

Asymptotic Properties for Weighted Averages of Longitudinal Dependent Data

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Abstract

This paper presents a set of normality general results for kernel weighted averages. We extend existing literature for independent data (Yao, 2007) to stationary dependent longitudinal data. The asymptotic properties of proposed weighted averages are investigated under α -mixing conditions. These results are useful for covariance function estimation based on nonparametric kernel method.

Keywords: longitudinal data, strong mixing data, weighted averages, covariance estimation, stationary process

1. Introduction

We present a set of asymptotic normality results of real-valued function that we assume to be formed by weighted averages of longitudinal data. Since it's well known that the most commonly nonparametric kernel type estimators are written as kernel weighted averages, our results take in account a large class of estimators (Nadaraya, 1964; Watson, 1964; Stone, 1977; Müller, 1984).

Recently, Yao (Yao, 2007) has given general normality results for some function of kernel averages formed by longitudinal independent data. He has applied his general result to covariance function estimator to derive its asymptotic distribution. Soro & Hili (Soro & Hili, 2012) have generalized the results of Yao (Yao, 2007) to three-dimensional context. The data were equally independent.

In this paper, we extend the two-dimensional general result (Yao, 2007) to dependent longitudinal data. Our main results are the asymptotic normality of a sample averages of some function that we suppose to be formed by longitudinal data. We suppose that the data are strongly mixing. The results we provide are applicable to covariance function kernel estimator to derive its asymptotic distribution under α -mixing conditions.

In Section 2, we introduce the model as well as assumptions that are necessary in deriving the main results of this paper. Section 3 presents main results of the paper.

2. Model and Some Assumptions

Let $\{(X_{ir}, U_{ir}, T_{ir}), 1 \leq i \leq n, 1 \leq r \leq N\}$ be $n \times N$ random variables, identically distributed as the random triple (X, U, T) with values in $\mathbb{R} \times \mathbb{R} \times \mathbb{T}$, where \mathbb{T} is such that $\mathbb{T} = [0, \mathcal{T}]$ with $\mathcal{T} < \infty$.

For the multi-index of integers $\lambda = (\lambda_1, \lambda_2)$ and $k = (k_1, k_2)$, let define $|\lambda| = \lambda_1 + \lambda_2$, $|k| = k_1 + k_2$; $\lambda! = \lambda_1! \lambda_2!$ and $k! = k_1! k_2!$.

We consider a model for repeated measurements, which is typically used for longitudinal data treatment :

$$U_{ir} = X_i(T_{ir}) + \epsilon_{ir}, \quad 1 \leq i \leq n, 1 \leq r \leq N. \quad (1)$$

In the model (1), U_{ir} is referred to the r -th observation of the random variable X_i , made at the random time T_{ir} . Assume that

- the number of observations $N(n)$ depends on the sample size n . For simplicity, $N(n)$ will be noted N .
- X takes values in a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ whereas U is a real random variable.
- the observation times T_{ir} are i.i.d. with a marginal density $f_0(t)$.

For $r \neq s$, and $(r, s) \neq (r', s')$, we define the joint probability densities as follow.

Let $f_1(t_1, t_2)$ be the joint density of (T_r, T_s) , $g(t, u)$ the density of (T_r, U_s) , $g_1(t_1, t_2, u_1, u_2)$ be the joint density of (T_r, T_s, U_r, U_s) and $g_2(t_1, t_2, t'_1, t'_2, u_1, u_2, u'_1, u'_2)$ be the joint density of $(T_r, T_s, T_{r'}, T_{s'}, U_r, U_s, U_{r'}, U_{s'})$.

We consider averages of longitudinal data of the form:

$$\begin{aligned} \hat{\Theta}_{n,\ell} &= \hat{\Theta}_{n,\ell}(t_1, t_2) \\ &= \frac{1}{nN(N-1)h_{n,K}^{|\lambda|+3}} \sum_{i=1}^n \sum_{1 \leq r \neq s \leq N} \psi_\ell(T_{ir}, T_{is}, U_{ir}, U_{is}) K\left(\frac{t_1 - T_{ir}}{h_{n,K}}, \frac{t_2 - T_{is}}{h_{n,K}}\right), \end{aligned} \tag{2}$$

for $1 \leq \ell \leq L$, where $h_{n,K}$ is a bandwidth, $K : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a kernel function and $\psi_\ell : \mathbb{R}^2 \rightarrow \mathbb{R}$ are real functions. Let

$$\vartheta_\ell(t_1, t_2) = \|K\|_2^2 \int_{\mathbb{R}^2} \psi_\ell^2(t_1, t_2, u_1, u_2) g_1(t_1, t_2, u_1, u_2) du_1 du_2, \tag{3}$$

$$\theta_\ell(t_1, t_2) = \frac{d^{|\lambda|}}{dt_1^{k_1} dt_2^{k_2}} \int_{\mathbb{R}^2} \psi_\ell(t_1, t_2, u_1, u_2) g_1(t_1, t_2, u_1, u_2) du_1 du_2, \tag{4}$$

$$\mathbf{B}(t_1, t_2) = \frac{(-1)^{|\lambda|}}{k!} \left\{ \int_{\mathbb{R}^2} v^{k_1} w^{k_2} K(v, w) dv dw \frac{d^{|\lambda|}}{dt_1^{k_1} dt_2^{k_2}} \int_{\mathbb{R}^2} \psi_\ell(t_1, t_2, u_1, u_2) g_1(t_1, t_2, u_1, u_2) du_1 du_2 \right\}, \text{ for } 1 \leq \ell \leq L. \tag{5}$$

Let $\mathcal{N}_{(t_1, t_2)}$ be a neighborhood of $(t_1, t_2) \in [0; \mathcal{T}]^2$.

Now, we introduce the following basic assumptions that are necessary in deriving the main result of this paper.

(H1) (i) The kernel K is symmetric with a compact support.

(ii) $\|K\|_2^2 = \int_{\mathbb{R}^2} K^2(v, w) dv dw < \infty$.

(iii) K is a kernel function of order $(|\lambda|, |k|)$, that is,

$$\int_{\mathbb{R}^2} v^{l_1} w^{l_2} K(v, w) dv dw = \begin{cases} 0, & 0 \leq |l| < |k|, |l| \neq |v|. \\ (-1)^{|v|} v!, & |l| = |v|, \\ C, & |l| = |k|. \end{cases} \tag{6}$$

where C is a non null constant.

(H2) The bandwidth $h_{n,K}$ satisfies,

(i) $h_{n,K} \rightarrow 0, nN(N-1)h_{n,K}^{|\lambda|+2} \rightarrow \infty, nN(N-1)h_{n,K}^{2|\lambda|+2} \rightarrow a^2$, where a is a positive constant, as $n \rightarrow +\infty$.

