{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}\right)}{2((\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_v(\beta_1+\lambda))}, \qquad i=2,3$$

By inspection, the largest of these eigenvalues in (5) and (6) will contain the positive radicals. Next we show that $\rho_0 \geqslant \rho_2$.

Assume $\rho_0 < \rho_2$, then:
$\Longrightarrow \rho_0\rho_3 < \rho_2\rho_3$
$\Longrightarrow \rho_0\rho_3 < \frac{\lambda^2}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_v(\beta_1+\lambda)}$

$$
\begin{aligned}
&\Longrightarrow \rho_0 \frac{\lambda\left(\theta+2\lambda+\beta_1+\beta_2+\mu_v-\sqrt{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}\right)}{2((\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_v(\beta_1+\lambda))} < \frac{\lambda^2}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_v(\beta_1+\lambda)} \\
&\Longrightarrow \frac{\rho_0\left(\theta+2\lambda+\beta_1+\beta_2+\mu_v-\sqrt{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}\right)}{2} < \lambda \\
&\Longrightarrow \frac{\rho_0\rho_1\left(\theta+2\lambda+\beta_1+\beta_2+\mu_v-\sqrt{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}\right)}{2} < \lambda\rho_1 \\
&\Longrightarrow \frac{\lambda^2\left(\theta+2\lambda+\beta_1+\beta_2+\mu_v-\sqrt{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}\right)}{2\mu_b\beta_1} < \lambda\rho_1 \\
&\Longrightarrow \frac{\lambda(\theta+2\lambda+\beta_1+\beta_2+\mu_v)}{2\mu_b\beta_1} - \rho_1 < \frac{\lambda\sqrt{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}}{2\mu_b\beta_1} \\
&\Longrightarrow \left(\frac{\lambda(\theta+2\lambda+\beta_1+\beta_2+\mu_v)}{2\mu_b\beta_1} - \frac{\lambda\left(\lambda+\mu_b+\beta_1+\beta_2-\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}\right)}{2\mu_b\beta_1}\right)^2 < \frac{\lambda^2(\theta-\beta_1-\beta_2+\mu_v)^2+4\lambda^2\beta_2\mu_v}{4\mu_b^2\beta_1^2} \\
&\Longrightarrow \left(\frac{\theta+\lambda+\mu_v-\mu_b+\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}}{2\mu_b\beta_1}\right)^2 < \frac{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}{4\mu_b^2\beta_1^2} \\
&\Longrightarrow \left(\frac{(\theta+\lambda+\mu_v-\mu_b)+\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}}{2\mu_b\beta_1}\right)^2 < \frac{(\theta-\beta_1-\beta_2+\mu_v)^2+4\lambda^2\beta_2\mu_v}{4\mu_b^2\beta_1^2} \\
&\Longrightarrow \frac{(\theta+\lambda+\mu_v-\mu_b)^2+2(\theta+\lambda+\mu_v-\mu_b)\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}+(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}{4\mu_b^2\beta_1^2} < \frac{(\theta-\beta_1-\beta_2+\mu_v)^2+4\beta_2\mu_v}{4\mu_b^2\beta_1^2} \\
&\Longrightarrow \frac{2(\theta+\lambda+\mu_v-\mu_b)\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}}{4\mu_b^2\beta_1^2} < \frac{(\theta-\beta_1-\beta_2+\mu_v)^2-(\lambda+\mu_b+\beta_1+\beta_2)^2-(\theta+\lambda+\mu_v-\mu_b)^2+4\beta_2\mu_v+4\mu_b\beta_1}{4\mu_b^2\beta_1^2} \\
&\Longrightarrow \frac{2(\theta+\lambda+\mu_v-\mu_b)\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}}{4\mu_b^2\beta_1^2} < \frac{(\theta-\beta_1-\beta_2+\mu_v)^2-(\theta+2\lambda+\beta_1+\beta_2+\mu_v)^2+2(\lambda+\mu_b+\beta_1+\beta_2)(\theta+\lambda+\mu_v-\mu_b)+4\beta_2\mu_v+4\mu_b\beta_1}{4\mu_b^2\beta_1^2} \\
&\Longrightarrow \frac{2(\theta+\lambda+\mu_v-\mu_b)\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}}{4\mu_b^2\beta_1^2} < \frac{-4(\lambda+\beta_1+\beta_2)(\theta+\lambda+\mu_v)+2(\lambda+\mu_b+\beta_1+\beta_2)(\theta+\lambda+\mu_v-\mu_b)+4\beta_2\mu_v+4\mu_b\beta_1}{4\mu_b^2\beta_1^2} \\
&\Longrightarrow \frac{(\theta+\lambda+\mu_v-\mu_b)\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}}{2\mu_b^2\beta_1^2} < \frac{(\mu_b-\beta_1+\beta_2-\lambda)(\theta+\lambda+\mu_v-\mu_b)-2\beta_2(\theta+\lambda)-2\lambda\mu_b}{2\mu_b^2\beta_1^2} \\
&\Longrightarrow \frac{(\theta+\lambda+\mu_v-\mu_b)\sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1}}{2\mu_b^2\beta_1^2} < \frac{(\mu_b-\beta_1+\beta_2-\lambda)(\theta+\lambda+\mu_v-\mu_b)}{2\mu_b^2\beta_1^2} \\
&\Longrightarrow \sqrt{(\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1} < (\mu_b-\beta_1+\beta_2-\lambda) \\
&\Longrightarrow (\lambda+\mu_b+\beta_1+\beta_2)^2-4\mu_b\beta_1 < (\mu_b-\beta_1+\beta_2-\lambda)^2 \\
&\Longrightarrow 4(\lambda\beta_2+\beta_1\beta_2+\lambda\mu_b) < 0 \longrightarrow\!\longleftarrow \\
&\\
&\Longrightarrow \rho_0 \geqslant \rho_2
\end{aligned}
$$

Thus, by Lemma 4.1 from (Patterson & Korzeniowski, 2018), **Q** is positive recurrent if and only if:
$\rho_0 < 1 \iff \beta_1(\mu_b-\lambda)-\lambda(\mu_b+\beta_2) > 0.$ $\square$

6. The Stationary Distribution

6.1 The Explicit Form of $\mathbf{R}^k$

To find an expression for $\mathbf{R}^k$, we utilize the block upper-triangular structure of the matrix $\mathbf{R}$ given in (4). To that end, we prove the following:

Lemma 6.1. *Given* $\mathbf{R} = \begin{bmatrix} \mathcal{A} & \mathcal{C} \\ \mathbf{0} & \mathcal{B} \end{bmatrix}$, *then* $\mathbf{R}^k = \begin{bmatrix} \mathcal{A}^k & \sum_{i=0}^{k-1} \mathcal{A}^i \mathcal{C} \mathcal{B}^{\mathbf{k-i-1}} \\ \mathbf{0} & \mathcal{B}^k \end{bmatrix}$.

