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Cubic Rank Transmuted Modified Burr III Pareto Distribution: Development, Properties, Characterizations and Applications

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Abstract

In this paper, a flexible lifetime distribution called Cubic rank transmuted modified Burr III-Pareto (CRTMBIII-P) is developed on the basis of the cubic ranking transmutation map. The density function of CRTMBIII-P is arc, exponential, left-skewed, right-skewed and symmetrical shaped. Descriptive measures such as moments, incomplete moments, inequality measures, residual life function and reliability measures are theoretically established. The CRTMBIII-P distribution is characterized via ratio of truncated moments. Parameters of the CRTMBIII-P distribution are estimated using maximum likelihood method. The simulation study for the performance of the maximum likelihood estimates (MLEs) of the parameters of the CRTMBIII-P distribution is carried out. The potentiality of CRTMBIII-P distribution is demonstrated via its application to the real data sets: tensile strength of carbon fibers and strengths of glass fibers. Goodness of fit of this distribution through different methods is studied.

Keywords: moments, reliability, characterizations, maximum likelihood estimation

1. Introduction

In recent decades, many continuous univariate distributions have been developed but various data sets from reliability, insurance, finance, climatology, biomedical sciences and other areas do not follow these distributions. Therefore, modified, extended and generalized distributions and their applications to the problems in these areas is a clear need of day.

The modified, extended and generalized distributions are obtained by the introduction of some transformation or addition of one or more parameters to the well-known baseline distributions. These new developed distributions provide better fit to the data than the sub and competing models.

Shaw and Buckley (2009) proposed ranking quadratic transmutation map to solve financial problems.

1.1 Quadratic Ranking Transmutation Map

Theorem 1.1: Let Z_1 and Z_2 be independent and identically distributed (i.i.d.) random variables with the common cumulative distribution function $G(z)$. Then, the ranking quadratic transmutation map is

$$F(x)=(1+\lambda)\,G(x)-\lambda G^2(x), \quad \lambda\in[-1,1]. \tag{1}$$

Proof

Let Z_1 and Z_2 be i.i.d. random variables with the common cumulative distribution function $G(z)$ of the parent distribution. Now, consider the following order:

$$Z_{1:2}=\min(Z_1,Z_2) \;\; and \;\; Z_{2:2}=\max(Z_1,Z_2),$$

let $Y\overset{d}{=}Z_{1:2}$, with probability π,

$Y\overset{d}{=}Z_{2:2}$, with probability $1-\pi$,

$where\ 0\le\pi\le1.$ The cumulative distribution function of Y is

$$F_Y(z)=\pi\Pr\left(\min(Z_1,Z_2)\le z\right)+(1-\pi)\Pr\left(\max(Z_1,Z_2)\le z\right).$$

Arnold et al. (1992) showed that

$$\Pr\left(\min\left(Z_1, Z_2\right) \le z\right) = 1 - \left[1 - G\left(z\right)\right]^2 ,$$

and

$$\Pr\left(\max\left(Z_1, Z_2\right) \le z\right) = \left[G\left(z\right)\right]^2 .$$

Now, the cumulative distribution function of Y becomes

$$F_Y\left(z\right) = \pi\left[1 - \left[1 - G\left(z\right)\right]^2\right] + \left(1 - \pi\right)\left[G\left(z\right)\right]^2,$$

$$F_Y\left(z\right) = 2\pi G\left(z\right) + \left(1 - 2\pi\right)\left[G\left(z\right)\right]^2. \tag{2}$$

If we take $\lambda = 2\pi - 1,$ the distribution in equation (2) is known as ranking quadratic transmutation map or transmuted distribution.

2. Cubic Ranking Transmutation Map

Theorem 2.1: Let Z_1, Z_2 and Z_3 be i.i.d. random variables with the common cumulative distribution function $G\left(z\right)$. Then, the cubic ranking transmutation map is

$$F\left(z\right) = \lambda_1 G\left(z\right) + \left(\lambda_2 - \lambda_1\right)G^2\left(z\right) + \left(1 - \lambda_2\right)G^3\left(z\right), \quad \lambda_1 \in [0,1], \quad \lambda_2 \in [-1,1]. \tag{3}$$

Proof

Consider the following order:

$$Z_{1:3} = \min\left(Z_1, Z_2, Z_3\right) \textit{ and } Z_{3:3} = \max\left(Z_1, Z_2, Z_3\right).$$

Let $Y \overset{d}{=} Z_{1:3}$, with probability π_1,

$Y \overset{d}{=} Z_{2:3}$, with probability π_2,

$Y \overset{d}{=} Z_{3:3}$, with probability π_3,

where $0 \le \pi_i \le 1,\ \pi_3 = 1 - \pi_1 - \pi_2$ *and* $\sum_{i=1}^{3} \pi_i = 1$. The cumulative distribution function of Y is

$$F_Y\left(z\right) = \pi_1 \Pr\left(\min\left(Z_1, Z_2, Z_3\right) \le z\right) + \pi_2 \Pr\left(Z_{2:3} \le z\right) + \left(1 - \pi_1 - \pi_2\right)\Pr\left(\max\left(Z_1, Z_2, Z_3\right) \le z\right).$$

Arnold et al. (1992) showed that $\Pr\left(\min\left(Z_1, Z_2, Z_3\right) \le z\right) = 1 - \left[1 - G\left(z\right)\right]^3$,

$\Pr\left(Z_{2;3} \le z\right) = 3G^2\left(z\right) - 2G^3\left(z\right)$ and $\Pr\left(\max\left(Z_1, Z_2, Z_3\right) \le z\right) = \left[G\left(z\right)\right]^3$.

Now, the cumulative distribution function of Y becomes

$$F_Y\left(z\right) = 3\pi_1 G\left(z\right) + 3\left(\pi_2 - \pi_1\right)\left[G\left(z\right)\right]^2 + \left(1 - \pi_2\right)\left[G\left(z\right)\right]^3. \tag{4}$$

If we take $\lambda_1 = 3\pi_1 - 2\pi_1$ *and* $\lambda_2 = \pi_2$ the distribution in equation (4) is known as Cubic ranking transmutation map or transmuted distribution of order 2.

Definition 2.1

The cumulative distribution function (cdf) and probability density function (pdf) for the cubic rank transmuted distribution are given, respectively, by

$$F\left(x\right) = \lambda_1 G\left(x\right) + \left(\lambda_2 - \lambda_1\right)G^2\left(x\right) + \left(1 - \lambda_2\right)G^3\left(x\right), \quad \lambda_1 \in [-1,1], \quad \lambda_2 \in [0,1], x \in \mathbb{R}, \tag{6}$$

and

$$f\left(x\right) = g\left(x\right)\left[\lambda_1 + 2\left(\lambda_2 - \lambda_1\right)G\left(x\right) + 3\left(1 - \lambda_2\right)G^2\left(x\right)\right], \quad x \in \mathbb{R}. \tag{7}$$

Afify et al. (2017) proposed the beta transmuted-H family of distributions. Al-Kadim and Mohammed (2017) presented the cubic transmuted Weibull distribution in terms of basic mathematical properties. Nofal et al. (2017) studied a generalized transmuted-G family of distributions. Alizadeh et al. (2017) developed generalized transmuted family of distributions. Bakouch et al. (2017) introduced a new family of transmuted distributions. Granzotto et al. (2017) proposed a cubic ranking transmutation map and studied different properties. They studied properties of Cubic rank transmuted

Weibull distribution and Cubic rank transmuted log logistic distribution. Yilmaz (2018) proposed a new family of distributions developed with polynomial rank transmutation.

In this paper, a flexible lifetime distribution with arc, exponential, left-skewed, right-skewed and symmetrical shaped density function called CRTMBIII-P is developed on the basis of the cubic ranking transmutation map.

The basic motivations for proposing the CRTMBIII-P distribution are: (i) to generate distributions with arc, exponential, left-skewed, right-skewed and symmetrical shaped; (ii) to serve as the best alternative model against the current models to explore and modeling real data in economics, life testing, reliability, survival analysis manufacturing and other areas of research and (iii) to provide better fits than other sub-models.

Our interest is to study CRTMBIII-P distribution in terms of its mathematical properties, applications and comparison to the other sub-models.

This paper is sketched into the following sections. In Section 2, CRTMBIII-P distribution is introduced. In Section 3, CRTMBIII-P distribution is studied in terms of the basic structural properties, sub-models and some plots. In Section 4, moments, incomplete moments, inequality measures, residual and reverse residual life function and some other properties are theoretically derived. In Section 5, stress-strength reliability and multicomponent stress-strength reliability of the model are studied. In Section 6, CRTMBIII-P distribution is characterized via truncated moments. In Section 7, the parameters of CRTMBIII-P are estimated using maximum likelihood method. In Section 8, the simulation study for the performance of the maximum likelihood estimates (MLEs) of the parameters of CRTMBIII-P distribution is carried out. In Section 9, the potentiality of CRTMBIII-P distribution is demonstrated via its application to the real data sets: failure times and strength of glass fiber. Goodness of fit of the probability distribution through different methods is studied. The concluding remarks are given in Section 10.