(ii) $h_{n,K}^{|\lambda|+2} \rightarrow \infty, nh_{n,K}^{|\lambda|+2} \rightarrow \infty$ and $N(N-1)h_{n,K}^{|\lambda|} \rightarrow \infty$, as $n \rightarrow \infty$.

(H3) (i) $\frac{d^{|\lambda|}}{du^{k_1} dv^{k_2} dw^{k_3}} f_1(v, w)$ exists and is continuous for $(v, w) \in \mathcal{N}_{(t_1, t_2)}$;

(ii) $g_1(v, w, u_1, u_2)$ is continuous for $(v, w) \in \mathcal{N}_{(t_1, t_2)}$, uniformly for $(u_1, u_2) \in \mathbb{R}^2$;

(H4) (i) $\frac{d^{|\lambda|}}{dv^{k_1} dw^{k_2}} g_1(v, w, u_1, u_2)$ exists and is continuous for $(v, w) \in \mathcal{N}_{(t_1, t_2)}$, uniformly for $(u_1, u_2) \in \mathbb{R}^2$;

(ii) $g_2(v, w, v', w', u_1, u_2, u'_1, u'_2)$ is continuous for $(v, w, v', w') \in \mathcal{N}_{(t_1, t_2)}^2$, uniformly for $(u_1, u_2) \in \mathbb{R}^2$.

(H5) The collection $\{\psi_\ell\}_{\ell=1, \dots, l}$ of real functions $\psi_\ell : \mathbb{R}^4 \rightarrow \mathbb{R}$ satisfies:

(i) $\psi_\ell(t_1, t_2, u_1, u_2)$ is continuous for (t_1, t_2) uniformly for $(u_1, u_2) \in \mathbb{R}^2$,

(ii) $\frac{d^{|\lambda|}}{dt_1^{k_1} dt_2^{k_2}} \psi_\ell(t_1, t_2, u_1, u_2)$ exists for all arguments $(t_1, t_2, u_1, u_2) \in \mathbb{R}^4$.

Let \mathcal{F}_a^b be the sigma algebra generated by the random variables $\{X_i, Y_i\}_{i=a}^b$.

The stationary process $\{X_i, Y_i\}_{i=a}^b$ is called strongly mixing (Rosenblatt, 1956) if

$$\alpha(j) = \sup_t \sup_{A \in \mathcal{F}_{-\infty}^t, B \in \mathcal{F}_{t+j}^\infty} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \rightarrow 0, \text{ as } j \rightarrow \infty.$$

(H6) The process $\{X_i, Y_i\}$ is strongly mixing with coefficient

$$\sum_{j=1}^\infty j^a [\alpha(j)]^{1-2/\delta} < \infty \text{ for some } a > 1 - 2/\delta, \text{ and some } \delta > 2. \tag{7}$$

Comments on the assumptions. Assumptions (H1) and (H2) are technical conditions for the proofs. Assumptions (H3) and (H4) are regularity conditions for joint probability densities. Assumption (H6) is mixing condition verified by the process.

3. Main Results: Consistency and Asymptotic Normality of Kernel Averages

Before establishing the main results of the paper, we first prove the consistency and the asymptotic normality of weighted averages (2).

3.1 Consistency of Kernel Averages

In this part of paper, we establish the consistency of (2). The result is given in the following theorem.

Theorem 1 *If assumptions (H1)-(H5) are satisfied, we have*

$$\hat{\Theta}_{n,\ell}(t_1, t_2) \xrightarrow{\mathbb{P}} \theta_\ell(t_1, t_2), \tag{8}$$

where $\theta_\ell(t_1, t_2)$ is defined in (4) for $\ell = 1, \dots, L$.

Proof. To establish the consistency of (8) we have to consider the following decomposition

$$\mathbb{E}\left\{\left(\hat{\Theta}_{n,\ell}(t_1, t_2) - \theta_\ell(t_1, t_2)\right)^2\right\} = \text{Var}\left\{\hat{\Theta}_{n,\ell}(t_1, t_2)\right\} + \left\{\mathbb{E}\left[\hat{\Theta}_{n,\ell}(t_1, t_2)\right] - \theta_\ell(t_1, t_2)\right\}^2. \tag{9}$$

We denote by $\xrightarrow{\mathbb{L}^2}$ (respectively $\xrightarrow{\mathbb{P}}$) the convergence in \mathbb{L}^2 (resp. in probability) and we also recall that when (9) goes to zero, we have

$$\hat{\Theta}_{n,\ell}(t_1, t_2) \xrightarrow{\mathbb{L}^2} \theta_\ell(t_1, t_2) \text{ implies } \hat{\Theta}_{n,\ell}(t_1, t_2) \xrightarrow{\mathbb{P}} \theta_\ell(t_1, t_2).$$

- Let prove that the second term in (9) goes to 0 when n goes to $+\infty$. We have

$$\begin{aligned} \mathbb{E}\hat{\Theta}_{n,\ell}(t_1, t_2) &= \frac{1}{nN(N-1)h_{n,K}^{|\nu|+2}} \mathbb{E}\left\{\sum_{i=1}^n \sum_{1 \leq r \neq s \leq N} \psi_\ell(T_{ir}, T_{is}, U_{ir}, U_{is}) K\left(\frac{t_1 - T_{ir}}{h_{n,K}}, \frac{t_2 - T_{is}}{h_{n,K}}\right)\right\} \\ &= \frac{1}{N(N-1)h_{n,K}^{|\nu|+2}} \mathbb{E}\left\{\sum_{1 \leq r \neq s \leq N} \psi_\ell(T_{ir}, T_{is}, U_{ir}, U_{is}) K\left(\frac{t_1 - T_{ir}}{h_{n,K}}, \frac{t_2 - T_{is}}{h_{n,K}}\right)\right\} \\ &= \frac{1}{h_{n,K}^{|\nu|+2}} \mathbb{E}\left\{\psi_\ell(T_{11}, T_{12s}, U_{11}, U_{12}) K\left(\frac{t_1 - T_{11}}{h_{n,K}}, \frac{t_2 - T_{12}}{h_{n,K}}\right)\right\} \\ &= \theta_\ell(t_1, t_2) + \frac{(-1)^{|\kappa|}}{k!} \left\{\int_{\mathbb{R}^2} v^{\kappa_1} w^{\kappa_2} K(v, w) dv dw\right. \\ &\quad \times \left.\frac{d^{|\kappa|}}{dt_1^{\kappa_1} dt_2^{\kappa_2}} \int_{\mathbb{R}^2} \psi_\ell(t_1, t_2, u_1, u_2) g_1(t_1, t_2, u_1, u_2) du_1 du_2 \times h_{n,K}^{|\kappa|-|\lambda|}\right\} + o(h_{n,K}^{|\kappa|-|\lambda|}) \\ &= \theta_\ell(t_1, t_2) + \mathbf{B}(t_1, t_2) \times h_{n,K}^{|\kappa|-|\lambda|} + o(h_{n,K}^{|\kappa|-|\lambda|}). \end{aligned} \tag{10}$$

Then,

$$\mathbb{E}\hat{\Theta}_{n,\ell}(t_1, t_2) - \theta_\ell(t_1, t_2) = \mathbf{B}(t_1, t_2) \times h_{n,K}^{|\kappa|-|\lambda|} + o(h_{n,K}^{|\kappa|-|\lambda|}).$$

And it follows that

$$\mathbb{E}\hat{\Theta}_{n,\ell}(t_1, t_2) - \theta_\ell(t_1, t_2) \longrightarrow 0. \tag{11}$$

- Now, we prove that $\text{Var}\left(\hat{\Theta}_{n,\ell}(t_1, t_2)\right) \longrightarrow 0$, as $n \longrightarrow \infty$.

