On a Family of Sequences Related to Prime Counting Function

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Abstract

In this study, we introduce a new family of integer sequences which are related to prime-counting function and we focus on some properties of these sequences. Sequence A316434 in OEIS is the fundamental member of solution family that we study. More precisely, we investigate the solutions of recurrence $a(n) = a(\pi(n)) + a(n - \pi(n))$ with some natural initial conditions where $\pi(n)$ is defined by A000720 in OEIS.

Keywords: slow sequences, prime-counting function

1. Introduction

The Hofstadter-Conway \$10000 sequence is recursively defined by the nested recurrence relation $c(n) = c(c(n-1)) + c(n-c(n-1))$ and initial values $c(1) = c(2) = 1$ (Mallows, 1991). This sequence has many amazing properties and a very intriguing generational structure (Alkan, Fox, & Aybar, 2017). One of the most important reasons behind its fascinating nature is the construction of parent spots which are $c(n-1)$ and $n - c(n-1)$ and the resulting symmetry that comes from here (Kubo & Vakil, 1996). If we rename $c(n-1) = x(n)$, we can easily observe that $c(n) = c(x(n)) + c(n-x(n))$ is the form of Conway's recurrence. At this point, it can be seen as natural to think that different $x(n)$ functions may also have some interesting results with this recurrence formula. In this study we will use the prime-counting function $\pi(n)$ (A000720 in OEIS) for this purpose (Sloane, 2018) due to its curious asymptotic form. $\pi(n)$ is a slow sequence by definition and there are many rigorous works on it in the literature. One of the most important result of it is Prime Number Theorem (PNT) which states that $\pi(n) \sim Li(n)$ where $Li(n) \sim \sum_{k=0}^{\infty} \frac{n \cdot k!}{(\ln n)^{(k+1)}}$ (Hadamard, 1896). Corresponding error term is improved by important studies and it is still very interesting in terms of different perspectives (Reyna & Toulisse, 2013).

This paper is structured as follows. In Section 2, we define and prove some properties of our sequence family. Then, in Section 3, we report a variety of interesting observations about asymptotic behaviors of the solutions and we obtain a conjectural result on this family. Finally, we provide some concluding remarks in Section 4.

2. Basic Properties of Family

We will define a sequence family similar with Newman generalization on Conway's sequence (Newman & Kleitman, 1991). This natural selection of initial conditions will provide many behavioral similarities and existence of infinitely many different slow solutions.

Definition 1. *Let $a_i(n) = a_i(\pi(n)) + a_i(n - \pi(n))$ for $n > i$, with the initial conditions $a_i(n) = 1$ for $1 \leq n \leq i$.*

See Table 1 in order to observe initial terms of $a_i(n)$ for $i \leq 10$. Definion 1 essentially guarantees that $a_i(n) \geq a_{i+1}(n)$ for all $n, i \geq 1$ based on selection of initial conditions by induction.

See also Table 2 in order to observe selected terms of $a_i(n)$ for $i \leq 5$. One can easily see that $a_2(n) + a_3(n) = n$ for $n > 1$ and $a_4(n) + a_5(n) = a_2(n)$ for $n > 2$ based on Table 1 and Table 2. In fact, this is also correct for $a_6(n) + a_7(n) = a_3(n)$ for $n > 3$.

Proposition 1. $a_1(n) = n$ *for all* $n \geq 1$.

Proof. We can see that this is true for small n and it is clear that $\pi(n) < n$ and $n - \pi(n) < n$ for all $n > 1$. So if $a_i(k) = k$ for all $k < n$, then $a_i(n) = a_i(\pi(n)) + a_i(n - \pi(n)) = \pi(n) + n - \pi(n) = n$ holds by induction for all n.

□

Proposition 2. $a_i(n) = a_{2 \cdot i}(n) + a_{2 \cdot i+1}(n)$ *for all* $n > i$ *and* $1 \leq i \leq 3$.

Table 1. Initial terms of first ten members of $a_i(n)$ generalization.