Proof.

Note that if $k = 1$, then $\mathbf{R}^1 = \begin{bmatrix} \mathcal{A}^1 & \sum_{i=0}^{0} \mathcal{A}^i \mathcal{C} \mathcal{B}^{\mathbf{1-i-1}} \\ \mathbf{0} & \mathcal{B}^1 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{C} \\ \mathbf{0} & \mathcal{B} \end{bmatrix}$

Next, assume $\mathbf{R}^k = \begin{bmatrix} \mathcal{A}^k & \sum_{i=0}^{k-1} \mathcal{A}^i \mathcal{C} \mathcal{B}^{\mathbf{k-i-1}} \\ \mathbf{0} & \mathcal{B}^k \end{bmatrix}$, and write $\mathbf{R}^{k+1}$ as follows:

$$
\begin{aligned}
\mathbf{R}^k\mathbf{R} &= \begin{bmatrix} \mathcal{A}^k & \sum_{i=0}^{k-1} \mathcal{A}^i \mathcal{C} \mathcal{B}^{\mathbf{k-i-1}} \\ \mathbf{0} & \mathcal{B}^k \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A}^k & \sum_{i=0}^{k-1} \mathcal{A}^i \mathcal{C} \mathcal{B}^{\mathbf{k-i-1}} \\ \mathbf{0} & \mathcal{B}^k \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{C} \\ \mathbf{0} & \mathcal{B} \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A}^{k+1} & \mathcal{A}^k \mathcal{C} + (\sum_{i=0}^{k-1} \mathcal{A}^i \mathcal{C} \mathcal{B}^{\mathbf{k-i-1}})\mathcal{B} \\ \mathbf{0} & \mathcal{B}^{k+1} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^k\mathcal{A} &= \begin{bmatrix} \frac{\rho_2\rho_3\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_\nu\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} & \frac{\rho_2\rho_3\left((\theta+\lambda+\mu_\nu)\left(\rho_3^k-\rho_2^k\right)-\lambda\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix} \begin{bmatrix} \frac{\rho_2\rho_3(\lambda+\beta_1+\beta_2)}{\lambda} & \frac{\mu_\nu\rho_2\rho_3}{\lambda} \\ \frac{\beta_2\rho_2\rho_3}{\lambda} & \frac{\rho_2\rho_3(\theta+\lambda+\mu_\nu)}{\lambda} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\rho_2^2\rho_3^2\left((\lambda+\beta_1+\beta_2)\left((\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)+\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)\right)+\beta_2\mu_\nu\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} & \frac{\rho_2^2\rho_3^2\left(\mu_\nu\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)\right)+\mu_\nu(\theta+\lambda+\mu_\nu)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2^2\rho_3^2\left(\rho_3^k-\rho_2^k\right)(\lambda+\beta_1+\beta_2)+\beta_2\rho_2^2\rho_3^2\left((\theta+\lambda+\mu_\nu)\left(\rho_3^k-\rho_2^k\right)-\lambda\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda^2(\rho_3-\rho_2)} & \frac{\mu_\nu\beta_2\rho_2^2\rho_3^2\left(\rho_3^k-\rho_2^k\right)+\rho_2^2\rho_3^2(\theta+\lambda+\mu_\nu)\left((\theta+\lambda+\mu_\nu)\left(\rho_3^k-\rho_2^k\right)-\lambda\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda^2(\rho_3-\rho_2)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\rho_2^2\rho_3^2\left(\lambda(\lambda+\beta_1+\beta_2)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+\left((\lambda+\beta_1+\beta_2)^2+\beta_2\mu_\nu\right)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} & \frac{\mu_\nu\rho_2^2\rho_3^2\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)+(\theta+\lambda+\mu_\nu)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2^2\rho_3^2(\theta+2\lambda+\beta_1+\beta_2+\mu_\nu)\left(\rho_3^k-\rho_2^k\right)-\lambda\beta_2\rho_2^2\rho_3^2\left(\rho_3^{k-1}-\rho_2^{k-1}\right)}{\lambda^2(\rho_3-\rho_2)} & \frac{\mu_\nu\beta_2\rho_2^2\rho_3^2\left(\rho_3^k-\rho_2^k\right)+\rho_2\rho_3(\theta+\lambda+\mu_\nu)\left(\rho_2\rho_3(\theta+\lambda+\mu_\nu)\left(\rho_3^k-\rho_2^k\right)-\lambda\rho_2\rho_3\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda^2(\rho_3-\rho_2)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\rho_2^2\rho_3^2\left(\lambda(\lambda+\beta_1+\beta_2)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+\left(\frac{\lambda\left((\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)-\lambda\right)}{\rho_2\rho_3}\right)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} & \frac{\mu_\nu\rho_2^2\rho_3^2\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\theta+2\lambda+\beta_1+\beta_2+\mu_\nu)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3(\lambda(\rho_3+\rho_2))\left(\rho_3^k-\rho_2^k\right)-\lambda\beta_2\rho_2^2\rho_3^2\left(\rho_3^{k-1}-\rho_2^{k-1}\right)}{\lambda^2(\rho_3-\rho_2)} & \frac{\mu_\nu\beta_2\rho_2^2\rho_3^2\left(\rho_3^k-\rho_2^k\right)+\rho_2\rho_3(\theta+\lambda+\mu_\nu)\left(\lambda\left(\rho_3^{k+1}-\rho_2^{k+1}\right)-\rho_2\rho_3(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\rho_2\rho_3\left(\rho_2\rho_3(\lambda+\beta_1+\beta_2)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+\left((\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)-\lambda\right)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_\nu\rho_2^2\rho_3^2\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+\frac{\lambda(\rho_3+\rho_2)}{\rho_2\rho_3}\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3(\rho_3+\rho_2)\left(\rho_3^k-\rho_2^k\right)-\beta_2\rho_2^2\rho_3^2\left(\rho_3^{k-1}-\rho_2^{k-1}\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_\nu\beta_2\rho_2^2\rho_3^2\left(\rho_3^k-\rho_2^k\right)+\left(\lambda(\rho_3+\rho_2)-\rho_2\rho_3(\lambda+\beta_1+\beta_2)\right)\left(\lambda\left(\rho_3^{k+1}-\rho_2^{k+1}\right)-\rho_2\rho_3(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\rho_2\rho_3\left(\rho_2\rho_3(\lambda+\beta_1+\beta_2)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)\left(\rho_3^k-\rho_2^k\right)+\lambda\left(\rho_2^k-\rho_3^k\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_\nu\rho_2\rho_3\left(\rho_2\rho_3\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\rho_3+\rho_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^{k+1}-\rho_2^{k+1}\right)+\beta_2\rho_2\rho_3\left(\rho_2\rho_3^k-\rho_3\rho_2^k\right)-\beta_2\rho_2^2\rho_3^2\left(\rho_3^{k-1}-\rho_2^{k-1}\right)}{\lambda(\rho_3-\rho_2)} & \frac{\lambda^2(\rho_3+\rho_2)\left(\rho_3^{k+1}-\rho_2^{k+1}\right)-\rho_2\rho_3\left(\lambda(\lambda+\beta_1+\beta_2)\left(\rho_3^{k+1}-\rho_2^{k+1}\right)+\lambda^2\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\rho_2\rho_3\left((\lambda+\beta_1+\beta_2)\left(\rho_2^k\rho_3-\rho_3^k\rho_2+(\rho_2+\rho_3)\left(\rho_3^k-\rho_2^k\right)\right)+\lambda\left(\rho_2^k-\rho_3^k\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_\nu\rho_2\rho_3\left(\rho_2\rho_3\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\rho_3+\rho_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^{k+1}-\rho_2^{k+1}\right)}{\lambda(\rho_3-\rho_2)} & \frac{\rho_2\rho_3\left(\rho_3^{k+1}-\rho_2^{k+1}\right)\left(\frac{\lambda(\rho_3+\rho_2)}{\rho_2\rho_3}-(\lambda+\beta_1+\beta_2)\right)-\lambda\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\rho_2\rho_3\left(\lambda\left(\rho_2^k-\rho_3^k\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^{k+1}-\rho_2^{k+1}\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_\nu\rho_2\rho_3\left(\rho_3^{k+1}-\rho_2^{k+1}\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^{k+1}-\rho_2^{k+1}\right)}{\lambda(\rho_3-\rho_2)} & \frac{\rho_2\rho_3\left((\theta+\lambda+\mu_\nu)\left(\rho_3^{k+1}-\rho_2^{k+1}\right)-\lambda\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix} = \mathcal{A}^{k+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{bmatrix} \mathcal{A}^{k+1} & \mathcal{A}^k\mathcal{C} + (\sum_{i=0}^{k-1}\mathcal{A}^i\mathcal{C}\mathcal{B}^{\mathbf{k-i}}) \\ \mathbf{0} & \mathcal{B}^{k+1} \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A}^{k+1} & \sum_{i=0}^{k}\mathcal{A}^i\mathcal{C}\mathcal{B}^{\mathbf{k-i}} \\ \mathbf{0} & \mathcal{B}^{k+1} \end{bmatrix} = \mathbf{R}^{k+1}
\end{aligned}
$$