Let

$$\Gamma_{i,r,s}^\ell = \psi_\ell(T_{ir}, T_{is}, U_{ir}, U_{is}) K\left(\frac{t_1 - T_{ir}}{h_{n,K}}, \frac{t_2 - T_{is}}{h_{n,K}}\right). \tag{12}$$

Using the definition of the variance, we have

$$\begin{aligned} \text{Var}(\hat{\Theta}_{n,\ell}(t_1, t_2)) &= \frac{1}{nN(N-1)h_{n,K}^{2|\lambda|+4}} \text{Var}(\Gamma_{1,1,2}^\ell) \\ &+ \frac{1}{\{nN(N-1)h_{n,K}^{|\lambda|+2}\}^2} \sum_{i=1}^n \sum_{i' \neq i}^n \sum_{1 \leq r \neq s' \leq N} \sum_{1 \leq r' \neq s' \leq N} \text{Cov}(\Gamma_{i,r,s}^\ell, \Gamma_{i',r',s'}^\ell) \\ &= \mathfrak{J}_1 + \mathfrak{J}_2. \end{aligned}$$

→ Concerning \mathfrak{J}_1 ,

$$\begin{aligned} \mathfrak{J}_1 &= \frac{1}{nN(N-1)h_{n,K}^{|\lambda|+2}} \\ &\times \left\{ \mathbb{E} \left[\frac{1}{h_{n,K}^{|\lambda|+2}} \psi_\ell^2(T_{11}, T_{12}, U_{11}, U_{12},) K^2 \left(\frac{t_1 - T_{11}}{h_{n,K}}, \frac{t_2 - T_{12}}{h_{n,K}} \right) \right] \right. \\ &- \left. \mathbb{E}^2 \left[\frac{1}{h_{n,K}^{|\lambda|+2}} \psi_\ell^2(T_{11}, T_{12}, U_{11}, U_{12},) K^2 \left(\frac{t_1 - T_{11}}{h_{n,K}}, \frac{t_2 - T_{12}}{h_{n,K}} \right) \right] \right\} \\ &= \frac{1}{nN(N-1)h_{n,K}^{|\lambda|+2}} \\ &\times \left\{ \frac{1}{h_{n,K}^{|\lambda|+2}} \int_{\mathbb{R}^4} g_1(t, t', u_1, u_2) \psi_\ell^2(t, t', u_1, u_2) K^2 \left(\frac{t_1 - t}{h_{n,K}}, \frac{t_2 - t'}{h_{n,K}} \right) dt dt' du_1 du_2 \right. \\ &- \left. \left[\frac{1}{h_{n,K}^{|\lambda|+2}} \int_{\mathbb{R}^4} g_1(t, t', u_1, u_2) \psi_\ell(t, t', u_1, u_2) K \left(\frac{t_1 - t}{h_{n,K}}, \frac{t_2 - t'}{h_{n,K}} \right) dt dt' du_1 du_2 \right]^2 \right\}. \end{aligned}$$

We have by changing variables

$$\begin{aligned} \mathfrak{J}_1 &= \frac{1}{nN(N-1)h_{n,K}^{|\lambda|+2}} \left\{ \frac{1}{h_{n,K}^{|\lambda|}} \int_{\mathbb{R}^4} g_1(t_1 - h_{n,K}v, t_2 - h_{n,K}w, u_1, u_2) \times \right. \\ &\psi_\ell^2(t_1 - h_{n,K}v, t_2 - h_{n,K}w, u_1, u_2) K^2(v, w) dv dw du_1 du_2 \\ &- h_{n,K}^2 \left[\frac{1}{h_{n,K}^{|\lambda|}} \int_{\mathbb{R}^4} g_1(t_1 - h_{n,K}v, t_2 - h_{n,K}w, u_1, u_2) \right. \\ &\psi_\ell(t_1 - h_{n,K}v, t_2 - h_{n,K}w, u_1, u_2) K(v, w) dv dw du_1 du_2 \left. \right]^2 \left. \right\} \\ &= \frac{1}{nN(N-1)h_{n,K}^{|\lambda|+2}} \{ \vartheta_\ell(t_1, t_2) + o(1) \} \\ &\rightarrow 0, n \rightarrow +\infty. \end{aligned} \tag{13}$$

→ Concerning \mathfrak{J}_2 ,

put

$$\Gamma_{i,r,s}^{\ell,1,2} = \psi_\ell(T_{ir}^{(1)}, T_{is}^{(2)}, U_{ir}^{(1)}, U_{is}^{(2)}) K \left(\frac{t_1 - T_{ir}^{(1)}}{h_{n,K}}, \frac{t_2 - T_{is}^{(2)}}{h_{n,K}} \right).$$

Given that triples $\{Y_{ij}, Y_{ik}, Y_{il}\}$ and $\{Y_{i'j'}, Y_{i'k'}, Y_{i'l'}\}$ are independent and equidistributed then we can write

$$\begin{aligned} \mathfrak{J}_2 &= \frac{\{N(N-1)\}^2}{\{nN(N-1)h_{n,K}^{|\lambda|+2}\}^2} \sum_{i=1}^n \sum_{i' \neq i}^n \text{Cov}(\Gamma_{i,1,1}^{\ell,1,2}; \Gamma_{i',2,2}^{\ell,1,2}) \\ &= \frac{1}{n^2 h_{n,K}^{2|\lambda|+4}} \sum_{i=1}^n \sum_{i' \neq i}^n \text{Cov}(\Gamma_{i,1,1}^{\ell,1,2}; \Gamma_{i',2,2}^{\ell,1,2}) \\ &= \frac{1}{n^2 h_{n,K}^{2|\lambda|+4}} \sum_{i=1}^n \sum_{i' \neq i}^n \text{Cov}(\Gamma_{i,\ell}; \Gamma_{i',\ell}). \end{aligned} \tag{14}$$

Let Λ_n be a positive sequence tending to ∞ to be specified later on. Define

$$S = \{(i, i') : 0 \leq |i - i'| < \Lambda_n, i, i' = 1, \dots, n, i \neq i'\}.$$