3. CRTMBIII-P Distribution and Its Structural Properties

Bhatti et al. (2018) studied Burr III-Pareto distribution and some of its properties. The cdf and pdf of the modified Burr III-Pareto (MBIII-Pareto) distribution are given, respectively, by

$$F(x)=\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}}, x\geq\theta, \tag{8}$$

and

$$f(x)=\alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}-1}, \quad x>\theta. \tag{9}$$

Here, the CRTBMIII-P distribution is introduced with the help of (6) and (7). The cdf and pdf of CRTMBIII-P distribution are given, respectively, by

$$F(x)=\left(\begin{array}{r}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}}+\left(\lambda_2-\lambda_1\right)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}}\\ +\left(1-\lambda_2\right)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}}\end{array}\right), x\geq\theta, \tag{10}$$

and

$$f(x)=\frac{\alpha\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left(\begin{array}{r}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}-1}+2\left(\lambda_2-\lambda_1\right)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}-1}\\ +3\left(1-\lambda_2\right)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}-1}\end{array}\right), \quad x>\theta, \tag{11}$$

where $\alpha>0,\beta>0,\gamma>0$, $\lambda_1\in[-1,1]$, $\lambda_2\in[0,1]$ are parameters.

3.1 Structural Properties of CRTMBIII-P Distribution

The survival, hazard, cumulative hazard and reverse hazard functions and the Mills ratio of a random variable X with CRTMBIII-P distribution are given, respectively, by

$$S(x)=1-\left(\begin{array}{r}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}}+(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}}\\ +(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}}\end{array}\right),\ x\geq\theta, \tag{12}$$

$$h(x)=-\frac{d}{dx}\ln\left[1-\left(\begin{array}{r}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}}+(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}}\\ +(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}}\end{array}\right)\right], \tag{13}$$

$$r(x)=\frac{\frac{\alpha\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left(\begin{array}{r}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-1}+2(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}-1}\\ +3(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}-1}\end{array}\right)}{\left[\lambda_1+(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}}+(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}}\right]}, \tag{14}$$

$$H(x)=-\ln\left[1-\left(\begin{array}{r}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}}+(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}}\\ +(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}}\end{array}\right)\right], \tag{15}$$

and

$$m(x)=\frac{1-\left(\begin{array}{r}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}}+(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}}\\ +(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}}\end{array}\right)}{\frac{\alpha\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left(\begin{array}{r}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}-1}+2(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}-1}\\ +3(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}-1}\end{array}\right)}. \tag{16}$$

The elasticity $e(x)=\frac{d\ln\mathrm{F}(x)}{d\ln x}=xr(x)$ for CRTMBIII-P distribution is

$$e(x)=\frac{d}{d\ln x}\ln\left(\begin{array}{r}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}}+(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}}\\ +(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}}\end{array}\right). \tag{17}$$

The elasticity of CRTMBIII-P distribution shows the behavior of the accumulation of probability in the domain of the random variable.

3.2 Shapes of the CRTMBIII-P Density

The following graphs show that shapes of CRTMBIII-P density are arc, exponential, positively skewed, negatively skewed and symmetrical (Fig. 1). The plots of hrf (Fig.2) are also given.

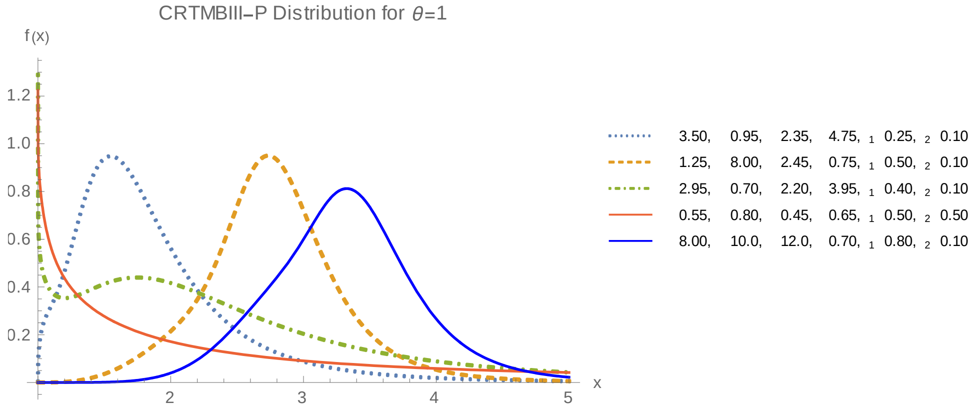

Figure 1. Plots of pdf of CRTMBIII-P distribution

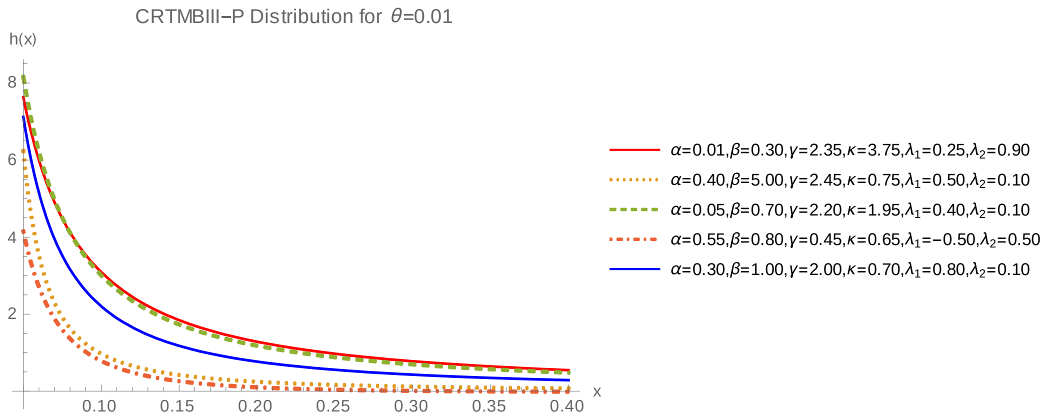

Figure 2. Plots of hrf of CRTMBIII-P distribution

3.3 Sub-Models

The CRTMBIII-P distribution has the following sub models.

Table 1. Sub-Models of CRTMBIII-P Distribution

Sr.No.	α	β	γ	κ	θ	λ_1	λ_2	Name of Distribution
1	α	β	γ	κ	θ	λ_1	λ_2	CRTMBIII-P distribution
2	α	β	1	κ	θ	λ_1	λ_2	CRTBIII-P distribution
3	1	β	1	κ	θ	λ_1	λ_2	CRTLL-P distribution
4	α	β	γ	κ	θ	λ_1	1	TMBIII-P distribution
5	α	β	1	κ	θ	λ_1	1	TBIII-P distribution
6	1	β	1	κ	θ	λ_1	1	TLL-P distribution
7	α	β	$\gamma \to 0$	κ	θ	λ_1	λ_2	CRTGIW-P distribution
8	α	β	$\gamma \to 0$	κ	θ	λ_1	1	TGIW-P distribution
9	α	β	$\gamma \to 0$	κ	θ	1	1	GIW-P distribution
10	α	β	γ	κ	θ	1	1	MBIII-P distribution
11	α	β	1	κ	θ	1	1	BIII-P distribution
12	1	β	1	κ	θ	1	1	LL-P distribution
13	α	1	γ	κ	θ	λ_1	λ_2	CRTMIL-P
14	α	1	1	κ	θ	λ_1	λ_2	CRTIL-P
15	α	1	γ	κ	θ	λ_1	1	TMIL-P
16	α	1	1	κ	θ	λ_1	1	TIL-P
17	α	1	γ	κ	θ	1	1	MIL-P
18	α	1	1	κ	θ	1	1	IL-P

3.4 Descriptive Measures Based Quantiles

The quantile function of CRTMBIII-P distribution is the solution of the following equation

$$x_q = \theta\left\{1+\left[\frac{1}{\gamma}\left(-\frac{B}{3A}-\frac{2^{1/3}(-B^2+3AC)}{3A\mathrm{M}}+\frac{\mathrm{M}}{32^{1/3}A}\right)^{-\frac{\gamma}{\alpha}}-\frac{1}{\gamma}\right]^{-\frac{1}{\beta}}\right\}^{\frac{1}{\kappa}},$$

where $A=(1-\lambda_2)$, $B=(\lambda_2-\lambda_1)$, $C=\lambda_1$ and

$$M=\left[-2B^3+9ABC+27A^2q+\sqrt{4(-B^2+3AC)^3+(-2B^3+9ABC+27A^2q)^2}\right]^{1/3}$$