$a_i(n)$	Initial terms of $a_i(n)$ sequence
$a_1(n)$	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 ...
$a_2(n)$	1, 1, 2, 2, 3, 4, 4, 4, 5, 6, 7, 7, 8, 8, 9, 10, 10, 11, 11, 11, 12, 12 ...
$a_3(n)$	1, 1, 1, 2, 2, 2, 3, 4, 4, 4, 4, 5, 5, 6, 6, 6, 7, 7, 8, 9, 9, 10, 10, 10 ...
$a_4(n)$	1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 6, 7, 8, 8 ...
$a_5(n)$	1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 7 ...
$a_6(n)$	1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 5, 5, 5, 5, 5, 5 ...
$a_7(n)$	1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 5, 5, 5, 5 ...
$a_8(n)$	1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 4, 5, 5 ...
$a_9(n)$	1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 4 ...
$a_{10}(n)$	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3 ...

Table 2. Values of $a_2(n)$, $a_3(n)$, $a_4(n)$, $a_5(n)$ where $n = 10^t$ for $0 \le t \le 9$.

n	$a_2(n)$	$a_3(n)$	$a_4(n)$	$a_5(n)$
1	1	1	1	1
10	6	4	3	3
10^2	59	41	31	28
10^3	588	412	315	273
10^4	5863	4137	3148	2715
10^5	58737	41263	31467	27270
10^6	587279	412721	314860	272419
10^7	5872093	4127907	3148391	2723702
10^8	58722632	41277368	31482786	27239846
10^9	587236948	412763052	314845574	272391374

Proof. We can see that this is true for small n where $1 \le i \le 3$. So if $a_i(k) = a_{2\cdot i}(k) + a_{2\cdot i+1}(k)$ for all $i < k < n$, then $a_i(n) = a_i(\pi(n)) + a_i(n - \pi(n)) = a_{2\cdot i}(\pi(n)) + a_{2\cdot i+1}(\pi(n)) + a_{2\cdot i}(n - \pi(n)) + a_{2\cdot i+1}(n - \pi(n)) = a_{2\cdot i}(\pi(n)) + a_{2\cdot i}(n - \pi(n)) + a_{2\cdot i+1}(\pi(n)) + a_{2\cdot i+1}(n - \pi(n)) = a_{2\cdot i}(n) + a_{2\cdot i+1}(n)$ holds by induction for all n.

□

Proposition 3. *$a_i(n + 1) - a_i(n) \in \{0, 1\}$ for all $n, i \ge 1$. In other words, $a_i(n)$ is slow for all $i \ge 1$.*

Proof. Initial conditions $a_i(k) = 1$ for $1 \le k \le i$ provide the basis for induction since $a_i(i+1) = 2$ and $a_i(k+1) - a_i(k) \in \{0, 1\}$ for $1 \le k \le i$. We must show that $a_i(n + 1) - a_i(n) \in \{0, 1\}$ for all $n \ge i + 1$.

$$\begin{aligned} a_i(n+1) &= a_i(\pi(n+1)) + a_i(n+1-\pi(n+1)) \\ a_i(n) &= a_i(\pi(n)) + a_i(n-\pi(n)). \end{aligned}$$

From above equations,

$$a_i(n+1) - a_i(n) = a_i(\pi(n+1)) - a_i(\pi(n)) + a_i(n+1-\pi(n+1)) - a_i(n-\pi(n))$$

Case 1. $\pi(n + 1) = \pi(n)$. At this case,

$$\begin{aligned} a_i(n+1) - a_i(n) &= a_i(\pi(n+1)) - a_i(\pi(n)) + a_i(n+1-\pi(n+1)) - a_i(n-\pi(n)) \\ &= a_i(n+1-\pi(n+1)) - a_i(n-\pi(n)) \in \{0, 1\}. \end{aligned}$$

Case 2. $\pi(n + 1) = \pi(n) + 1$. At this case,

$$\begin{aligned} a_i(n+1) - a_i(n) &= a_i(\pi(n+1)) - a_i(\pi(n)) + a_i(n+1-\pi(n+1)) - a_i(n-\pi(n)) \\ &= a_i(\pi(n+1)) - a_i(\pi(n)) \in \{0, 1\}. \end{aligned}$$

This completes the induction about slowness of $a_i(n)$ for all $i \geq 1$.