Split (14) into two separate summations over index in S and not in S . That is

$$\mathfrak{J}_2 = \mathfrak{J}_{21} + \mathfrak{J}_{22}, \tag{15}$$

where

$$\mathfrak{J}_{21} = \frac{1}{n^2 h_{n,K}^{2|\lambda|+4}} \left\{ \sum_{i,i'=1}^n \sum_{(i,i') \in S} Cov(\Gamma_{i,\ell}; \Gamma_{i',\ell}) \right\} \tag{16}$$

$$\mathfrak{J}_{22} = \frac{1}{n^2 h_{n,K}^{2|\lambda|+4}} \left\{ \sum_{i,i'=1}^n \sum_{(i,i') \notin S} Cov(\Gamma_{i,\ell}; \Gamma_{i',\ell}) \right\}. \tag{17}$$

For (16), using Hölder’s inequality, $|Cov(\Gamma_{i,\ell}, \Gamma_{i',\ell})| \leq (\mathbb{E}\Gamma_{i,\ell}^2 \mathbb{E}\Gamma_{i',\ell}^2)^{1/2} + (\mathbb{E}|\Gamma_{i',\ell}|)^2$, it follows that

$$\begin{aligned} |\mathfrak{J}_{21}| &\leq \frac{1}{n^2 h_{n,K}^{|\lambda|+2}} \sum_{i,i'=1}^n \sum_{(i,i') \in S} \left\{ \frac{1}{n^2 h_{n,K}^{|\lambda|+2}} (\mathbb{E}\Gamma_{i,\ell}^2 \mathbb{E}\Gamma_{i',\ell}^2)^{1/2} + \frac{1}{n^2 h_{n,K}^{|\lambda|+2}} (\mathbb{E}|\Gamma_{i',\ell}|)^2 \right\} \\ &= \frac{1}{n^2 h_{n,K}^{|\lambda|+2}} \sum_{i,i'=1}^n \sum_{(i,i') \in S} \{\vartheta_\ell(t_1, t_2) + o(1)\}. \end{aligned}$$

Since $Card(S) \leq n\Lambda_n$, then

$$\begin{aligned} |\mathfrak{J}_{21}| &\leq \frac{n\Lambda_n}{n^2 h_{n,K}^{|\lambda|+2}} \{\vartheta_\ell(t_1, t_2) + o(1)\} \\ &\leq \frac{\Lambda_n}{n h_{n,K}^{|\lambda|+2}} \{\vartheta_\ell(t_1, t_2) + o(1)\}. \end{aligned} \tag{18}$$

Clearly, if taking $\Lambda_n = (\ln \ln n)^2 \ln n$, $h_{n,K} = \frac{\ln \ln n}{\ln n}$ in (18), one obtain

$$\Lambda_n \rightarrow \infty, h_{n,K}^{|\lambda|+2} \rightarrow 0, n h_{n,K}^{|\lambda|+2} \rightarrow \infty \text{ and } \frac{\Lambda_n}{n h_{n,K}^{|\lambda|+2}} \rightarrow 0, \tag{19}$$

so

$$\mathfrak{J}_{21} \rightarrow 0, \text{ as } n \rightarrow +\infty. \tag{20}$$

Turn to \mathfrak{J}_{22} . Applying Davydov’s Lemma (see Hall & Heyde, Corollary A.2), and assumption (H6) we have

$$\begin{aligned} |Cov(\Gamma_{i,\ell}, \Gamma_{i',\ell'})| &\leq 8 (\mathbb{E}|\Gamma_{i,\ell}|^\delta)^{2/\delta} (\alpha(|i - i'|))^{1-2/\delta} \\ &\leq 8 Const [h_{n,K}^{|\lambda|+2}]^{2/\delta} [\alpha(|i - i'|)]^{1-2/\delta}. \end{aligned} \tag{21}$$

Using (21)

$$\begin{aligned} |\mathfrak{J}_{22}| &\leq \frac{8 Const. [h_{n,K}^{|\lambda|+2}]^{2/\delta}}{n^2 h_{n,K}^{2|\lambda|+4}} \sum_{i,i'=1}^n \sum_{(i,i') \notin S} [\alpha(|i - i'|)]^{1-2/\delta} \\ &\leq \frac{8 Const.}{n^2 h_{n,K}^{(|\lambda|+2)(2(1-1/\delta))}} \sum_{i,i'=1}^n \sum_{(i,i') \notin S} [\alpha(|i - i'|)]^{1-2/\delta}. \end{aligned}$$

Reducing the double sum above to a single sum, it follows that

$$\begin{aligned} |\mathfrak{J}_{22}| &\leq \frac{8 Const.}{n^2 h_{n,K}^{(|\lambda|+2)(2(1-1/\delta))}} \sum_{\ell=\Lambda_n+1}^n \ell^\alpha [\alpha(\ell)]^{1-2/\delta} \\ &\leq \frac{8n Const.}{n^2 h_{n,K}^{(|\lambda|+2)(2(1-1/\delta))}} \sum_{\ell=\Lambda_n+1}^n \ell^\alpha [\alpha(\ell)]^{1-2/\delta} \\ &\leq \frac{8 Const.}{n h_{n,K}^{(|\lambda|+2)(2(1-1/\delta))}} \sum_{\ell=\Lambda_n+1}^\infty \ell^\alpha [\alpha(\ell)]^{1-2/\delta}. \end{aligned} \tag{22}$$

Since $\delta > 2$, it is easy to see that $(2 - 1/\delta) > 0$, by (H2) $nh_{n,K}^{|\lambda|+2} \rightarrow \infty$ and applying (H6) in (22) it follows that

$$\mathfrak{J}_{22} \rightarrow 0, \text{ as } n \rightarrow +\infty. \tag{23}$$

Finally $\text{Var}(\hat{\Theta}_{n,\ell}(t_1, t_2)) \rightarrow 0$, as $n \rightarrow \infty$ via (13), (19) and (23).

This concludes the proof of theorem 1 \square

3.2 Asymptotic Normality of Kernel Averages

Here, we give the asymptotic normality of (2) in the following theorem.

Theorem 2 *If assumptions (H1)- (H5) and (H6) are satisfied, we have*

$$\sqrt{nN(N-1)h_{n,K}^{|\lambda|+2}}(\hat{\Theta}_{n,\ell}(t_1, t_2) - \mathbb{E}\hat{\Theta}_{n,\ell}(t_1, t_2)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \vartheta_\ell(t_1, t_2)), \tag{24}$$

where $\xrightarrow{\mathcal{D}}$ denote the convergence in distribution.

Proof. We will establish the asymptotic normality of $\hat{\Theta}_{n,\ell}(t_1, t_2) - \mathbb{E}\hat{\Theta}_{n,\ell}(t_1, t_2)$ suitably normalized.