Median of CRTMBIII-P distribution is the solution of the following

$$x_{Med} = \theta\left\{1+\left[\frac{1}{\gamma}\left(-\frac{B}{3A}-\frac{2^{1/3}(-B^2+3AC)}{3A\tilde{\mathrm{M}}}+\frac{\tilde{\mathrm{M}}}{32^{1/3}A}\right)^{-\frac{\gamma}{\alpha}}-\frac{1}{\gamma}\right]^{-\frac{1}{\beta}}\right\}^{\frac{1}{\kappa}},$$

where $\tilde{\mathrm{M}}=\left[-2B^3+9ABC+13.5A^2+\sqrt{4(-B^2+3AC)^3+(-2B^3+9ABC+13.5A^2)^2}\right]^{1/3}$

The random number generator of CRTMBIII-P distribution is the solution of the following

$$X = \theta\left\{1+\left[\frac{1}{\gamma}\left(-\frac{B}{3A}-\frac{2^{1/3}(-B^2+3AC)}{3A\mathrm{M}_Z}+\frac{\mathrm{M}_Z}{32^{1/3}A}\right)^{-\frac{\gamma}{\alpha}}-\frac{1}{\gamma}\right]^{-\frac{1}{\beta}}\right\}^{\frac{1}{\kappa}},$$

where $\mathrm{M}_Z=\left[-2B^3+9ABC+27A^2Z+\sqrt{4(-B^2+3AC)^3+(-2B^3+9ABC+27A^2Z)^2}\right]^{1/3}$ and the random variable Z has uniform distribution on $(0,1)$. Some measures based on quartiles for location, dispersion, skewness and kurtosis for the CRTMBIII-P distribution respectively are: Median M=Q (0.5); Quartile deviation $Q.D.=\frac{Q_{\frac{3}{4}}-Q_{\frac{1}{4}}}{2}$; Bowley's skewness measure $S_q=\frac{Q_{\frac{3}{4}}-2Q_{\frac{1}{2}}+Q_{\frac{1}{4}}}{Q_{\frac{3}{4}}-Q_{\frac{1}{4}}}$ and Moors kurtosis measure based on Octiles $K=\frac{Q_{\frac{7}{8}}-Q_{\frac{5}{8}}+Q_{\frac{3}{8}}-Q_{\frac{1}{8}}}{Q_{\frac{6}{8}}-Q_{\frac{2}{8}}}$. The quantile based measures exist even for distributions that have no moments. The quantile based measures are less sensitive to the outliers.

4. Moments

Moments, incomplete moments, inequality measures, residual and reverse residual life function and some other properties are theoretically derived in this section.

4.1 Moments About the Origin

The r^{th} ordinary moment of CRTMBIII-P distribution is,

$$E\left(X^r\right)=\int_{\theta}^{\infty}x^r f(x)\,dx,$$

$$\mu_r^{/}=E\left(X^r\right)=\int_\theta^\infty x^r\frac{\alpha\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left(\begin{array}{r}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}-1}+2\left(\lambda_2-\lambda_1\right)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}-1}\\+3\left(1-\lambda_2\right)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}-1}\end{array}\right)dx.$$

Letting $\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}=y, x^r=\theta^r\left[\left(\frac{y}{\gamma}\right)^{-\frac{1}{\beta}}+1\right]^{\frac{r}{\kappa}}=\theta^r\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\gamma^{\frac{\ell}{\beta}}y^{-\frac{\ell}{\beta}}$, then

$$E\left(X^r\right)=\alpha\theta^r\gamma^{\frac{\ell}{\beta}-1}\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\left[\begin{array}{r}\lambda_1B\left(1-\frac{\ell}{\beta},\frac{\alpha}{\gamma}+\frac{\ell}{\beta}\right)+2\left(\lambda_2-\lambda_1\right)B\left(1-\frac{\ell}{\beta},\frac{2\alpha}{\gamma}+\frac{\ell}{\beta}\right)\\+3\left(1-\lambda_2\right)B\left(1-\frac{\ell}{\beta},\frac{3\alpha}{\gamma}+\frac{\ell}{\beta}\right)\end{array}\right],$$

$$\mu_r^{/}=E\left(X^r\right)=\theta^r\gamma^{\frac{\ell}{\beta}}\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\left[\begin{array}{r}\lambda_1A_1\left(\ell;\alpha,\beta,\gamma\right)+\left(\lambda_2-\lambda_1\right)A_2\left(\ell;\alpha,\beta,\gamma\right)\\+\left(1-\lambda_2\right)A_3\left(\ell;\alpha,\beta,\gamma\right)\end{array}\right],\quad r=1,2,3,4,... \tag{18}$$

where $A_i\left(\ell;\alpha,\beta,\gamma\right)=\dfrac{\Gamma\left(1-\frac{\ell}{\beta}\right)\Gamma\left(\frac{i\alpha}{\gamma}+\frac{\ell}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)}, i=1,2,3$.

Mean and Variance of CRTMBIII-P distribution are

$$E\left(X\right)=\theta\gamma^{\frac{\ell}{\beta}}\sum_{\ell=0}^{\frac{1}{\kappa}}\binom{\frac{1}{\kappa}}{\ell}\left[\begin{array}{r}\lambda_1A_1\left(\ell;\alpha,\beta,\gamma\right)+\left(\lambda_2-\lambda_1\right)A_2\left(\ell;\alpha,\beta,\gamma\right)\\+\left(1-\lambda_2\right)A_3\left(\ell;\alpha,\beta,\gamma\right)\end{array}\right], \tag{19}$$

and

$$Var\left(X\right)=\theta^2\gamma^{\frac{\ell}{\beta}}\left(\begin{array}{l}\sum_{\ell=0}^{\frac{2}{\kappa}}\binom{\frac{2}{\kappa}}{\ell}\left[\begin{array}{r}\lambda_1A_1\left(\ell;\alpha,\beta,\gamma\right)+\left(\lambda_2-\lambda_1\right)A_2\left(\ell;\alpha,\beta,\gamma\right)\\+\left(1-\lambda_2\right)A_3\left(\ell;\alpha,\beta,\gamma\right)\end{array}\right]-\\ \quad\gamma^{\frac{\ell}{\beta}}\left\{\sum_{\ell=0}^{\frac{1}{\kappa}}\binom{\frac{1}{\kappa}}{\ell}\left[\begin{array}{r}\lambda_1A_1\left(\ell;\alpha,\beta,\gamma\right)+\left(\lambda_2-\lambda_1\right)A_2\left(\ell;\alpha,\beta,\gamma\right)\\+\left(1-\lambda_2\right)A_3\left(\ell;\alpha,\beta,\gamma\right)\end{array}\right]\right\}^2\end{array}\right). \tag{20}$$

The factorial moments for CRTMBIII-P distribution are given by

$$E[X]_n=\sum_{r=1}^{n}\varphi_rE(X^r),$$

$$E[X]_n=\sum_{r=1}^{n}\sum_{\ell=0}^{\frac{r}{\kappa}}\varphi_r\gamma^{\frac{\ell}{\beta}}\theta^r\binom{\frac{r}{\kappa}}{\ell}\left[\begin{array}{r}\lambda_1A_1\left(\ell;\alpha,\beta,\gamma\right)+\left(\lambda_2-\lambda_1\right)A_2\left(\ell;\alpha,\beta,\gamma\right)\\+\left(1-\lambda_2\right)A_3\left(\ell;\alpha,\beta,\gamma\right)\end{array}\right], \tag{21}$$

where $[X]_i=X\left(X+1\right)\left(X+2\right)......\left(X+i-1\right)$ and φ_r is Stirling number of the first kind.

The Mellin transform helps to determine moments for a probability distribution. The Mellin transform of X with CRTMBIII-P distribution is

$$\mathrm{M}\left\{f\left(x\right);\mathrm{s}\right\}=f^*\left(s\right)=E(X^{s-1}),$$

$$M\{f(x);s\}=\theta^{(s-1)}\gamma^{\frac{\ell}{\beta}}\sum_{\ell=0}^{\frac{s-1}{\kappa}}\binom{\frac{s-1}{\kappa}}{\ell}\begin{bmatrix}\lambda_1 A_1(\ell;\alpha,\beta,\gamma)+(\lambda_2-\lambda_1)A_2(\ell;\alpha,\beta,\gamma)\\ +(1-\lambda_2)A_3(\ell;\alpha,\beta,\gamma)\end{bmatrix}. \tag{22}$$

The k^{th} moment about mean of X is determined from the relationship

$$\mu_k=E\left[X-E(X)\right]^k=\sum_{j=1}^{k}\binom{k}{j}(-1)^j\mu'_j\mu'_{(k-j)}.$$

The Pearson's measure of skewness γ_1, Kurtosis β_2, moment generating function and cumulants can be calculated from

$$\gamma_1=\frac{\mu_3}{(\mu_2)^{\frac{3}{2}}},\ \beta_2=\frac{\mu_4}{(\mu_2)^2},M_X(t)=\sum_{r=1}^{\infty}\frac{t^r}{r!}E(X)^r\ and\ k_r=E\left(X^r\right)-\sum_{c=1}^{r-1}\binom{r-1}{c-1}k_cE\left(X^{r-c}\right).$$

The numerical measures of the median, mean, variance, skewness and kurtosis of the CRTMBIII-P distribution for selected values of the parameters to illustrate their effect on these measures.