□

Proposition 6.2. *Using* $\mathbf{R} = \begin{bmatrix} \mathcal{A} & \mathcal{C} \\ \mathbf{0} & \mathcal{B} \end{bmatrix}$ *from (4),*

where $\mathcal{A} = \begin{bmatrix} \frac{\lambda(\lambda+\beta_1+\beta_2)}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_\nu(\beta_1+\lambda)} & \frac{\lambda\mu_\nu}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_\nu(\beta_1+\lambda)} \\ \frac{\beta_2\lambda}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_\nu(\beta_1+\lambda)} & \frac{\lambda(\theta+\lambda+\mu_\nu)}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_\nu(\beta_1+\lambda)} \end{bmatrix}$

$= \frac{\lambda}{(\lambda+\beta_1+\beta_2)(\theta+\lambda)+\mu_\nu(\beta_1+\lambda)} \begin{bmatrix} \lambda+\beta_1+\beta_2 & \mu_\nu \\ \beta_2 & \theta+\lambda+\mu_\nu \end{bmatrix}$

$= \frac{\rho_2\rho_3}{\lambda} \begin{bmatrix} \lambda+\beta_1+\beta_2 & \mu_\nu \\ \beta_2 & \theta+\lambda+\mu_\nu \end{bmatrix}$, *we find:*

$$
\mathcal{A}^k = \begin{bmatrix} \frac{\rho_2\rho_3\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_\nu\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} & \frac{\rho_2\rho_3\left((\theta+\lambda+\mu_\nu)\left(\rho_3^k-\rho_2^k\right)-\lambda\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix}
$$