We have

$$\begin{aligned} \sqrt{nN(N-1)h_{n,K}^{|\lambda|+2}}(\hat{\Theta}_{n,\ell}(t_1, t_2) - \mathbb{E}\hat{\Theta}_{n,\ell}(t_1, t_2)) &= \frac{\sqrt{nN(N-1)h_{n,K}^{|\lambda|+2}}}{nN(N-1)h_{n,K}^{|\lambda|+2}} \sum_{i=1}^n \\ &\sum_{1 \leq r \neq s \leq N} \left[\psi_\ell(T_{ir}, T_{is}, U_{ir}, U_{is}) K\left(\frac{t_1 - T_{ir}}{h_{n,K}}, \frac{t_2 - T_{is}}{h_{n,K}}\right) \right. \\ &\left. - \mathbb{E}\psi_\ell(T_{ir}, T_{is}, U_{ir}, U_{is}) K\left(\frac{t_1 - T_{ir}}{h_{n,K}}, \frac{t_2 - T_{is}}{h_{n,K}}\right) \right] \\ &= \frac{1}{\sqrt{nN(N-1)h_{n,K}^2}} \sum_{i=1}^n \sum_{1 \leq r \neq s \leq N} [\Gamma_{i,r,s}^\ell - \mathbb{E}\Gamma_{i,r,s}^\ell] \\ &= \sum_{i=1}^n \sum_{1 \leq r \neq s \leq N} \frac{1}{\sqrt{nN(N-1)h_{n,K}^2}} [\Gamma_{i,r,s}^\ell - \mathbb{E}\Gamma_{i,r,s}^\ell], \end{aligned}$$

where $\Gamma_{i,r,s}^\ell$ is defined in (12).

Denote

$$\Xi_{ijkl} = \frac{1}{\sqrt{nN(N-1)h_{n,K}^2}} \Gamma_{i,r,s}^\ell.$$

Then

$$\begin{aligned} \sqrt{nN(N-1)h_{n,K}^{|\lambda|+2}}(\hat{\Theta}_{n,\ell}(t_1, t_2) - \mathbb{E}\hat{\Theta}_{n,\ell}(t_1, t_2)) &= \sum_{i=1}^n \sum_{1 \leq j \neq k \neq l \leq N} (\Xi_{ijkl} - \mathbb{E}\Xi_{ijkl}) \\ &= \sum_{i=1}^n \Xi_{n,i}. \end{aligned} \tag{25}$$

We now introduce Bernstein's big-block and small-block decomposition. We partition the set $\{1, 2, \dots, n\}$ into $2k_n + 1$ subsets with large blocks of size u_n and small blocks of size v_n and we set $k_n = \lfloor \frac{n}{u_n+v_n} \rfloor$, where $u_n = \lfloor nN(N-1)h_{n,K}^{|\lambda|+2} \rfloor$ and $v_n = o(nN(N-1)h_{n,K}^{|\lambda|+2})$. The symbol $\lfloor \cdot \rfloor$ is integer part. Using (H2), one has

$$\frac{v_n}{u_n} \rightarrow 0, \frac{u_n}{n} \rightarrow 0, \frac{nN(N-1)}{u_n h_{n,K}^2} \rightarrow 0, \frac{n}{u_n} \alpha(v_n) \rightarrow 0, \text{ as } n \rightarrow +\infty. \tag{26}$$

Let U_m, V_m and W_m be defined as follows:

$$U_m = \sum_{i=m(u_n+v_n)+1}^{m(u_n+v_n)+u_n} \Xi_{n,i}, \quad 0 \leq m \leq k_n - 1 \tag{27}$$

$$V_m = \sum_{i=m(u_n+v_n)+u_n+1}^{(m+1)(u_n+v_n)} \Xi_{n,i}, \quad 0 \leq m \leq k_n - 1 \tag{28}$$

$$W_m = \sum_{i=k_n(u_n+v_n)+1}^n \Xi_{n,i}. \tag{29}$$

Then, we obtain the decomposition

$$Z_n = \sum_{i=1}^n \Xi_{n,i} = \sum_{m=0}^{k_n-1} U_m + \sum_{m=0}^{k_n-1} V_m + W_m \tag{30}$$

$$= S_{n,1} + S_{n,2} + S_{n,3}. \tag{31}$$

Now, let start the proof of theorem 2. The main idea is to show that as $n \rightarrow \infty$,

$$\mathbb{E}[S_{n,2}^2] \rightarrow 0 \tag{32}$$

$$\mathbb{E}[S_{n,3}^2] \rightarrow 0 \tag{33}$$

$$\left| \mathbb{E}[\exp(iuS_{n,1})] - \prod_{m=0}^{k_n-1} \mathbb{E}[\exp(iuU_m)] \right| \rightarrow 0 \tag{34}$$

$$\mathbb{E}[U_m^2] \rightarrow \vartheta_\ell(t_1, t_2) \tag{35}$$

$$\sum_{m=0}^{k_n-1} \mathbb{E}[U_m^2 I\{|U_m| > \varepsilon \vartheta_\ell(t_1, t_2)\}] \rightarrow 0, \text{ for every } \varepsilon > 0. \tag{36}$$

Remark: Relations (32) and (33) imply that $S_{n,2}$ and $S_{n,3}$ are asymptotically negligible; (34) and (35) show that the summands $\{U_m\}$ in $S_{n,1}$ are asymptotically independent, verifying that the sum of their variances tends to $\vartheta_\ell(t_1, t_2)$. Expression (36) is the Lindeberg-Feller's condition for asymptotic normality of $S_{n,1}$ under dependence. Asymptotic normality of Z_n is a consequence of equations (34)-(35):

$$Z_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \vartheta_\ell(t_1, t_2)). \tag{37}$$

• **Proof of (32)**

$$\begin{aligned} \mathbb{E}[S_{n,2}^2] &= \text{Var} \left(\sum_{m=0}^{k_n-1} V_m \right) \\ &= \sum_{m=0}^{k_n-1} \text{Var}(V_m) + \sum_{\substack{m=0 \\ m \neq m'}}^{k_n-1} \sum_{m'=0}^{k_n-1} \text{Cov}(V_m, V_{m'}) \\ &= \mathfrak{J}_1 + \mathfrak{J}_2. \end{aligned} \tag{38}$$

↪ **Concerning \mathfrak{J}_1 ,**
we have

$$\begin{aligned} \text{Var}(V_m) &= \text{var} \left(\sum_{i=m(u_n+v_n)+u_n+1}^{(m+1)(u_n+v_n)} \Xi_{n,i} \right) \\ &= \sum_{i=m(u_n+v_n)+u_n+1}^{(m+1)(u_n+v_n)} \text{Var}(\Xi_{n,i}) + \sum_{\substack{i=m(u_n+v_n)+u_n+1 \\ i \neq i'}}^{(m+1)(u_n+v_n)} \sum_{i'=m(u_n+v_n)+u_n+1}^{(m+1)(u_n+v_n)} \text{Cov}(\Xi_{n,i}, \Xi_{n,i'}). \end{aligned} \tag{39}$$