Table 2. Median, mean, standard deviation, skewness and Kurtosis of the CRTMBIII-P Distribution

Parameters $\alpha,\beta,\gamma,\kappa,\lambda_1,\lambda_2,\theta$	Median	Mean	Standard Deviation	Skewness	Kurtosis
1,1,1,1,0.8,0.5,0.5	1.61036	65.7235	35476.7	748.019	582122.0667
1,1,1,1,0.8,1,1	1.81711	67.7487	36758	750.832	586974.385
1.5,1.5,0.5,0.5,0.8,0.5,0.5	2.2263	3.49459	5.95006	116.649	42022.3
1.5,1.5,1.5,1.5,0.8,0.5,1	1.33952	1.658	1.57969	74.607	15727.1
1.5,1.5,1.5,1.5,0.8,0.8,1	1.37936	1.68996	1.57127	65.3279	11774.8
2.5,2.5,2.5,5,0.8,0.5,1	1.09436	1.11177	0.0739985	2.4123	15.4914
2.5,2.5,2.5,3,0.8,3.5,0.25	1.22764	1.25534	0.11829	3.54321	36.2057
2.95,0.7,2.20,3.95,0.4,0.1,1	1.01107	1.06867	0.235427	15.8608	727.687
3,3,2,0.5,-0.9,1,1	2.10523	2.14782	0.492799	0.379771	2.67321
3,3,4,2.5,0.5,0.9,1	1.20566	1.24052	0.161448	4.52189	62.1364
3,3,3,3,-0.5,0.5,6	1.13119	1.13261	0.0468075	0.14014	2.6245
5,5,0.5,3,0.3,0.3,3	**1.16313**	**1.16287**	**0.0369379**	**0.000334117**	**3.13759**
5,3.5,0.5,3.5,0.5,0.5,5	1.11904	1.12075	0.390521	0.237948	0.237948
5,5,0.1,3,0.3,0.3,3	1.16278	1.16214	0.0360593	-0.0769682	3.05655
5,5,0.3,3,0.3,0.3,3	3.48886	3.48752	0.109464	-0.0382699	3.09349
5,5,0.3,3,-0.8,5	5.88415	5.86576	0.138629	-0.789373	3.77744
4.5,4.5,4.5,4.5,0.9,0.5,1	1.11873	1.12575	0.0453069	1.36909	7.78144
5,5,0.5,0.5,0.5,0.5,1	2.56042	2.60084	0.516872	0.443135	3.23449
5,5,5,5,0.5,0.5,1	1.10151	1.10552	0.0321254	1.33553	8.67252

4.2 Incomplete Moments

Incomplete moments are used to study mean inactivity life, mean residual life function and other inequality measures. The lower incomplete moments of a random variable X with CRTMBIII-P distribution are

$$M'_r(z)=E_{X\le z}\left(X^r\right)=\int_0^z x^r\frac{\alpha\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\begin{pmatrix}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}-1}+2(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}-1}\\ +3(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}-1}\end{pmatrix}dx.$$

Letting $\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}=y\ ,x^r=\theta^r\left[\left(\frac{y}{\gamma}\right)^{-\frac{1}{\beta}}+1\right]^{\frac{r}{\kappa}}=\theta^r\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\gamma^{\frac{\ell}{\beta}}y^{-\frac{\ell}{\beta}}$, then

$$E_{X\le z}\left(X^r\right)=\alpha\theta^r\gamma^{\frac{\ell}{\beta}-1}\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\left\{\begin{array}{l}\lambda_1\left[B\left(1-\frac{\ell}{\beta},\frac{\alpha}{\gamma}+\frac{\ell}{\beta}\right)-B_{w(z)}\left(1-\frac{\ell}{\beta},\frac{\alpha}{\gamma}+\frac{\ell}{\beta}\right)\right]+\\ 2\left(\lambda_2-\lambda_1\right)\left[B\left(1-\frac{\ell}{\beta},\frac{2\alpha}{\gamma}+\frac{\ell}{\beta}\right)-B_{w(z)}\left(1-\frac{\ell}{\beta},\frac{2\alpha}{\gamma}+\frac{\ell}{\beta}\right)\right]+\\ 3\left(1-\lambda_2\right)\left[B\left(1-\frac{\ell}{\beta},\frac{3\alpha}{\gamma}+\frac{\ell}{\beta}\right)-B_{w(z)}\left(1-\frac{\ell}{\beta},\frac{3\alpha}{\gamma}+\frac{\ell}{\beta}\right)\right]\end{array}\right\},$$

$$E_{X\le z}\left(X^r\right)=\alpha\theta^r\gamma^{\frac{\ell}{\beta}-1}\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\left[\begin{array}{r}\lambda_1D_1\left(\ell,\alpha,\beta,\gamma\right)+2\left(\lambda_2-\lambda_1\right)D_2\left(\ell,\alpha,\beta,\gamma\right)\\ +3\left(1-\lambda_2\right)D_3\left(\ell,\alpha,\beta,\gamma\right)\end{array}\right], \tag{23}$$

where $D_i\left(\ell,\alpha,\beta,\gamma\right)=B\left(1-\frac{\ell}{\beta},\frac{i\alpha}{\gamma}+\frac{\ell}{\beta}\right)-B_{w(z)}\left(1-\frac{\ell}{\beta},\frac{i\alpha}{\gamma}+\frac{\ell}{\beta}\right),\ i=1,2,3$,

$w(z)=\gamma\left[\left(\frac{z}{\theta}\right)^{\kappa}-1\right]^{-\beta}$ and $B_{w(z)}(.,.)$ is the incomplete beta function.

The upper incomplete moments for the random variable X with CRTMBIII-P distribution are

$$E_{X>z}\left(X^r\right)=\int_z^{\infty}x^r\frac{\alpha\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left(\begin{array}{r}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}-1}+2\left(\lambda_2-\lambda_1\right)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}-1}\\ +3\left(1-\lambda_2\right)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}-1}\end{array}\right)dx,$$

Letting $\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}=y\,,x^r=\theta^r\left[\left(\frac{y}{\gamma}\right)^{-\frac{1}{\beta}}+\ 1\right]^{\frac{r}{\kappa}}=\theta\ \sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\gamma^{\frac{\ell}{\beta}}y^{-\frac{\ell}{\beta}}$, then

$$E_{X>z}\left(X^r\right)=\alpha\theta^r\gamma^{\frac{\ell}{\beta}-1}\sum_{\ell=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{\ell}\left[\begin{array}{r}\lambda_1B_{w(z)}\left(1-\frac{\ell}{\beta},\frac{\alpha}{\gamma}+\frac{\ell}{\beta}\right)+2\left(\lambda_2-\lambda_1\right)B_{w(z)}\left(1-\frac{\ell}{\beta},\frac{2\alpha}{\gamma}+\frac{\ell}{\beta}\right)\\ +3\left(1-\lambda_2\right)B_{w(z)}\left(1-\frac{\ell}{\beta},\frac{3\alpha}{\gamma}+\frac{\ell}{\beta}\right)\end{array}\right], \tag{24}$$

The mean deviation about mean is $MD_{\bar{X}}=E\left|X-\mu_1^1\right|=2\mu_1^1F\left(\mu_1^1\right)-2\mu_1^1M_1'\left(\mu_1^1\right)$ and the mean deviation about median is $MD_M=E\left|X-M\right|=2MF\left(M\right)-2MM_1'\left(M\right)$, where $\mu_1^/=E\left(X\right)$ and $M=Q(0.5)$. Bonferroni and Lorenz curves for a specified probability p are computed from $\mathrm{B}(p)=M_1^/\left(q\right)/p\mu_1^1$ and $L(p)=M_1^/\left(q\right)/\mu_1^1$ where $q=Q\left(p\right)$.