Proof.
We use Mathematical Induction by noting that:

$$
\begin{aligned}
\mathcal{A}^1 &= \begin{bmatrix} \frac{\rho_2\rho_3\left(\lambda\left(\rho_2^0-\rho_3^0\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^1-\rho_2^1\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_\nu\rho_2\rho_3\left(\rho_3^1-\rho_2^1\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^1-\rho_2^1\right)}{\lambda(\rho_3-\rho_2)} & \frac{\rho_2\rho_3\left((\theta+\lambda+\mu_\nu)\left(\rho_3^1-\rho_2^1\right)-\lambda\left(\rho_3^0-\rho_2^0\right)\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\rho_2\rho_3(\lambda+\beta_1+\beta_2)}{\lambda} & \frac{\mu_\nu\rho_2\rho_3}{\lambda} \\ \frac{\beta_2\rho_2\rho_3}{\lambda} & \frac{\rho_2\rho_3\left((\theta+\lambda+\mu_\nu)\left(\rho_3^1-\rho_2^1\right)\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\rho_2\rho_3(\lambda+\beta_1+\beta_2)}{\lambda} & \frac{\mu_\nu\rho_2\rho_3}{\lambda} \\ \frac{\beta_2\rho_2\rho_3}{\lambda} & \frac{\rho_2\rho_3(\theta+\lambda+\mu_\nu)}{\lambda} \end{bmatrix} = \mathcal{A}
\end{aligned}
$$

and assume the result for $\mathcal{A}^k$, write $\mathcal{A}^{k+1}$:

Remark. Three substitutions were needed in this derivation; namely:

$(\lambda+\beta_1+\beta_2)^2+\beta_2\mu_\nu = \frac{\lambda((\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)-\lambda)}{\rho_2\rho_3}$, and
$\theta+2\lambda+\beta_1+\beta_2+\mu_\nu = \frac{\lambda(\rho_3+\rho_2)}{\rho_2\rho_3}$, and
$\lambda(\rho_3+\rho_2)-\rho_2\rho_3(\lambda+\beta_1+\beta_2) = \rho_2\rho_3(\theta+\lambda+\mu_\nu)$.
These can readily be verified from (6).

□

Proposition 6.3. *Using the block-matrix form of* $\mathbf{R} = \begin{bmatrix} \mathcal{A} & \mathcal{C} \\ \mathbf{0} & \mathcal{B} \end{bmatrix}$, *we find by Lemma 6.1:*

$$\mathbf{R}^k = \begin{bmatrix} \mathcal{A}^k & \sum_{i=0}^{k-1} \mathcal{A}^i \mathcal{C} \mathcal{B}^{\mathbf{k-i-1}} \\ \mathbf{0} & \mathcal{B}^k \end{bmatrix} = \begin{bmatrix} \mathcal{A}^k & \mathcal{C}(k) \\ \mathbf{0} & \mathcal{B}^k \end{bmatrix} \tag{7}$$

where:

$$\mathcal{A}^k = \begin{bmatrix} \frac{\rho_2\rho_3\left(\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)+(\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda(\rho_3-\rho_2)} & \frac{\mu_\nu\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} \\ \frac{\beta_2\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)} & \frac{\rho_2\rho_3\left((\theta+\lambda+\mu_\nu)\left(\rho_3^k-\rho_2^k\right)-\lambda\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda(\rho_3-\rho_2)} \end{bmatrix} \text{ (by 6.2)}$$

$$\mathcal{B}^k = \begin{bmatrix} \frac{(\beta_1\rho_0-\lambda)\rho_0^k+\rho_1^k(\lambda-\beta_1\rho_1)}{\beta_1(\rho_0-\rho_1)} & \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)} \\ \frac{\lambda(\lambda+\beta_2)\left(\rho_0^k-\rho_1^k\right)}{\mu_b\beta_1(\rho_0-\rho_1)} & \frac{(\lambda-\beta_1\rho_1)\rho_0^k+\rho_1^k(\beta_1\rho_0-\lambda)}{\beta_1(\rho_0-\rho_1)} \end{bmatrix} \text{ and if:}$$

$$\delta_1 = \frac{\rho_2\left(\lambda^2-\rho_3(\lambda(\theta+\lambda)+\beta_1\mu_\nu\rho_2)\right)}{\lambda\mu_b\beta_1(\rho_0-\rho_1)(\rho_2-\rho_3)},\quad \delta_2 = \frac{\rho_3\left(\lambda^2-\rho_2(\lambda(\theta+\lambda)+\beta_1\mu_\nu\rho_3)\right)}{\lambda\mu_b\beta_1(\rho_0-\rho_1)(\rho_2-\rho_3)},\quad \delta_3 = \frac{\rho_3\left(\lambda^2-\rho_2\left(\lambda^2+\beta_1\rho_3(\theta+\lambda+\mu_\nu)\right)\right)}{\lambda(\rho_0-\rho_1)(-\rho_2+\rho_3)},$$
$$\delta_4 = \frac{\rho_2\left(\lambda^2-\rho_3\left(\lambda^2+\beta_1\rho_2(\theta+\lambda+\mu_\nu)\right)\right)}{\lambda(\rho_0-\rho_1)(-\rho_2+\rho_3)}$$

then $\mathcal{C}(k) = \begin{bmatrix} c_{11}(k) & c_{12}(k) \\ c_{21}(k) & c_{22}(k) \end{bmatrix}$, *where:*