Using the second-order stationarity and the fact that $\{\Xi_{irs}\}$ and $\{\Xi_{ir's'}\}$ are independent, (39) becomes

$$\begin{aligned} \text{Var}(V_m) &= \sum_{i=1}^{v_n} \text{Var}(\Xi_{n,i}) + \sum_{\substack{i=1 \\ i \neq i'}}^{v_n} \sum_{i'=1}^{v_n} \text{Cov}(\Xi_{n,i}, \Xi_{n,i'}) \\ &= v_n \text{Var}(\Xi_{n,1}) + \sum_{\substack{i=1 \\ i \neq i'}}^{v_n} \sum_{i'=1}^{v_n} \text{Cov}(\Xi_{n,i}, \Xi_{n,i'}). \end{aligned} \tag{40}$$

First, we have

$$\begin{aligned} \text{Var}(\Xi_{n,1}) &= \text{Var}\left(\sum_{1 \leq r \neq s \leq N} (\Xi_{1rs} - \mathbb{E}\Xi_{1rs})\right) \\ &= \text{Var}\left(\sum_{1 \leq r \neq s \leq N} (\Xi_{rs} - \mathbb{E}\Xi_{rs})\right) \\ &= \sum_{1 \leq r \neq s \leq N} \text{var}(\Xi_{rs} - \mathbb{E}\Xi_{rs}) \\ &= N(N-1) \text{Var}(\Xi_{11} - \mathbb{E}\Xi_{11}) \\ &= N(N-1) \left\{ \mathbb{E}(\Xi_{11} - \mathbb{E}\Xi_{11})^2 \right\} \\ &= N(N-1) \left\{ \frac{\vartheta_\ell(t_1, t_2)}{nN(N-1)} (1 + o(1)) \right\} \\ &= \frac{\vartheta_\ell(t_1, t_2)}{n} (1 + o(1)). \end{aligned} \tag{41}$$

Secondly,

$$\begin{aligned} |\text{Cov}(\Xi_{n,i}, \Xi_{n,i'})| &\leq \frac{N(N-1)}{n} \left| \frac{1}{h_{n,K}^{|\lambda|+2}} \text{Cov}(\Gamma_{i,1,1}^\ell, \Gamma_{i',1,1}^{\ell'}) \right| \\ &\leq \frac{N(N-1)}{n} \left\{ \vartheta_\ell^2(t_1, t_2) + o(1) \right\} \\ \sum_{\substack{i=1 \\ i \neq i'}}^{v_n} \sum_{i'=1}^{v_n} |\text{Cov}(\Xi_{n,i}, \Xi_{n,i'})| &\leq \frac{v_n}{n} \{v_n N(N-1) [\vartheta_\ell(t_1, t_2) + o(1)]\} \\ &= \frac{v_n}{n} \{o(n)\} \\ &= v_n o(1). \end{aligned} \tag{42}$$

Thirdly, replacing (41) and (42) in (40), it follows

$$\begin{aligned} \text{Var}(V_m) &= v_n \frac{\vartheta_\ell(t_1, t_2)}{n} (1 + o(1)) + v_n o(1) \\ &= v_n \left\{ \frac{\vartheta_\ell(t_1, t_2)}{n} (1 + o(1)) + o(1) \right\}. \end{aligned} \tag{43}$$

At the end, \mathfrak{J}_1 in (38) is such that

$$\begin{aligned}
 |\mathfrak{J}_1| &\leq \sum_{m=0}^{k_n-1} v_n \left\{ \frac{\vartheta_\ell(t_1, t_2)}{n} (1 + o(1)) + o(1) \right\} \\
 &= k_n v_n \left\{ \frac{\vartheta_\ell(t_1, t_2)}{n} (1 + o(1)) + o(1) \right\} \\
 &= k_n v_n v_n \left\{ \frac{\vartheta_\ell(t_1, t_2)}{n} (1 + o(1)) \right\} \\
 &= k_n \frac{v_n}{n} \vartheta_\ell(t_1, t_2) (1 + o(1)) \\
 &= \left[\frac{n}{u_n + v_n} \right] \frac{v_n}{n} \vartheta_\ell(t_1, t_2) (1 + o(1)) \\
 &\sim \frac{n}{u_n} \frac{v_n}{n} \vartheta_\ell(t_1, t_2) (1 + o(1)) \\
 &= \frac{v_n}{u_n} \vartheta_\ell(t_1, t_2) (1 + o(1)) \\
 &\rightarrow 0, \text{ by (26).}
 \end{aligned} \tag{44}$$

↪ **Concerning \mathfrak{J}_2 ,**

$$\begin{aligned}
 \mathfrak{J}_2 &= \sum_{\substack{m=0 \\ m \neq m'}}^{k_n-1} \sum_{m'=0}^{k_n-1} Cov(V_m; V_{m'}) \\
 &= \sum_{\substack{m=0 \\ m \neq m'}}^{k_n-1} \sum_{m'=0}^{k_n-1} \sum_{\substack{i=m(u_n+v_n)+u_n+1 \\ i \neq i'}}^{(m+1)(u_n+v_n)} \sum_{\substack{i=m'(u_n+v_n)+u_n+1}}^{(m'+1)(u_n+v_n)} Cov(\Xi_{n,i}; \Xi_{n,i'}) \\
 &= \sum_{\substack{m=0 \\ m \neq m'}}^{k_n-1} \sum_{m'=0}^{k_n-1} \sum_{\substack{i=1 \\ i \neq i'}}^{v_n} \sum_{i=1}^{v_n} Cov(\Xi_{n,m(u_n+v_n)+u_n+i}; \Xi_{n,m'(u_n+v_n)+u_n+i'}) \\
 &= \sum_{\substack{m=0 \\ m \neq m'}}^{k_n-1} \sum_{m'=0}^{k_n-1} \sum_{\substack{i=1 \\ i \neq i'}}^{v_n} \sum_{i=1}^{v_n} Cov(\Xi_{n,\eta_m+i}; \Xi_{n,\eta_{m'}+i'})
 \end{aligned}$$

since $|\eta_m - \eta_{m'} + i - i'| \geq u_n$ then we reduce the sums and we write

$$\begin{aligned}
 |\mathfrak{J}_2| &\leq \sum_{\substack{i=1 \\ |i-i'| \geq u_n}}^n \sum_{i=1}^n |Cov(\Xi_{n,i}, \Xi_{n,i'})| \\
 &\leq \frac{N(N-1)}{nh_{n,K}^{|\lambda|+2}} 8C[h_{n,K}^{|\lambda|+2}]^{2/\delta} \sum_{\ell=1}^{\infty} \ell^\alpha [\alpha(\ell)]^{1-2/\delta} \\
 &= \frac{8CN(N-1)}{nh_{n,k}^{(|\lambda|+2)(1-2/\delta)}} \sum_{\ell=1}^{\infty} \ell^\alpha [\alpha(\ell)]^{1-2/\delta} \\
 &= o(1) \text{ by (26)}.
 \end{aligned}$$

Therefore $\mathfrak{J}_2 \rightarrow 0$, as $n \rightarrow +\infty$. (45)

Combining (44) and (45), it follows that $\mathbb{E}[S_{n,2}^2] \rightarrow 0$ and

$$S_{n,2} \rightarrow 0 \text{ in probability.}$$

This achieves the proof of (32).