4.3 Residual Life Functions

The residual life, say $m_n\left(z\right)$, of X with CRTMBIII-P distribution has the n^th^ moment

$$m_n\left(z\right)=E\left[\left(X-z\right)^n\middle|X>z\right]=\frac{1}{S\left(z\right)}\int_z^{\infty}\left(x-z\right)^sf(x)dx,$$

$$m_n\left(z\right)=\frac{1}{S\left(z\right)}\sum_{s=0}^{n}\binom{n}{s}\left(-z\right)^{n-s}E_{X>z}\left(X^s\right),$$

$$m_n\left(z\right)=\frac{\alpha}{S\left(z\right)}\sum_{s=0}^{n}\sum_{\ell=0}^{\frac{s}{\kappa}}\binom{n}{s}\binom{\frac{s}{\kappa}}{\ell}\left(-z\right)^{n-s}\theta^s\gamma^{\frac{\ell}{\beta}-1}\left[\begin{array}{r}\lambda_1B_{w(z)}\left(1-\frac{\ell}{\beta},\frac{\alpha}{\gamma}+\frac{\ell}{\beta}\right)+2\left(\lambda_2-\lambda_1\right)B_{w(z)}\left(1-\frac{\ell}{\beta},\frac{2\alpha}{\gamma}+\frac{\ell}{\beta}\right)\\ +3\left(1-\lambda_2\right)B_{w(z)}\left(1-\frac{\ell}{\beta},\frac{3\alpha}{\gamma}+\frac{\ell}{\beta}\right)\end{array}\right], \tag{25}$$

The average remaining lifetime of a component at time z, say $m_1\left(z\right)$, or life expectancy known as mean residual life (MRL) function is given by

$$m_1(z)=\frac{\alpha}{S(z)}\sum_{s=0}^{1}\sum_{\ell=0}^{\frac{s}{\kappa}}\binom{1}{s}\binom{\frac{s}{\kappa}}{\ell}(-z)^{1-s}\theta^s\gamma^{\frac{\ell}{\beta}-1}\left[\begin{array}{r}\lambda_1 B_{w(z)}\left(1-\frac{\ell}{\beta},\frac{\alpha}{\gamma}+\frac{\ell}{\beta}\right)+2(\lambda_2-\lambda_1)B_{w(z)}\left(1-\frac{\ell}{\beta},\frac{2\alpha}{\gamma}+\frac{\ell}{\beta}\right)\\ +3(1-\lambda_2)B_{w(z)}\left(1-\frac{\ell}{\beta},\frac{3\alpha}{\gamma}+\frac{\ell}{\beta}\right)\end{array}\right]. \tag{26}$$

The reverse residual life, say, $M_n(z)$ of X with CRTMBIII-P distribution having n[th] moment is

$$M_n(z)=E\left[(z-X)^n / X\le z\right]=\frac{1}{F(z)}\int_a^z (z-x)^n f(x)dx,$$

$$M_n(z)=\frac{1}{F(z)}\sum_{s=0}^{n}(-1)^s\binom{n}{s}z^{n-s}E_{X\le z}\left(X^s\right),$$

$$M_n(z)=\frac{\alpha}{F(z)}\sum_{s=0}^{n}\sum_{\ell=0}^{\frac{s}{\kappa}}\binom{n}{s}\binom{\frac{s}{\kappa}}{\ell}(-1)^s z^{n-s}\theta^s\gamma^{\frac{\ell}{\beta}-1}\left[\begin{array}{r}\lambda_1 D_1(\ell,\alpha,\beta,\gamma)+2(\lambda_2-\lambda_1)D_2(\ell,\alpha,\beta,\gamma)\\ +3(1-\lambda_2)D_3(\ell,\alpha,\beta,\gamma)\end{array}\right]. \tag{27}$$

The waiting time z for failure of a component has passed with condition that this failure had happened in the interval [0, z] is called mean waiting time (MWT) or mean inactivity time. The waiting time z for failure of a component of X having CRTMBIII-P distribution is defined by

$$M_1(z)=\frac{\alpha}{F(z)}\sum_{s=0}^{1}\sum_{\ell=0}^{\frac{s}{\kappa}}\binom{1}{s}\binom{\frac{s}{\kappa}}{\ell}(-1)^s z^{1-s}\theta^s\gamma^{\frac{\ell}{\beta}-1}\left[\begin{array}{r}\lambda_1 D_1(\ell,\alpha,\beta,\gamma)+2(\lambda_2-\lambda_1)D_2(\ell,\alpha,\beta,\gamma)\\ +3(1-\lambda_2)D_3(\ell,\alpha,\beta,\gamma)\end{array}\right]. \tag{28}$$

5. Reliability Measures

In this section, reliability measures are studied.

5.1 Stress-Strength Reliability for CRTMBIII-P Distribution

Let $X_1 \sim CRTMBIII-P(\alpha_1,\beta,\gamma,\theta,\lambda_1,\lambda_2), X_2 \sim CRTMBIII-P(\alpha_2,\beta,\gamma,\theta,\lambda_1,\lambda_2)$ such that X_1 represents strength and X_2 represents stress, then $R=\Pr(X_2<X_1)=\int_0^\infty f_{x_1}(x)F_{x_2}(x)dx$ is the characteristic of the distribution of X_1 and X_2. Then reliability of the component for CRTMBIII-P distribution is computed as

$$R=\int_0^\infty \frac{\alpha_1\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left[\begin{array}{c}\left(\begin{array}{r}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha_1}{\gamma}-1}+2(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha_1}{\gamma}-1}\\ +3(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha_1}{\gamma}-1}\end{array}\right)\times\\ \left(\begin{array}{r}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha_2}{\gamma}}+(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha_2}{\gamma}}\\ +(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha_2}{\gamma}}\end{array}\right)\end{array}\right]dx,$$

$$R=\int_0^\infty \frac{\alpha_1\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left(\begin{array}{l} \lambda_1^2\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha_1}{\gamma}-\frac{\alpha_2}{\gamma}-1}+2\lambda_1(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha_1}{\gamma}-\frac{\alpha_2}{\gamma}-1}+ \\ 3(\lambda_1-\lambda_1\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha_1}{\gamma}-\frac{\alpha_2}{\gamma}-1}+\left(\lambda_1\lambda_2-\lambda_1^2\right)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha_1}{\gamma}-\frac{2\alpha_2}{\gamma}-1}+ \\ 2(\lambda_2-\lambda_1)^2\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha_1}{\gamma}-\frac{2\alpha_2}{\gamma}-1}+3(1-\lambda_2)(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha_1}{\gamma}-\frac{2\alpha_2}{\gamma}-1}+ \\ +(\lambda_1-\lambda_1\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha_1}{\gamma}-\frac{3\alpha_2}{\gamma}-1}+2(\lambda_2-\lambda_1)(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha_1}{\gamma}-\frac{3\alpha_2}{\gamma}-1}+ \\ 3(1-\lambda_2)^2\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha_1}{\gamma}-\frac{3\alpha_2}{\gamma}-1} \end{array}\right),$$

$$R=\left\{\begin{array}{l} \frac{\lambda_1^2\alpha_1}{(\alpha_1+\alpha_2)}+\frac{2\alpha_1\left(\lambda_1\lambda_2-\lambda_1^2\right)}{(2\alpha_1+\alpha_2)}+\frac{3\alpha_1(\lambda_1-\lambda_1\lambda_2)}{(3\alpha_1+\alpha_2)}+ \\ \frac{\alpha_1(\lambda_2-\lambda_1)^2}{(\alpha_1+2\alpha_2)}+\frac{2\alpha_1\left(\lambda_1\lambda_2-\lambda_1^2\right)}{(2\alpha_1+2\alpha_2)}+\frac{3\alpha_1(1-\lambda_2)(\lambda_2-\lambda_1)}{(3\alpha_1+2\alpha_2)}+ \\ \frac{\alpha_1(\lambda_1-\lambda_1\lambda_2)}{(\alpha_1+3\alpha_2)}+\frac{2\alpha_1(\lambda_2-\lambda_1)(1-\lambda_2)}{(2\alpha_1+3\alpha_2)}+\frac{3\alpha_1(1-\lambda_2)^2}{(3\alpha_1+3\alpha_2)} \end{array}\right\}. \qquad (29)$$

Therefore (i) R is independent of β,γ,λ_1 *and* λ_2 (ii) for $\alpha_1=\alpha_2$, R=0.5, it means that X_1 and X_2 are i.i.d. and there is equal chance that X_1 is bigger than X_2.

6. Characterizations

In order to develop a stochastic function in a certain problem, it is necessary to know whether the selected function fulfills the requirements of the specific underlying probability distribution. To this end, it is required to study characterizations of the specific probability distribution. Certain characterizations of CRTMBIII-P distribution are presented in this section.