$$c_{11}(k) = \delta_1\left(\frac{\rho_0(\lambda-\mu_b\rho_1)\left(\rho_0^k-\rho_2^k\right)}{\rho_1(\rho_0-\rho_2)} - \frac{\rho_1(\lambda-\mu_b\rho_0)\left(\rho_1^k-\rho_2^k\right)}{\rho_0(\rho_1-\rho_2)}\right) + \delta_2\left(\frac{\rho_1(\lambda-\mu_b\rho_0)\left(\rho_1^k-\rho_3^k\right)}{\rho_0(\rho_1-\rho_3)} - \frac{\rho_0(\lambda-\mu_b\rho_1)\left(\rho_0^k-\rho_3^k\right)}{\rho_1(\rho_0-\rho_3)}\right)$$
$$c_{12}(k) = \delta_1\left(\frac{\mu_b\rho_0\left(\rho_0^k-\rho_2^k\right)}{\rho_0-\rho_2} - \frac{\mu_b\rho_1\left(\rho_1^k-\rho_2^k\right)}{\rho_1-\rho_2}\right) + \delta_2\left(\frac{\mu_b\rho_1\left(\rho_1^k-\rho_3^k\right)}{\rho_1-\rho_3} - \frac{\mu_b\rho_0\left(\rho_0^k-\rho_3^k\right)}{\rho_0-\rho_3}\right)$$
$$c_{21}(k) = \delta_3\left(\frac{\rho_0^2(\lambda-\mu_b\rho_1)\left(\rho_0^k-\rho_3^k\right)}{\lambda^2(\rho_0-\rho_3)} - \frac{\rho_1^2(\lambda-\mu_b\rho_0)\left(\rho_1^k-\rho_3^k\right)}{\lambda^2(\rho_1-\rho_3)}\right) + \delta_4\left(\frac{\rho_1^2(\lambda-\mu_b\rho_0)\left(\rho_1^k-\rho_2^k\right)}{\lambda^2(\rho_1-\rho_2)} - \frac{\rho_0^2(\lambda-\mu_b\rho_1)\left(\rho_0^k-\rho_2^k\right)}{\lambda^2(\rho_0-\rho_2)}\right)$$
$$c_{22}(k) = \delta_3\left(\frac{\rho_0\left(\rho_0^k-\rho_3^k\right)}{\beta_1(\rho_0-\rho_3)} - \frac{\rho_1\left(\rho_1^k-\rho_3^k\right)}{\beta_1(\rho_1-\rho_3)}\right) + \delta_4\left(\frac{\rho_1\left(\rho_1^k-\rho_2^k\right)}{\beta_1(\rho_1-\rho_2)} - \frac{\rho_0\left(\rho_0^k-\rho_2^k\right)}{\beta_1(\rho_0-\rho_2)}\right)$$

Proof. The above result is the consequence of previous works, specifically Lemma 6.1, Proposition 6.2, and the observation that $\mathcal{B}$ is entry-wise identical to the matrix $\mathbf{R}$ from (Patterson & Korzeniowski, 2018) after letting $\mu = \mu_b$. The rest is merely the computation of $C(k) = \sum_{i=0}^{k-1} \mathcal{A}^i \mathcal{C} \mathcal{B}^{\mathbf{k-i-1}}$ which is tedious but straightforward. □

6.2 The Initial Terms of $\boldsymbol{\pi}$

Turning attention to the computation of $\mathbf{B}[\mathbf{R}] = \begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{C}} \\ \hat{\mathbf{B}} & \mathbf{A}+\mathbf{RB} \end{bmatrix}$, and a positive vector $(x_0, \mathbf{x_1})$, such that $(x_0, \mathbf{x_1})\mathbf{B}[\mathbf{R}] = 0$, we have:

$$(x_0, \mathbf{x_1})\mathbf{B}[\mathbf{R}] = (x_0, \mathbf{x_1}) \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ 0 & -\theta-\lambda-\mu_\nu & \mu_\nu & \theta+\lambda & 0 \\ \beta_1 & \beta_2 & -\beta_1-\beta_2-\lambda & \lambda & 0 \\ 0 & 0 & 0 & -\mu_b & \mu_b \\ \beta_1 & 0 & 0 & \beta_2+\lambda & -\beta_1-\beta_2-\lambda \end{bmatrix} = \mathbf{0} \tag{8}$$

$$\Longrightarrow \begin{cases} x_0 = 1 \\ x_{10} = \frac{\lambda(\lambda+\beta_1+\beta_2)}{\beta_2(\theta+\lambda)+\beta_1(\theta+\lambda+\mu_\nu)+\lambda(\theta+\lambda+\mu_\nu)} \\ x_{11} = \frac{\lambda\mu_\nu}{\beta_2(\theta+\lambda)+\beta_1(\theta+\lambda+\mu_\nu)+\lambda(\theta+\lambda+\mu_\nu)} \\ x_{12} = \frac{\lambda(\lambda+\beta_1+\beta_2)(\beta_1(\theta+\lambda)+\beta_2(\theta+\lambda)+\lambda(\theta+\lambda+\mu_\nu))}{\beta_1\mu_b(\beta_2(\theta+\lambda)+\beta_1(\theta+\lambda+\mu_\nu)+\lambda(\theta+\lambda+\mu_\nu))} \\ x_{13} = \frac{\lambda(\beta_1(\theta+\lambda)+\beta_2(\theta+\lambda)+\lambda(\theta+\lambda+\mu_\nu))}{\beta_1(\beta_2(\theta+\lambda)+\beta_1(\theta+\lambda+\mu_\nu)+\lambda(\theta+\lambda+\mu_\nu))} \end{cases}$$

We normalize the solution in order to generate the first three terms of π:

$$K(\mathbf{x_0} + \mathbf{x_1}(\mathbf{I}-\mathbf{R})^{-1}\mathbf{e}) = 1$$

$$\Longrightarrow K = \frac{(\beta_1(\mu_b-\lambda)-\lambda(\mu_b+\beta_2))(\theta\beta_2+\beta_1(\theta+\mu_v))}{\beta_1((\theta+\lambda)(\beta_1+\beta_2)\mu_b+((\beta_1(\mu_b-\lambda)-\lambda(\mu_b+\beta_2))+\lambda\mu_b)\mu_v)} \tag{9}$$

$$(\pi_{00},\pi_{10},\pi_{11},\pi_{12},\pi_{13}) = K(x_0,\mathbf{x_1}) \Longrightarrow \begin{cases} \pi_{00} = K \\ \pi_{10} = \frac{K\rho_2\rho_3(\lambda+\beta_1+\beta_2)}{\lambda} \\ \pi_{11} = \frac{K\mu_v\rho_2\rho_3}{\lambda} \\ \pi_{12} = \frac{K(\lambda+\beta_1+\beta_2)\left(\lambda^2-\beta_1\mu_v\rho_2\rho_3\right)}{\lambda\beta_1\mu_b} \\ \pi_{13} = \frac{K\left(\lambda^2-\beta_1\mu_v\rho_2\rho_3\right)}{\lambda\beta_1} \end{cases}$$

Remark. Observe that the condition given by Corollary 5 for positive recurrence: $\beta_1(\mu_b-\lambda)-\lambda(\mu_b+\beta_2) > 0$ is equivalent to $K > 0$.