- **Proof of (33)** Using the same arguments as in the proof of (32), one has

$$\begin{aligned}
 \mathbb{E}[S_{n,3}^2] &= \text{Var}\left(\sum_{m=0}^{k_n-1} U_m\right) \\
 &\leq \frac{u_n + v_n}{n} \{\vartheta_\ell(t_1, t_2) + o(1)\}. \\
 &\sim \frac{u_n}{n} \{\vartheta_\ell(t_1, t_2) + o(1)\}. \\
 &\rightarrow 0.
 \end{aligned}
 \tag{46}$$

• **Proof of (34)**

We make use of Volkonskii & Rozanov’s (1959) Lemma (see the appendix in Masry (2005)).

Note that U_m is $\{\mathcal{F}_{i_1, \dots, i_{u_n}}\}$ -measurable with $i_1 = m(u_n + v_n) + 1$ and

$i_{u_n} = m(u_n + v_n) + u_n$. Note that using that $V_m = \exp(iuU_m)$ as in the Lemma of Volkonskii & Rozanov, we have

$$\begin{aligned}
 \left| \mathbb{E}\left\{\exp(iuS_{n,1})\right\} - \prod_{m=0}^{k_n-1} \mathbb{E}\left\{\exp(iuU_m)\right\} \right| &\leq 16k_n\alpha(v_n + 1) \\
 &\sim 16\frac{n}{u_n}\alpha(v_n + 1) \\
 &\rightarrow 0 \text{ by (26)}
 \end{aligned}
 \tag{47}$$

as n goes to infinity.

• **Proof of (35)**

Replacing u_n by v_n we have

$$\begin{aligned}
 \text{Var}(U_m) &= \text{var}\left(\sum_{i=m(u_n+v_n)+1}^{m(u_n+v_n)+u_n} \Xi_{n,i}\right) \\
 &= \sum_{i=m(u_n+v_n)+1}^{m(u_n+v_n)+u_n} \text{Var}(\Xi_{n,i}) + \sum_{\substack{i=m(u_n+v_n)+1 \\ i \neq i'}}^{m(u_n+v_n)+u_n} \sum_{i'=m(u_n+v_n)+1}^{m(u_n+v_n)+u_n} \text{Cov}(\Xi_{n,i}, \Xi_{n,i'}) \\
 &= u_n\vartheta_\ell(t_1, t_2)(1 + o(1)).
 \end{aligned}
 \tag{48}$$

So

$$\begin{aligned}
 \sum_{m=0}^{k_n-1} \mathbb{E}[U_m^2] &= k_n\frac{u_n}{n}\vartheta_\ell(t_1, t_2)(1 + o(1)) \\
 &\sim \frac{u_n}{n}\vartheta_\ell(t_1, t_2)(1 + o(1)) \\
 &\rightarrow \vartheta_\ell(t_1, t_2).
 \end{aligned}$$

• **Proof of (36)**

We first establish the asymptotic normality (37) for the particular case where ψ_ℓ is bounded. The case of ψ_ℓ possibly unbounded is then establish by using a truncation argument. Let τ_n be a fixed truncation point. We can replace $\psi_\ell(T_{ir}, T_{is}, U_{ir}, U_{is})$ with the truncated process

$\psi_\ell(T_{ir}, T_{is}, U_{ir}, U_{is})\mathbb{I}\left\{\left|\psi_\ell(T_{ir}, T_{is}, U_{ir}, U_{is})\right| \leq \tau_n\right\}$ in (U_{ir}, U_{is}) . Denote

$$\begin{aligned}
 \Xi_{irs}^{\tau_n} &= \frac{1}{\sqrt{nN(N-1)h_{n,K}^{|\lambda|+2}}} \psi_\ell(T_{ir}, T_{is}, U_{ir}, U_{is})\mathbb{I}\left\{\left|\psi_\ell(T_{ir}, T_{is}, U_{ir}, U_{is})\right| \leq \tau_n\right\} \\
 &\times K\left(\frac{t_1 - T_{ir}}{h_{n,K}}, \frac{t_2 - T_{is}}{h_{n,K}}\right), \\
 \Xi_{n,i}^{\tau_n} &= \sum_{1 \leq r \neq s \leq N} (\Xi_{irs}^{\tau_n} - \mathbb{E}\Xi_{irs}^{\tau_n}).
 \end{aligned}$$

Define $Z_n^{\tau_n} = \sum_{i=1}^n \Xi_{n,i}^{\tau_n}$ and

$$Z_n^{*\tau_n} = \sum_{i=1}^n (\Xi_{n,i} - \Xi_{n,i}^{\tau_n}) = \sum_{i=1}^n \Xi_{n,i} \mathbb{I}\left\{|\psi_\ell(T_{ir}, T_{is}, U_{ir}, U_{is})| > \tau_n\right\}. \tag{49}$$

Since $|\psi_\ell(T_{ir}, T_{is}, U_{ir}, U_{is})| \leq \tau_n$ and from (H.5), it follows that

$$|\Xi_{n,i}^{\tau_n}| \leq 2C \frac{N(N-1)\tau_n}{\sqrt{nN(N-1)h_{n,K}^{|\lambda|+2}}}$$

and

$$\max_{0 \leq m \leq k_n-1} |U_m^{\tau_n}| \leq 2C \frac{N(N-1)u_n\tau_n}{\sqrt{nN(N-1)h_{n,K}^{|\lambda|+2}}}.$$

Therefore if we take τ_n and u_n such that

$$u_n\tau_n = \frac{n^{1/2}h_{n,K}^{|\lambda|+3}}{(N(N-1))^{1/2}},$$

then,

$$\max_{0 \leq m \leq k_n-1} |U_m^{\tau_n}| \leq 2C \frac{N(N-1)u_n\tau_n}{\sqrt{nN(N-1)h_{n,K}^{|\lambda|+2}}} \rightarrow 0.$$

Hence, for n sufficiently large, the set $\{|U_m^{\tau_n}| > \varepsilon\vartheta_\ell(t_1, t_2)\}$ becomes empty for all $\varepsilon > 0$. Thus, $\mathbb{P}(|U_m^{\tau_n}| > \varepsilon\vartheta_\ell(t_1, t_2)) \rightarrow 0$ for large n , for all $\varepsilon > 0$. So

$$\sum_{m=0}^{k_n-1} \mathbb{E}\left[U_m^2 \mathbb{I}\{|U_m| > \varepsilon\vartheta_\ell(t_1, t_2)\}\right] = 0, \text{ for all } \varepsilon > 0.$$