6.1 Characterization Through Ratio of Truncated Moments

The CRTMBIII-P distribution is characterized using Theorem 1 (Glänzel; 1987) on the basis of a simple relationship between two truncated moments of functions of X. Theorem 1 is given in Appendix A.

Proportion 6.1.1. Let $X:\Omega\to(0,\infty)$ be a continuous random variable. Let

$$h_1(x)=\frac{1}{\alpha}\left[\begin{array}{r} \lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}-1}+2(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}-1} \\ +3(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}-1} \end{array}\right]^{-1}$$

and $$h_2(x)=\frac{2}{\alpha}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\left[\begin{array}{r} \lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}-1}+2(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}-1} \\ +3(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}-1} \end{array}\right]^{-1},\ x>\theta.$$

The pdf of X is (11), if and only if $q(x)$ (in Theorem 1) has the form $q(x)=\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{\beta},\ x>0$.

Proof. If X has pdf (11), then

$$\left(1-F(x)\right)E\left(h_1(x)\middle|X\geq x\right)=\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta},\ x>\theta,$$

and $\left(1-F(x)\right)E\left(h_2(x)\middle|X\geq x\right)=\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-2\beta},\ x>\theta$.

After simplification, we have $\dfrac{E[h_1(X)|X\geq x]}{E[h_2(X)|X\geq x]}=q(x)=\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{\beta}$ and $q'(x)=\frac{\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{\beta-1}$.

The differential equation $s'(x)=\dfrac{q'(x)h_2(x)}{q(x)h_2(x)-h_1(x)}=\dfrac{2\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-1}$ has solution $s(x)=\ln\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{2\beta}$.

Therefore according to theorem 1, X has pdf (11).

Corollary 6.1.1. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let

$$h_2(x)=\frac{2}{\alpha}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\left\{\begin{array}{c}\lambda_1\left[1+\gamma\left(\left(\frac{x}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}-1}+2(\lambda_2-\lambda_1)\left[1+\gamma\left(\left(\frac{x}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{2\alpha}{\gamma}-1}\\+3(1-\lambda_2)\left[1+\gamma\left(\left(\frac{x}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{3\alpha}{\gamma}-1}\end{array}\right\}^{-1},x>\theta.$$

The pdf of X is (11) if and only if functions $q(x)$ and $h_1(x)$ satisfy the equation

$$\frac{q'(x)h_2(x)}{q(x)h_2(x)-h_1(x)}=\frac{2\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-1}.$$

*Remark 6.1.1. T*he general solution of the above differential equation is

$$q(x)=\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{2\beta}\int\left[-\frac{\alpha\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left(\begin{array}{c}\lambda_1\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}-1}+2(\lambda_2-\lambda_1)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{2\alpha}{\gamma}-1}\\+3(1-\lambda_2)\left\{1+\gamma\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\frac{3\alpha}{\gamma}-1}\end{array}\right)h_1(x)\right]dx+D,$$

where D is a constant.

6.2 Characterization via Doubly Truncated Moment

Here CRTMBIII-P distribution is characterized via doubly truncated moment.

Proposition 6.2.1. Let $X:\Omega\to(0,+\infty)$ be a continuous random variable. Then, X has pdf (11) if and only if

$$E\left(\left\{\begin{array}{c}\lambda_1+2(\lambda_2-\lambda_1)\left[1+\gamma\left(\left(\frac{X}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}\\+3(1-\lambda_2)\left[1+\gamma\left(\left(\frac{X}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{2\alpha}{\gamma}}\end{array}\right\}^{-1}\middle|x<X<y\right)$$

$$=\frac{\left[1+\gamma\left(\left(\frac{y}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}-\left[1+\gamma\left(\left(\frac{x}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}}{\left[\bar{F}(x)-\bar{F}(y)\right]},x>0,y>0.\ (30)$$

Proof.

For random variable X with pdf (11), we have

$$E\left(\left\{\begin{array}{l}\lambda_1+2(\lambda_2-\lambda_1)\left[1+\gamma\left(\left(\frac{X}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}\\ \quad +3(1-\lambda_2)\left[1+\gamma\left(\left(\frac{X}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{2\alpha}{\gamma}}\end{array}\right\}^{-1}\middle| x<X<y\right)$$

$$=\frac{\int_y^x\left[\lambda_1+2(\lambda_2-\lambda_1)\left[1+\gamma\left(\left(\frac{u}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}+3(1-\lambda_2)\left[1+\gamma\left(\left(\frac{u}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{2\alpha}{\gamma}}\right]^{-1}f(u)\,du}{\left[\bar F(x)-\bar F(y)\right]},$$

$$=\frac{\int_y^x\left[\begin{array}{l}\lambda_1+2(\lambda_2-\lambda_1)\left[1+\gamma\left(\left(\frac{u}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}\\ \quad +3(1-\lambda_2)\left[1+\gamma\left(\left(\frac{u}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{2\alpha}{\gamma}}\end{array}\right]^{-1}\frac{\alpha\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left[1+\gamma\left(\left(\frac{u}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}-1}\left[\begin{array}{l}\lambda_1+2(\lambda_2-\lambda_1)\left[1+\gamma\left(\left(\frac{u}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}\\ \quad +3(1-\lambda_2)\left[1+\gamma\left(\left(\frac{u}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{2\alpha}{\gamma}}\end{array}\right]du}{\left[\bar F(x)-\bar F(y)\right]},$$

$$=\frac{\int_y^x\frac{\alpha\beta\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left[1+\gamma\left(\left(\frac{u}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}-1}du}{\left[\bar F(x)-\bar F(y)\right]},$$

$$=\frac{\left[1+\gamma\left(\left(\frac{y}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}-\left[1+\gamma\left(\left(\frac{x}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}}{\left[\bar F(x)-\bar F(y)\right]}.$$

Conversely, if (30) holds, then

$$\frac{\int_y^x\left[\begin{array}{l}\lambda_1+2(\lambda_2-\lambda_1)\left[1+\gamma\left(\left(\frac{u}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}\\ \quad +3(1-\lambda_2)\left[1+\gamma\left(\left(\frac{u}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{2\alpha}{\gamma}}\end{array}\right]^{-1}f(u)\,du}{\left[\bar F(x)-\bar F(y)\right]}=\frac{\left[1+\gamma\left(\left(\frac{y}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}-\left[1+\gamma\left(\left(\frac{x}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}}{\left[\bar F(x)-\bar F(y)\right]},$$

$$\int_y^x\left[\begin{array}{l}\lambda_1+2(\lambda_2-\lambda_1)\left[1+\gamma\left(\left(\frac{u}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}\\ \quad +3(1-\lambda_2)\left[1+\gamma\left(\left(\frac{u}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{2\alpha}{\gamma}}\end{array}\right]^{-1}f(u)\,du=\left[1+\gamma\left(\left(\frac{y}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}-\left[1+\gamma\left(\left(\frac{x}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}.$$

Differentiating with respect to y, we have

$$\left\{\begin{array}{c}\lambda_1+2(\lambda_2-\lambda_1)\left[1+\gamma\left(\left(\frac{y}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}} \\ +3(1-\lambda_2)\left[1+\gamma\left(\left(\frac{y}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{2\alpha}{\gamma}}\end{array}\right\}^{-1} f(y)=\frac{\alpha\beta\kappa}{\theta}\left(\frac{y}{\theta}\right)^{\kappa-1}\left[1+\gamma\left(\left(\frac{y}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}-1},$$

or

$$f(y)=\frac{\alpha\beta\kappa}{\theta}\left(\frac{y}{\theta}\right)^{\kappa-1}\left\{\begin{array}{c}\lambda_1\left[1+\gamma\left(\left(\frac{y}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}-1}+2(\lambda_2-\lambda_1)\left[1+\gamma\left(\left(\frac{y}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{2\alpha}{\gamma}-1} \\ +3(1-\lambda_2)\left[1+\gamma\left(\left(\frac{y}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{3\alpha}{\gamma}-1}\end{array}\right\},$$

which is pdf of CRTMBIII-P distribution.