Proposition 6.4. *The remaining elements* $\{(\pi_{k0},\pi_{k1},\pi_{k2},\pi_{k3}) \mid k \geqslant 2\}$ *of the stationary distribution satisfying* $(\pi_{k0},\pi_{k1},\pi_{k2},\pi_{k3}) = (\pi_{10},\pi_{11},\pi_{12},\pi_{13})\mathbf{R}^{k-1}$ *and* $\pi_{00}+\pi_{10}+\pi_{11}+\pi_{12}+\pi_{13}+\sum_{k=2}^{\infty}(\pi_{k0}+\pi_{k1}+\pi_{k2}+\pi_{k3}) = 1$ *are given by:*

$$\begin{cases} \pi_{k0} = \frac{K\rho_2\rho_3\left((\lambda+\beta_1+\beta_2)\left(\rho_3^k-\rho_2^k\right)-\lambda\left(\rho_3^{k-1}-\rho_2^{k-1}\right)\right)}{\lambda(\rho_3-\rho_2)} \\ \pi_{k1} = \frac{K\mu_v\rho_2\rho_3\left(\rho_2^k-\rho_3^k\right)}{\lambda(\rho_2-\rho_3)} \\ \pi_{k2} = K\left(\frac{\mu_b(\delta_1-\delta_2)\left(\beta_1\left(\rho_0^{k+1}-\rho_1^{k+1}\right)-\lambda\left(\rho_0^k-\rho_1^k\right)\right)}{\lambda} + \frac{\rho_2\rho_3(\lambda+\beta_1+\beta_2)}{\lambda}\left(\delta_1\left(\frac{\rho_0(\lambda-\mu_b\rho_1)\left(\rho_0^{k-1}-\rho_2^{k-1}\right)}{\rho_1(\rho_0-\rho_2)} - \frac{\rho_1(\lambda-\mu_b\rho_0)\left(\rho_1^{k-1}-\rho_2^{k-1}\right)}{\rho_0(\rho_1-\rho_2)}\right) \right.\right. \\ \qquad \left. +\delta_2\left(\frac{\rho_1(\lambda-\mu_b\rho_0)\left(\rho_1^{k-1}-\rho_3^{k-1}\right)}{\rho_0(\rho_1-\rho_3)} - \frac{\rho_0(\lambda-\mu_b\rho_1)\left(\rho_0^{k-1}-\rho_3^{k-1}\right)}{\rho_1(\rho_0-\rho_3)}\right)\right) \\ \qquad \left. + \frac{\mu_v\rho_2\rho_3}{\lambda^3}\left(\delta_4\left(\frac{\rho_1^2(\lambda-\mu_b\rho_0)\left(\rho_1^{k-1}-\rho_2^{k-1}\right)}{\rho_1-\rho_2} - \frac{\rho_0^2(\lambda-\mu_b\rho_1)\left(\rho_0^{k-1}-\rho_2^{k-1}\right)}{\rho_0-\rho_2}\right) + \delta_3\left(\frac{\rho_0^2(\lambda-\mu_b\rho_1)\left(\rho_0^{k-1}-\rho_3^{k-1}\right)}{\rho_0-\rho_3} - \frac{\rho_1^2(\lambda-\mu_b\rho_0)\left(\rho_1^{k-1}-\rho_3^{k-1}\right)}{\rho_1-\rho_3}\right)\right)\right) \\ \pi_{k3} = K\left(\mu_b(\delta_1-\delta_2)\left(\rho_0^k-\rho_1^k\right) + \frac{\mu_b(\lambda+\beta_1+\beta_2)\rho_2\rho_3}{\lambda}\left(\delta_1\left(\frac{\rho_0\left(\rho_0^{k-1}-\rho_2^{k-1}\right)}{\rho_0-\rho_2} - \frac{\rho_1\left(\rho_1^{k-1}-\rho_2^{k-1}\right)}{\rho_1-\rho_2}\right)\right.\right. \\ \qquad \left. +\delta_2\left(\frac{\rho_1\left(\rho_1^{k-1}-\rho_3^{k-1}\right)}{\rho_1-\rho_3} - \frac{\rho_0\left(\rho_0^{k-1}-\rho_3^{k-1}\right)}{\rho_0-\rho_3}\right)\right) \\ \qquad \left. - \frac{\mu_v\rho_2\rho_3}{\lambda\beta_1}\left(\frac{\delta_4\left((\rho_1-\rho_2)\rho_0^k-(\rho_0-\rho_2)\rho_1^k+(\rho_0-\rho_1)\rho_2^k\right)}{(\rho_0-\rho_2)(\rho_1-\rho_2)} - \frac{\delta_3\left((\rho_1-\rho_3)\rho_0^k-(\rho_0-\rho_3)\rho_1^k-(\rho_1-\rho_0)\rho_3^k\right)}{(\rho_0-\rho_3)(\rho_1-\rho_3)}\right)\right) \end{cases} \tag{10}$$

Proof. We begin by noting that:

$$\begin{aligned}(\pi_{k0},\pi_{k1},\pi_{k2},\pi_{k3}) = (\pi_{10},\pi_{11},\pi_{12},\pi_{13})\mathbf{R}^{k-1} &\iff (\pi_{k0},\pi_{k1},\pi_{k2},\pi_{k3}) = (\pi_{10},\pi_{11},\pi_{12},\pi_{13})\mathbf{R}^{k-2}\mathbf{R} \\ &\iff (\pi_{k0},\pi_{k1},\pi_{k2},\pi_{k3}) = (\pi_{k-1,0},\pi_{k-1,1},\pi_{k-1,2},\pi_{k-1,3})\mathbf{R}\end{aligned}$$

Furthermore, we use an alternative form of $\mathbf{R}$ given below. This is entry-by-entry identical to that defined by (4), but is merely expressed in terms of $\rho_0, \rho_1, \rho_2, \rho_3, \delta_1, \delta_2, \delta_3$ and δ_4 whenever possible.