□

Proposition 4. *$a_i(n)$ hits every positive integer for all $i \geq 1$.*

Proof. Let us assume that $a_i(n) = K_i$ is the maximum value of sequence and N_i is the first occurence K_i. So, $a_i(N_i+t_i) = K_i$ for all $t_i \geq 0$. At this case $a_i(N_i + t_i) = a_i(\pi(N_i + t_i)) + a_i(N_i + t_i - \pi(N_i + t_i))$ for all $t_i \geq 0$. If we choose t_i such that $\pi(N_i + t_i) = N_i$ then $K_i = K_i + a_i(t_i)$ and this is contradiction. Since $a_i(n)$ is slow, $a_i(n)$ must hit every positive integer.

□

Proposition 5. *Both $(a_i(\pi(n)))_{n>1}$ and $(a_i(n-\pi(n)))_{n>1}$ are slow sequences for all $i \geq 1$ and they hit every positive integer.*

Proof. Since we show that $a_i(n)$ is slow and unbounded, $\pi(n)$ and $n - \pi(n)$ slow parent spots give this result similar with above propositions for $n > 1$ and for all $i \geq 1$. □

Definition 2. *Let $f_i(n) = \frac{a_i(n-\pi(n))}{a_i(\pi(n))}$ for $n > 1$.*

We know that $f_1(n) = \frac{n-\pi(n)}{\pi(n)}$ since $a_1(n) = n$. See Figure 1 in order to observe $f_i(n)$ for $i \leq 7$. Figure 1 suggests that $f_i(n) \sim \frac{n-\pi(n)}{\pi(n)}$ for $i \leq 7$. More detailed empirical investigation also confirms this suggestion, at least for $i \leq 100$.

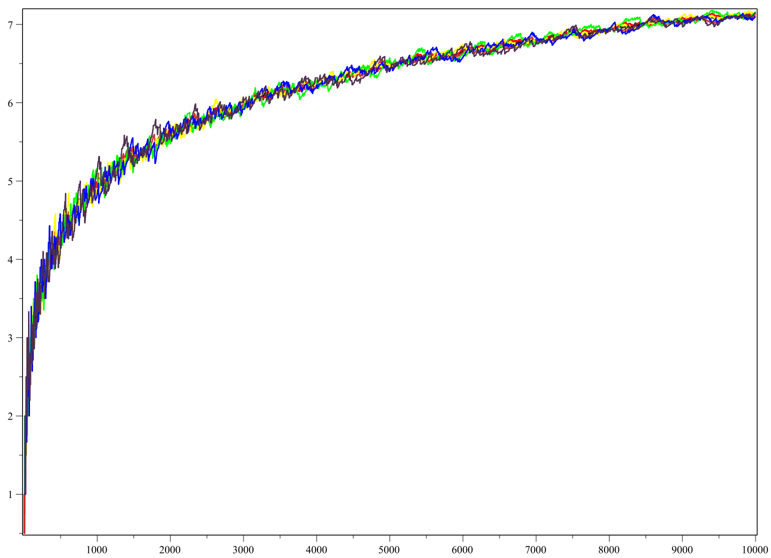

Figure 1. $f_1(n)$: black, $f_2(n)$: red, $f_3(n)$: orange, $f_4(n)$: yellow, $f_5(n)$: green, $f_6(n)$: blue, $f_7(n)$: violet.

3. Experiments On Behaviour of $\frac{a_i(n)}{n}$

Based on previous section, it is natural to think that $a_i(n)$ sequences have considerable similarities in terms of proportion between $a_i(\pi(n))$ and $a_i(n - \pi(n))$. This is particularly interesting since $f_i(n)$ conjecturally generalizes the distributional behavior of $\frac{n-\pi(n)}{\pi(n)}$, at least in our experimental range. Additionally, oscillations of $\frac{a_i(n)}{n}$ provide a curious collection of conjectural constants. Statistical analysis based on successive selected intervals suggests that $lim_{n\to\infty}\frac{a_2(n)}{n} < 4^{1/3} - 1$, if it exists. This empirical investigation is also determinative for $lim_{n\to\infty}\frac{a_3(n)}{n}$ since $a_2(n) + a_3(n) = n$ for $n > 1$, that is, $lim_{n\to\infty}\frac{a_3(n)}{n} > 2 - 4^{1/3}$, if it exists and previous observation is correct. Similar analysis can be done for $\frac{a_5(n)}{a_2(n)}$ based on $a_4(n) + a_5(n) = a_2(n)$ for $n > 2$. Figure 2 shows that $\frac{a_5(n)}{a_2(n)}$ oscillates around $\frac{e}{e+\pi}$ in our investigation range. Although such results are observed in limited range that this study focuses on, exact analysis is also related to corresponding frequency sequences which are relatively complicated to investigate in detail. On the other hand, for more general perspective, see Figure 3 and Figure 4 in order to observe order signs in asymptotic behavior of $a_i(n)$ family. Based on these empirical evidences and analysis in previous section we can conjecture the general property about these sequences.