7. Maximum Likelihood Estimation

In this section, parameter estimates are derived using maximum likelihood method. The log- likelihood function for CRTMBIII-P distribution with the vector of parameters $\Phi=(\alpha,\beta,\gamma,\kappa,\theta,\lambda_1,\lambda_2)$ is

$$\begin{aligned}\ln L(x_i;\Phi) &= n\ln\alpha+n\ln\beta+n\ln\kappa-n\kappa\ln\theta+(\kappa-1)\sum\ln x_i-(\beta+1)\sum\ln\left[\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right] \\ &\quad-\left(\frac{\alpha}{\gamma}+1\right)\sum\ln\left[1+\gamma\left(\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}-1} \\ &\quad+\sum\ln\left\{\begin{array}{c}\lambda_1+2(\lambda_2-\lambda_1)\left[1+\gamma\left(\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}} \\ +3(1-\lambda_2)\left[1+\gamma\left(\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right)^{-\beta}\right]^{-\frac{2\alpha}{\gamma}}\end{array}\right\}. \quad (31)\end{aligned}$$

In order to estimate the parameters of CRTMBIII-P distribution, the following nonlinear equations must be solved simultaneously:

$$\frac{\partial}{\partial\alpha}(\ln L(x_i;\Phi))=\left\{\begin{array}{l}\frac{n}{\alpha}-\frac{1}{\gamma}\sum\ln\left(1+\gamma x_i^{-\beta}\right)- \\ \frac{2}{\gamma}\sum\frac{\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}\ln\left(1+\gamma x_i^{-\beta}\right)\left[(\lambda_2-\lambda_1)+3(1-\lambda_2)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}\right]}{\left[\lambda_1+2(\lambda_2-\lambda_1)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}+3(1-\lambda_2)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}\right]}\end{array}\right\}=0, \quad (32)$$

$$\frac{\partial}{\partial\beta}\left(\ln L\left(x_i;\Phi\right)\right)=\left(\begin{array}{l}\frac{n}{\beta}-\sum\ln x_i+\left(\frac{\alpha}{\gamma}+1\right)\sum\frac{\gamma x_i^{-\beta}\ln x_i}{\left(1+\gamma x_i^{-\beta}\right)}+2\sum x_i^{-\beta}\ln x_i\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}-1}\times\\ \left\{\frac{\alpha\left(\lambda_2-\lambda_1\right)+3\alpha\left(1-\lambda_2\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}}{\lambda_1+2\left(\lambda_2-\lambda_1\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}+3\left(1-\lambda_2\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}}\right\}\end{array}\right)=0,\ (33)$$

$$\frac{\partial}{\partial\gamma}\left(\ln L\left(x_i;\Phi\right)\right)=\left(\begin{array}{l}\frac{\alpha}{\gamma^2}\sum\ln\left(1+\gamma x_i^{-\beta}\right)-\left(1+\frac{\alpha}{\gamma}\right)\sum x_i^{-\beta}\left(1+\gamma x_i^{-\beta}\right)^{-1}+\\ \frac{2\alpha}{\gamma}\sum\left(\frac{\left\{\frac{\left(\lambda_2-\lambda_1\right)}{\left(1+\gamma x_i^{-\beta}\right)^{\frac{\alpha}{\gamma}}}\left[\frac{1}{\gamma}\ln\left(1+\gamma x_i^{-\beta}\right)-\frac{x_i^{-\beta}}{\left(1+\gamma x_i^{-\beta}\right)}\right]+\frac{3\left(1-\lambda_2\right)}{\left(1+\gamma x_i^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}}\left[\frac{\alpha}{\gamma^2}\ln\left(1+\gamma x_i^{-\beta}\right)-\frac{x_i^{-\beta}}{\left(1+\gamma x_i^{-\beta}\right)}\right]\right\}}{\lambda_1+2\left(\lambda_2-\lambda_1\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}+3\left(1-\lambda_2\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}}\right)\end{array}\right)=0,(34)$$

$$\frac{\partial}{\partial\lambda_1}\left(\ln L\left(x_i;\Phi\right)\right)=\sum\left[\frac{1-2\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}}{\lambda_1+2\left(\lambda_2-\lambda_1\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}+3\left(1-\lambda_2\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}}\right]=0,\quad (35)$$

$$\frac{\partial}{\partial\lambda_2}\left(\ln L\left(x_i;\Phi\right)\right)=\sum\left[\frac{2\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}-3\left(1+\gamma x_i^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}}{\lambda_1+2\left(\lambda_2-\lambda_1\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{\alpha}{\gamma}}+3\left(1-\lambda_2\right)\left(1+\gamma x_i^{-\beta}\right)^{-\frac{2\alpha}{\gamma}}}\right]=0.\ (36)$$

8. Simulation Study

In this section, we perform the simulation study to illustrate the performance of MLE. We consider the CRTMBIII-P distribution with $\alpha = 2.95,\ \beta = 0.7, \gamma = 2.20, k = 3.95,\ \lambda_1 = 0.4, \lambda_2 = 0.1, \theta = 1.$ We generate 1000 samples of sizes 20, 50, 200. The simulation results are reported in Table 3. In the table, it reports the average estimated $\alpha,\ \beta, \gamma, k,$ λ_1, λ_2 and the standard deviation of the estimates within the parenthesis. From this Table, we observe that the MLE estimates approach true values as the sample size increases whereas the standard deviations of the estimates decrease, as expected.

Table 3. MLE simulation

Sample size	α	β	γ	k	λ_1	λ_2	θ
20	4.1394 (0.7279)	0.6876 (0.0373)	2.2206 (0.4541)	3.5892 (0.0875)	0.3740 (0.0418)	0.0759 (0.0502)	1.0076 (0.0026)
50	3.2909 (0.6665)	0.7475 (0.0610)	2.3937 (0.8290)	3.8474 (0.1169)	0.3854 (0.0420)	0.0872 (0.0428)	1.0075 (0.0023)
200	3.0703 (0.5578)	0.7340 (0.0668)	1.9048 (0.8527)	3.9018 (0.0887)	0.3887 (0.0340)	0.0914 (0.0398)	1.0073 (0.0021)

9. Applications

In this section, the CRTMBIII-P distribution is compared with TMBIII-P, MBIII-P, BIII-P, IL-P, LL-P distributions. Different goodness fit measures like Cramer-von Mises (W), Anderson Darling (A), Kolmogorov- Smirnov (K-S) statistics with p-values, and likelihood ratio statistics are computed using R-package for tensile strength of carbon fibers and strengths of glass fibers.

The better fit corresponds to smaller W, A, K-S (p-value), AIC, CAIC, BIC, HQIC and $-\ell$ value. The maximum likelihood estimates (MLEs) of unknown parameters and values of goodness of fit measures are computed for CRTMBIII-P distribution and its sub-models. The MLEs**,** their standard errors (in parentheses) and goodness-of-fit statistics like W, A, K-S (p-values) are given in table 4 and 6. Table 5 and 7 displays goodness-of-fit values.

9.1 Application I: Strengths of Carbon Fibers

The data of 100 observations about tensile strength of carbon fibers (Nicholas and Padgett (2006) are : 3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

Table 4. MLEs and their standard errors (in parentheses) and Goodness-of-fit statistics for data set I

Model	θ	α	β	γ	κ	λ_1	λ_2
CRTMBIII-P	0.39	719.9295 (523.3879620)	0.9732829 (1.8953695)	500.1105 (408.361968)	4.117140 (8.0763895)	0.7742347 (0.389845)	1.0000e-10 (0.8599415)
TMBIII-P	0.39	719.8619233 (509.782044)	0.7427825 (1.072550)	500.1927347 (379.177167)	5.0462072 (7.304020)	0.7473355 (0.388087)	--------------
MBIII-P	0.39	236.99999990(107.36391667)	0.05177609 (0.04879421)	108.00000016 (68.36992397)	60.00000145 (56.68837095)	--------------	--------------
BIII-P	0.39	43.4307596 (10.2832402)	0.1518039 (0.4298325)	--------------	15.2231468 (43.1097656)	--------------	--------------
IL-P	0.39	37.276213 (9.6291355)	--------------	--------------	2.248313 (0.1666808)	--------------	--------------
LL-P	0.39	--------------	5.4603307 (0.466840215)	--------------	0.3758895 (0.008629736)	--------------	--------------

Table 5. Goodness-of-fit statistics for data set I

Model	W	A	K-S (p-value)	AIC	CAIC	BIC	HQIC	$-\ell$
CRTMBIII-P	**0.1944189**	**1.016396**	**0.0833** **(0.4937)**	**290.5319**	**291.445**	**306.1027**	**296.8319**	**139.266**
TMBIII-P	0.2952717	1.553695	0.0986 (0.291)	293.9015	294.5467	306.8771	299.1515	141.9508
MBIII-P	0.3747326	2.013152	0.1179 (0.1275)	297.706	298.1315	308.0865	301.9059	144.853
BIII-P	0.5889363	3.291431	0.1447 (0.03163)	311.9804	312.233	319.7657	315.1304	152.9902
IL-P	0.6321082	3.546179	0.152 (0.0206)	313.0601	313.1851	318.2504	315.1601	154.5301
LL-P	0.3579362	1.915628	0.102 (0.2548)	291.962	292.087	297.1523	294.062	143.981

The CRTMBIII-P distribution is best fitted model than the other sub-models because the values of all criteria of goodness of fit are significantly smaller for CRTMBIII-P distribution.