$$\mathbf{R} = \begin{bmatrix} \frac{(\lambda+\beta_1+\beta_2)\rho_2\rho_3}{\lambda} & \frac{\mu_v\rho_2\rho_3}{\lambda} & (\lambda+\beta_1+\beta_2)(\delta_1-\delta_2)(\rho_0-\rho_1) & (\delta_1-\delta_2)\mu_b(\rho_0-\rho_1) \\ \frac{\beta_2\rho_2\rho_3}{\lambda} & \frac{\lambda(\rho_2+\rho_3)-(\lambda+\beta_1+\beta_2)\rho_2\rho_3}{\lambda} & \frac{(\lambda+\beta_1+\beta_2)(\delta_3-\delta_4)\rho_0(\rho_0-\rho_1)\rho_1}{\lambda^2} & \frac{(\delta_3-\delta_4)(\rho_0-\rho_1)}{\beta_1} \\ 0 & 0 & \frac{\beta_1(\rho_0+\rho_1)-\lambda}{\beta_1} & \frac{\lambda}{\beta_1} \\ 0 & 0 & \frac{\lambda(\rho_0+\rho_1)-(\beta_1+\mu_b)\rho_0\rho_1}{\lambda} & \frac{\lambda}{\beta_1} \end{bmatrix} \tag{11}$$

Next, define: $(\pi_{k-1,0},\pi_{k-1,1},\pi_{k-1,2},\pi_{k-1,3})\mathbf{R} = (a,b,c,d)$

Then:

$$\begin{aligned} a &= \frac{K\rho_2\rho_3\left((\lambda+\beta_1+\beta_2)\left(\rho_3^{k-1}-\rho_2^{k-1}\right)-\lambda\left(\rho_3^{k-2}-\rho_2^{k-2}\right)\right)}{\lambda(\rho_3-\rho_2)}\frac{(\lambda+\beta_1+\beta_2)\rho_2\rho_3}{\lambda} + \frac{K\mu_v\rho_2\rho_3\left(\rho_2^{k-1}-\rho_3^{k-1}\right)}{\lambda(\rho_2-\rho_3)}\frac{\beta_2\rho_2\rho_3}{\lambda} \\ &= \frac{K\rho_2^2\rho_3^2\left(\left((\lambda+\beta_1+\beta_2)^2+\beta_2\mu_v\right)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)-\lambda(\lambda+\beta_1+\beta_2)\left(\rho_2^{k-2}-\rho_3^{k-2}\right)\right)}{\lambda^2(\rho_2-\rho_3)} \\ &= \frac{K\rho_2^2\rho_3^2\left(\left(\frac{\lambda((\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)-\lambda)}{\rho_2\rho_3}\right)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)-\lambda(\lambda+\beta_1+\beta_2)\left(\rho_2^{k-2}-\rho_3^{k-2}\right)\right)}{\lambda^2(\rho_2-\rho_3)} \\ &= \frac{K\rho_2\rho_3\left((\lambda((\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)-\lambda))\left(\rho_2^{k-1}-\rho_3^{k-1}\right)-\lambda(\lambda+\beta_1+\beta_2)\rho_2\rho_3\left(\rho_2^{k-2}-\rho_3^{k-2}\right)\right)}{\lambda^2(\rho_2-\rho_3)} \\ &= \frac{K\rho_2\rho_3\left(\left(\lambda(\lambda+\beta_1+\beta_2)(\rho_2+\rho_3)-\lambda^2\right)\left(\rho_2^{k-1}-\rho_3^{k-1}\right)-\lambda(\lambda+\beta_1+\beta_2)\rho_2\rho_3\left(\rho_2^{k-2}-\rho_3^{k-2}\right)\right)}{\lambda^2(\rho_2-\rho_3)} \\ &= \frac{K\rho_2\rho_3\left(\lambda(\lambda+\beta_1+\beta_2)\left(\rho_2^k-\rho_3^k\right)-\lambda^2\left(\rho_2^{k-1}-\rho_3^{k-1}\right)\right)}{\lambda^2(\rho_2-\rho_3)} \end{aligned}$$

$= \frac{K\rho_2\rho_3\left((\lambda+\beta_1+\beta_2)\left(\rho_2^k-\rho_3^k\right)-\lambda\left(\rho_2^{k-1}-\rho_3^{k-1}\right)\right)}{\lambda(\rho_2-\rho_3)}$ and,

$$
\begin{aligned}
b &= \frac{K\rho_2\rho_3\left((\lambda+\beta_1+\beta_2)\left(\rho_3^{k-1}-\rho_2^{k-1}\right)-\lambda\left(\rho_3^{k-2}-\rho_2^{k-2}\right)\right)}{\lambda(\rho_3-\rho_2)}\frac{\mu_\nu\rho_2\rho_3}{\lambda}+\frac{K\mu_\nu\rho_2\rho_3\left(\rho_2^{k-1}-\rho_3^{k-1}\right)}{\lambda(\rho_2-\rho_3)}\frac{\lambda(\rho_2+\rho_3)-(\lambda+\beta_1+\beta_2)\rho_2\rho_3}{\lambda}\\
&= \frac{K\mu_\nu\rho_2^2\rho_3^2\left((\lambda+\beta_1+\beta_2)\left(\rho_3^{k-1}-\rho_2^{k-1}\right)-\lambda\left(\rho_3^{k-2}-\rho_2^{k-2}\right)\right)+K\mu_\nu\rho_2\rho_3\left(\rho_3^{k-1}-\rho_2^{k-1}\right)(\lambda(\rho_2+\rho_3)-(\lambda+\beta_1+\beta_2)\rho_2\rho_3)}{\lambda^2(\rho_3-\rho_2)}\\
&= \frac{K\mu_\nu\rho_2^2\rho_3^2\left(-\lambda\left(\rho_3^{k-2}-\rho_2^{k-2}\right)\right)+K\mu_\nu\rho_2\rho_3\left(\rho_3^{k-1}-\rho_2^{k-1}\right)(\lambda(\rho_2+\rho_3))}{\lambda^2(\rho_3-\rho_2)}\\
&= \frac{K\mu_\nu\rho_2\rho_3\left(\lambda(\rho_2+\rho_3)\left(\rho_3^{k-1}-\rho_2^{k-1}\right)-\lambda\rho_2\rho_3\left(\rho_3^{k-2}-\rho_2^{k-2}\right)\right)}{\lambda^2(\rho_3-\rho_2)}\\
&= \frac{K\mu_\nu\rho_2\rho_3\left(\lambda\left(\rho_3^k-\rho_2^k\right)\right)}{\lambda^2(\rho_3-\rho_2)}\\
&= \frac{K\mu_\nu\rho_2\rho_3\left(\rho_3^k-\rho_2^k\right)}{\lambda(\rho_3-\rho_2)}
\end{aligned}
$$