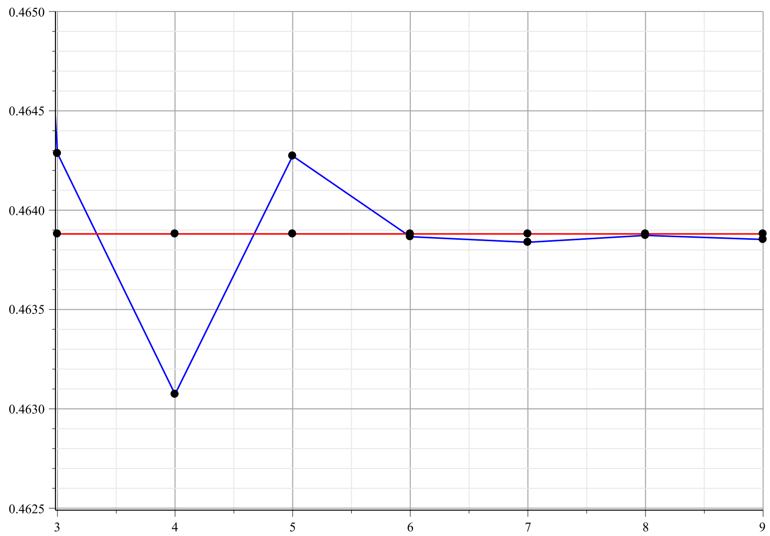

Figure 2. $\frac{e}{e+\pi}$: red, $\frac{a_5(n)}{a_2(n)}$: blue where $n = 10^t$ for $3 \leq t \leq 9$.

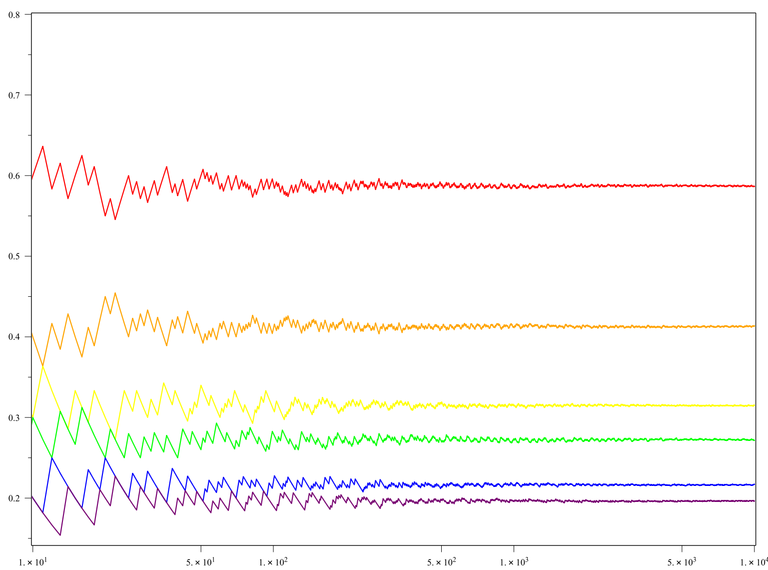

Figure 3. $\frac{a_2(n)}{n}$: red, $\frac{a_3(n)}{n}$: orange, $\frac{a_4(n)}{n}$: yellow, $\frac{a_5(n)}{n}$: green, $\frac{a_6(n)}{n}$: blue, $\frac{a_7(n)}{n}$: purple.