Hence

$$Z_n^{\tau_n} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \vartheta_{\ell, \tau_n}(t_1, t_2)). \tag{50}$$

In order to complete the proof, namely to establish (37) for the general case, it suffices to show that as first $n \rightarrow +\infty$ and $\tau_n \rightarrow +\infty$ (see Masry, 2005 or Fan & Masry, 1992) we have

$$\text{Var}(Z_n^{*\tau_n}) \rightarrow 0. \tag{51}$$

Indeed,

$$\begin{aligned} & \left| \mathbb{E} \exp\{iuZ_n\} - \exp\{-u^2\vartheta_\ell(t_1, t_2)/2\} \right| \\ &= \left| \mathbb{E} \exp\{iu(Z_n^{\tau_n} + Z_n^{*\tau_n})\} - \exp\{-u^2\vartheta_{\ell, \tau_n}(t_1, t_2)/2\} \right| \\ & \quad + \left| \exp\{-u^2\vartheta_{\ell, \tau_n}(t_1, t_2)/2\} - \exp\{-u^2\vartheta_\ell(t_1, t_2)/2\} \right| \\ &\leq \left| \mathbb{E} \exp\{iuZ_n^{\tau_n}\} - \exp\{-u^2\vartheta_{\ell, \tau_n}(t_1, t_2)/2\} \right| + \left| \mathbb{E} \exp\{iuZ_n^{*\tau_n}\} - 1 \right| \\ & \quad + \left| \exp\{-u^2\vartheta_{\ell, \tau_n}(t_1, t_2)/2\} - \exp\{-u^2\vartheta_\ell(t_1, t_2)/2\} \right|. \end{aligned}$$

Letting $n \rightarrow +\infty$, the first term goes to zero by (50), for every $\tau_n > 0$; the second term converges to zero by (51), because first $n \rightarrow +\infty$ and then $\tau_n \rightarrow +\infty$; the third term goes to zero as $\tau_n \rightarrow +\infty$ by the dominated convergence theorem.

Therefore, it remains to prove (51). Note that by (50), $Z_n^{*\tau_n}$ has the same structure as $Z_n^{\tau_n}$ except that $\Xi_{n,i}^{\tau_n}$ is replaced by $(\Xi_{n,i} - \Xi_{n,i}^{\tau_n})$. Applying the Lemma 2.3 in Fan & Masry (1992) or using the same arguments as in Masry (2005) we conclude that, for all fixed $\tau_n > 0$, one has (51).

Then, it suffices to choose τ_n sufficiently large, such that the non-truncated part becomes asymptotically negligible. \square

The following theorem provides the first main result of the paper.

Theorem 3 Under assumptions of theorems 1 and 2, we have

$$\sqrt{nN(N-1)h_{n,K}^{|\ell|+2}} [\hat{\Theta}_{n,\ell}(t_1, t_2) - \theta_\ell(t_1, t_2)] \xrightarrow{\mathcal{D}} \mathcal{N}(\mathbf{B}(t_1, t_2), \vartheta_\ell(t_1, t_2)). \tag{52}$$

Proof. According to the Proposition 2, p 243 (Müller & Song, 1993), Theorem 3 is a consequence of theorem 1 and theorem 2. □

Now, we extend the general result (Theorem 2 of Yao, 2007) under alpha-mixing conditions. Then, additional assumptions on the bandwidth are given in assumptions (H.2).

Let $H : \mathbb{R}^L \rightarrow \mathbb{R}$ be a function with continuous second order derivatives. If we denote the gradient vector $(\frac{\partial H}{\partial x_1}(v), \dots, \frac{\partial H}{\partial x_L}(v))^T$ by $DH(v)$, and let

$$\begin{aligned} \tilde{\mathbf{B}}(t_1, t_2) &= \frac{(-1)^{|\ell|} a}{k!} \sum_{\ell=1}^L \left\{ \int_{\mathbb{R}^2} v^{k_1} w^{k_2} K(v, w) dv dw \frac{d^{|\ell|}}{dt_1^{k_1} dt_2^{k_2}} \int_{\mathbb{R}^2} \psi_\ell(t_1, t_2, u_1, u_2) g_1(t_1, t_2, u_1, u_2) du_1 du_2 \right\} \\ &\times \left\{ \frac{\partial H}{\partial \theta_L}(\theta_1, \dots, \theta_L)^T \right\} \end{aligned}$$

and

$$\begin{aligned} \vartheta_{\ell,\ell'}(t_1, t_2) &= \|K\|_2^2 \int_{\mathbb{R}^4} \psi_\ell(t_1, t_2, u_1, u_2) \psi_{\ell'}(t_1, t_2, u_1, u_2) g_2(t_1, t_2, t_1, t_2, u_1, u_2, u'_1, u'_2), du_1 du_2 du'_1 du'_2, \\ V &= (\vartheta_{\ell,\ell'}(t_1, t_2))_{1 \leq \ell, \ell' \leq L} \text{ the variance-covariance matrix,} \end{aligned}$$

then the second main result of the paper follows.

Theorem 4 Assume that assumptions of theorem 3 hold. Then

$$\sqrt{nN(N-1)h_{n,K}^{|\ell|+2}} [H(\hat{\Theta}_{1n}, \dots, \hat{\Theta}_{Ln}) - H(\theta_1, \dots, \theta_L)] \xrightarrow{\mathcal{D}} \mathcal{N}(\tilde{\mathbf{B}}(t_1, t_2), [DH(\theta_1, \dots, \theta_L)]^T V [DH(\theta_1, \dots, \theta_L)]). \tag{53}$$

Proof. A L -dimensional Taylor expansion of H around $(m_1, \dots, m_L)^T$ of order 1 combined with (8) gives

$$\sqrt{nN(N-1)h_{n,K}^{|\ell|+2}} [H(\mathbb{E}\hat{\Theta}_{1n}, \dots, \mathbb{E}\hat{\Theta}_{Ln}) - H(\theta_1, \dots, \theta_L)] \xrightarrow{\mathbb{P}} \tilde{\mathbf{B}}(t_1, t_2). \tag{54}$$

Applying the Cramér-Wold device to (24) it comes

$$\sqrt{nN(N-1)h_{n,K}^{|\ell|+2}} (H(\hat{\Theta}_{1n}, \dots, \hat{\Theta}_{Ln}) - H(\mathbb{E}\hat{\Theta}_{1n}, \dots, \mathbb{E}\hat{\Theta}_{Ln})) \rightarrow \mathcal{N}(0, [DH(\theta_1, \dots, \theta_L)]^T V [DH(\theta_1, \dots, \theta_L)]). \tag{55}$$

Finally, (54) and (55) lead to (53). □

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