Since the verification of c and d are similar in nature to a and b, but are too lengthy to provide the step-by-step details, they are omitted. Similarly, one verifies that (10) recovers $(\pi_{00},\pi_{10},\pi_{11},\pi_{12},\pi_{13})$ when $k=1$, and consequently: $\pi_{00}+\sum_{k=1}^{\infty}(\pi_{k0}+\pi_{k1}+\pi_{k2}+\pi_{k3})=1$. These steps are omitted as well.

□

7. Special Cases

We can now recover, as a special case, the stationary distribution of the model studied in (Patterson & Korzeniowski, 2018).

Proposition 7.1. *Let $\theta\to\infty$ with $\mu_b=\mu$, then (10) recovers the stationary distribution of the model studied in (Patterson & Korzeniowski, 2018). Consequently, we also get the special cases of the previous model as follows:*

- *Let $\theta\to\infty$ with $\mu_b=\mu$, and*
  - *i. $\beta_1\to\infty$ with $0\leqslant\beta_2<\infty$ results in the classical $M/M/1$ queue.*
  - *ii $0<\beta_1<\infty$ with $\beta_2=0$ and $\mu=\beta_1$ results in an $M/E_2/1$ queue, where E_2 refers to an 'Erlang' service time distribution with shape 2 and rate μ.*
  - *iii $0<\beta_1<\infty$ with $\beta_2=0$ and $\mu>\beta_1$ results in an $M/HE/1$ queue, where HE refers to a hypoexponential service time distribution $\sim f(t)=\frac{\mu\beta_1(e^{-\beta_1 t}-e^{-\mu t})}{\mu-\beta_1}$* (Ross, 2006).

Proof. We obtain the stationary distribution from the previous model by substituting $\mu_b=\mu$, computing K, $\{\rho_i\}_{i=0,1,2,3}$ and $\{\delta_j\}_{j=1,2,3,4}$, and letting $\theta\to\infty$. Namely:

$\Longrightarrow \lim_{\theta\to\infty} K=\frac{\beta_1(\mu-\lambda)-\lambda(\beta_2+\mu)}{\beta_1\mu}$

$\Longrightarrow \lim_{\theta\to\infty}\{\rho_0,\rho_1,\rho_2,\rho_3\}=\{\rho_0,\rho_1,\frac{\lambda}{\beta_1+\beta_2+\lambda},0\}$

$\Longrightarrow \lim_{\theta\to\infty}\{\delta_1,\delta_2,\delta_3,\delta_4\}=\{\frac{\lambda}{\mu\beta_1(\rho_0-\rho_1)},0,0,-\frac{\lambda}{\rho_0-\rho_1}\}$

Care is needed when substituting these values into (10) due to indeterminate expression 0^0 arising from ρ_3^k and ρ_3^{k-1} terms, leading to three different cases.

<u>k=0</u>

$\boldsymbol{\pi}_k=(K,0,0,0)$

$\Longrightarrow \lim_{\theta\to\infty}\boldsymbol{\pi}_k=K(1,0,0,0)$

<u>k=1</u>

$$
\begin{aligned}
\Longrightarrow \lim_{\theta\to\infty}\boldsymbol{\pi}_k &= \lim_{\theta\to\infty}\left(\frac{K\rho_2\rho_3(\lambda+\beta_1+\beta_2)}{\lambda},\frac{K\mu_\nu\rho_2\rho_3}{\lambda},\frac{K(\lambda+\beta_1+\beta_2)\left(\lambda^2-\beta_1\mu_\nu\rho_2\rho_3\right)}{\lambda\beta_1\mu_b},\frac{K\left(\lambda^2-\beta_1\mu_\nu\rho_2\rho_3\right)}{\lambda\beta_1}\right)\\
&= K\left(0,0,\frac{\lambda(\lambda+\beta_1+\beta_2)}{\beta_1\mu_b},\frac{\lambda}{\beta_1}\right)
\end{aligned}
$$

<u>$k\geqslant 2$</u>

$$
\begin{aligned}
\Longrightarrow \lim_{\theta\to\infty}\boldsymbol{\pi}_k &= K\left(0,0,\frac{\mu_b\delta_1\left(\beta_1\left(\rho_0^{k+1}-\rho_1^{k+1}\right)-\lambda\left(\rho_0^k-\rho_1^k\right)\right)}{\lambda},\mu_b\delta_1\left(\rho_0^k-\rho_1^k\right)\right)\\
&= K\left(0,0,\frac{\beta_1\left(\rho_0^{k+1}-\rho_1^{k+1}\right)-\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)},\frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}\right)
\end{aligned}
$$

$$= K\left(0, 0, \frac{\rho_0^{k+1}-\rho_1^{k+1}}{\rho_0-\rho_1} - \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}, \frac{\lambda\left(\rho_0^k-\rho_1^k\right)}{\beta_1(\rho_0-\rho_1)}\right)$$

We recall that the first two entries in $\boldsymbol{\pi}_k$ are from states where $J(t) = 0$, i.e., where the server is undergoing a working vacation (see Definition 2.1). By taking $\theta \to \infty$, we take the expected working vacation duration to 0. Thus for $k \geqslant 1$, π_{k0} and π_{k1} are 0. It is still possible, however, to visit the vacation state $(0, 0, 0)$ when the queue is empty. □

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