Conjecture 1. *$lim_{n\to\infty}\frac{a_i(n)}{n}$ exists and it is constant c_i with $c_i > c_{i+1}$ for all positive integer $i \geq 1$. In other words, $\frac{a_i(n-\pi(n))}{a_i(\pi(n))} \sim (\sum_{k=0}^{\infty} \frac{k!}{(\ln n)^{(k+1)}})^{-1}$ for all $i \geq 1$ as a result of PNT.*

At this point it would be nice to remember similar analysis on Newman generalization of Conway's sequence gives the result that is completely soluble by the largest root of the characteristic polynomial that sequences correspond (Kubo & Vakil, 1996) based on their well-behaved generational strucutures (Dalton, Rahman, & Tanny, 2011). On the other hand, Figure 4 suggests a remarkably predictable behaviour despite the complicated nature of $\pi(n)$.

4. Conclusion

Many slow solutions and corresponding recurrences are investigated in the literature thanks to concept of meta-Fibonacci sequences. However, $a_i(n)$ sequences have different and much more complicated structure due to nature of $\pi(n)$. It remains as an open question what are the complete structures of these sequences and values of c_i constants for $i > 1$ with their distribution characteristics. Additionally, different slow sequences can be constructed such as A316942 and A316388 in OEIS thanks to similar approaches (Sloane, 2018). In other words, slow sequence families can be extended curiously with the help of certain natural mathematical functions.

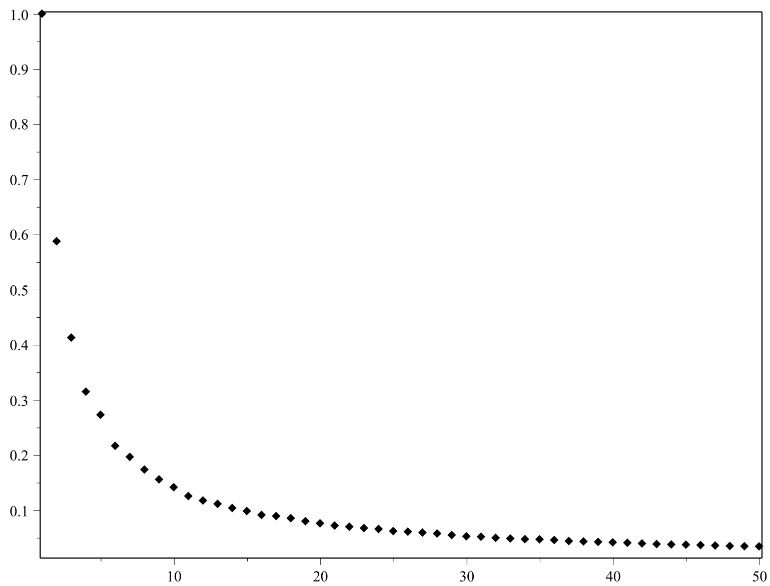

Figure 4. Plot of $\frac{a_i(k)}{k}$ for $1 \leq i \leq 50$ and $k = 10^5$.

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Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

Alkan, A., Fox, N., & Aybar, O. O. (2017). On Hofstadter Heart Sequences. Complexity, 2017, 1-8. https://doi.org/10.1155/2017/2614163.

Dalton, B., Rahman, M., & Tanny, S. (2011). Spot-Based Generations for Meta-Fibonacci Sequences. *Experimental Mathematics, 20*(2), 129-137. https://doi.org/10.1080/10586458.2011.544565.

Hadamard, J. (1896). Sur la distribution des zros de la fonction $zeta(s)$ et ses consquences arithmtiques. Bulletin De La Socit Mathmatique De France, *2*, 199-220. https://doi.org/10.24033/bsmf.545.

Newman, D., & Kleitman, D. J. (1991). E3274. *The American Mathematical Monthly, 98*(10), 958. https://doi.org/10.2307/2324158.

Kubo, T., & Vakil, R. (1996). On Conways recursive sequence. *Discrete Mathematics, 152*(1-3), 225-252. https://doi.org/10.1016/0012-365x(94)00303-z.

Mallows, C. L. (1991). Conways Challenge Sequence. *The American Mathematical Monthly, 98*(1), 5-20. https://doi.org/10.1080/00029890.1991.11995694.

Reyna, J. A., & Toulisse, J. (2013). The n-th prime asymptotically. *Journal De Thorie Des Nombres De Bordeaux, 25*(3), 521-555. https://doi.org/10.5802/jtnb.847.

Sloane, N. J. A. (2018). OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences. https://oeis.org.