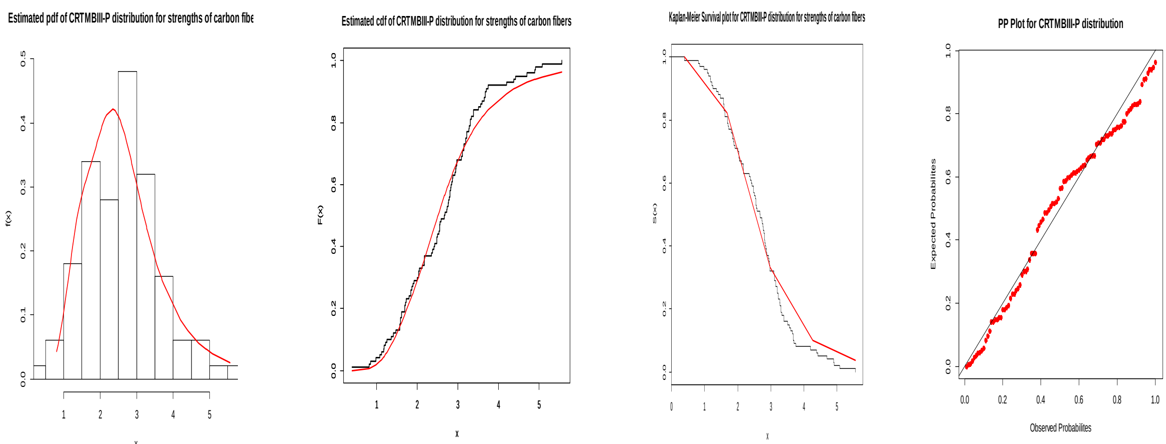

Figure 3. Fitted pdf, cdf, survival and pp plots of the CRTMBIII-P distribution for carbon fibers

We can also perceive that the CRTMBIII-P distribution is best fitted model than other sub-models because the CRTMBIII-P distribution offers the closer fit to empirical data (Fig. 3).

9.2 Application II: Strengths of Glass Fibers

The data of strengths of 1.5cm glass fibers (Smith and Naylor; 1987) are :0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.0, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

Table 6. MLEs and their standard errors (in parentheses) and Goodness-of-fit statistics for data set II

Model	θ	α	β	γ	κ	λ_1	λ_2
CRTMBIII-P	0.55	73.87911 (3.364499e+03)	14.71889 (6.209595)	497.0174 (23061.72)	0.8096296 (1.689077)	0.5221054 (0.5534387)	1.000000e-10 (1.182675)
TMBIII-P	0.55	74.3642653 (601.5807783)	16.0453413 (4.9856028)	497.8138888 (3991.182102)	0.7697337 (0.2948280)	0.4117638 (0.2694987)	
MBIII-P	0.55	144.0176794 (752.6882294)	16.6280692 (4.9243447)	773.7621945 (4002.818096)	0.7651168 (0.1911755)		
BIII-P	0.55	23.55444876 (5.4988241)	0.09262152 (0.2572815)	1	38.76318442 (107.7000457)		
IL-P	0.55	17.079603 (4.5058600)	1	1	3.351204 (0.3126531)		
LL-P	0.55	1	5.5523815 (0.63158387)	1	0.6811163 (0.01865733)		

Table 7. Goodness-of-fit statistics for data set II

Model	W	A	K-S (p-value)	AIC	CAIC	BIC	HQIC	$-\ell$
CRTMBIII-P	**0.04680253**	**0.2783179**	**0.087 (0.7361)**	**24.01893**	**25.5462**	**36.78173**	**29.02992**	**6.009463**
TMBIII-P	0.1080799	0.6156135	0.1226 (0.3089)	26.16174	27.23317	36.79742	30.33758	8.080872
MBIII-P	0.1580743	0.8983662	0.1514 (0.1167)	27.00258	27.70434	35.51112	30.34325	9.501291
BIII-P	1.005342	5.58068	0.2404 (0.001548)	77.20884	77.62263	83.59024	79.71434	35.60442
IL-P	1.128209	6.209134	0.258 (0.005188)	81.43704	81.64043	85.69131	83.10738	38.71852
LL-P	0.6565834	3.688252	0.173 (0.04898)	52.8925	53.09589	57.14677	54.56283	24.44625

The CRTMBIII-P distribution is best fitted model than the other sub-models because the values of all criteria of goodness of fit are significantly smaller for CRTMBIII-P distribution.

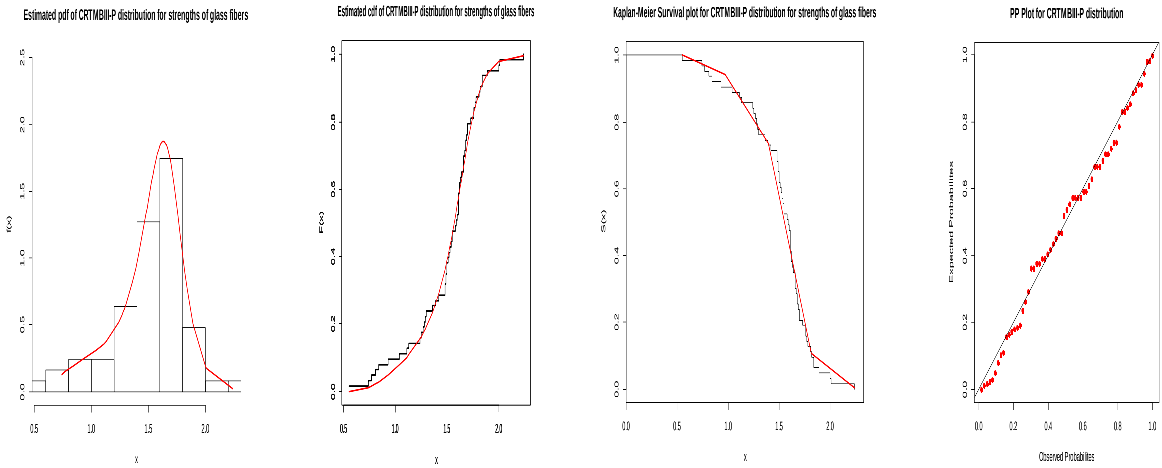

Figure 4. Fitted pdf, cdf, survival and pp plots of the CRTMBIII-P distribution

We can also perceive that the CRTMBIII-P distribution is best fitted model than other sub-models because the CRTMBIII-P distribution offers the closer fit to empirical data (Fig. 4).

10. Concluding Remarks

We have developed a more flexible distribution on the basis of the cubic transmuted mapping that is suitable for applications in survival analysis, reliability and actuarial science. The important properties of the proposed CRTMBIII-P distribution such as survival function, hazard function, reverse hazard function, cumulative hazard function, mills ratio, elasticity, quantile function, moments about the origin, incomplete moments, inequality measures and stress-strength reliability measures are presented. The proposed distribution is characterized via ratio of truncated moments and doubly truncated moment. Maximum likelihood estimates are computed. The simulation study for the performance of the MLEs of parameters for the new distribution is carried out. Applications of the proposed model to tensile strength of carbon fibers and strengths of glass fibers are presented to show its significance and flexibility. Goodness of fit shows that the new distribution is a better fit. We have demonstrated that proposed distribution is empirically better for tensile strength of carbon fibers and strengths of glass fibers data.

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Appendix A

Theorem 1: Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a probability space for given interval $[d_1, d_2]$ with $-\infty < d_1 < d_2 < d_2 + \infty$. Let $X : \Omega \to [d_1, d_2]$ be a continuous random variable with distribution function F. Let real functions h_1 and h_2 be continuous on $[d_1, d_2]$ such that $\frac{E[\mathrm{h}_1(X) | \mathrm{X} \geq \mathrm{x}]}{E[\mathrm{h}_2(X) | \mathrm{X} \geq \mathrm{x}]} = q(x)$ is real function in simple form. Assume that $h_1,\ h_2 \in C([d_1, d_2])$, $q(x) \in C^2([d_1, d_2])$ and F is two times continuously differentiable and strictly monotone function on $[d_1, d_2]$ $([d_1, d_2])$.

Finally, assume that the equation $h_2 q(x) = h_1$ has no real solution in $[\mathrm{d}_1, \mathrm{d}_2]$. Then $F(x) = \int_{\ln k}^{x} K \left| \frac{q'(t)}{q(t) h_2(t) - h_1(t)} \right| \exp(-s(t)) dt$ is obtained from the functions h_1, h_2, $q(t)$ and $s(t)$, where $s(t)$ is obtained from equation $s'(t) = \frac{q'(t) h_2(t)}{q(t) h_2(t) - h_1(t)}$ and K is a constant, adopted such that $\int_{d_1}^{d_2} dF = 1.$