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Examining the Efficiency of American Put Option Pricing by Monte Carlo Methods with Variance Reduction

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Abstract

We apply the Monte Carlo simulation algorithm developed by Broadie and Glasserman (1997) and the control variate technique first introduced to asset pricing via simulation by Boyle (1977) to examine the efficiency of American put option pricing via this combined method. The importance and effectiveness of variance reduction is clearly demonstrated in our simulation results. We also found that the control variates technique does not work as well for deep-in-the-money American put options. This is because deep-in-the-money American options are more likely to be exercised early, thus the value of the American options are less in line (or less correlated) with those of their European counterparts.

Keywords: option pricing, american put option, monte carlo simulation, variance reduction

1. Introduction

It is well known that there is no closed-form solution for pricing an American option on a stock. However, numerical methods have been developed to calculate the value of the option and the optimal exercise boundary; for example, binomial methods by Cox, Ross, and Rubinstein (1979); the method of Richardson extrapolation by Geske and Johnson (1984); Quasi-analytical solutions by Barone-Adesi and Whaley (1987); the quadratic method of Barone-Adesi and Whaley (1987); the accelerated binomial method of Breen (1991); multinomial methods of Boyle (1988); lower and upper bounds methods by Johnson (1983) and Broadie and Detemple (1996); finite difference methods by Brennan and Schwartz (1997, 1998); randomization technique by Carr (1998); and the multipiece exponential function by Ju (1998).

In practice, many security pricing models involve three or more state variables. For example, options on foreign currencies and differential swaps involve modeling the uncertainty of exchange rates and the term structures of both domestic and foreign interest rates. For models with multiple state variables there are few, if any, analytical solutions for pricing these securities. However, numerical methods, in particular, simulation methods, are very viable approaches to analyzing these models. This is because simulation methods not only allow greater flexibility in the modeling of the state variables but also easy tracking of the complex path dependencies for early exercise decision in American-style options.

Broadie and Glasserman (1996) developed a simulation method for estimating the prices of American-style securities. Their proposed algorithm is especially attractive in cases where there are multiple state variables and opportunities for early exercise. Because their method uses random sampling, rather than the enumeration implicit in lattice and finite-difference methods, it can be easily applied to models with multiple state variables and possible path dependencies.

Since Broadie and Glasserman (1996), there have been studies proposing various simulation methods for pricing American-style contingent claims. All these numerical methods have a dual objective of accuracy and speed of computation, the latter of which is becoming a lesser issue these days due to the technological advance in computing power. Readers interested in the related literature review can refer to the survey article by Musshoff and Hirschauer (2010).

The rest of the paper is organized as follows. Section 1 provides brief introduction to the related literature. Section 2 describes the method developed by Broadie and Glasserman. Section 3 discusses our implementation of Broadie and Glasserman's method in conjunction with the control variate technique. Section 4 presents the numerical results and conclusion.

2. Description of the Method

For the pricing of a European call option, the typical approach is to simulate the following expectation:

$$C = E[e^{-rT} max(S_T-K,0)] \text{ under the risk-neutral measure} \quad (1)$$

While for pricing an American call option, we are to find

$$C = Max_t E[e^{-rt} max(S_t-K,0)] \text{ over all stopping times } t <= T \quad (2)$$

The main question is how to calculate this American option value based on the path of the stock price. This problem would be trivial if the optimal stopping policy were known. In that case, the option value would simply be $e^{-rt}max(S_t-K,0)$. Unfortunately, the optimal stopping policy is not known, so it must also be determined along with the simulated paths. Therefore, the problem with the Monte Carlo estimate for American-style option is that the estimate is based on the "perfect hindsight" and is prone to overestimating the true value of the option.

Broadie and Glasserman (1997) circumvent the aforementioned problem by generating two estimates of the option price based on simulations of future projections and increasingly fine-tuned approximations to the early exercise decisions. They create two estimates which are both asymptotically unbiased and converge to the true price. One estimate is biased high while the other is biased low. By combining these two estimates, they obtain a valid confidence interval for the true price of the option:

Their high estimator Θ is defined recursively by

$$\Theta_T^{i_1 \dots i_T} = f_t(S_T) \text{ and}$$

$$\Theta_t^{i_1 \dots i_t} = max\,[h_t(S_t^{i_1 \dots i_t}), (1/b) \textstyle\sum_{j=1}^{b} exp(-R_{t+1}^{i_1 \dots i_t, j})\, \Theta_{t+1}^{i_1 \dots i_t, j}] \quad \text{for } t = 0, \dots, T\text{-}1 \quad (3)$$

where i's denote different assets

$h_t(s)$ is the payoff from exercise at time t in state s.

b = the number of branches at each node.

Their low estimator θ is defined recursively by

$$\theta_T^{i_1 \dots i_T} = f_t(S_T) \text{ and}$$

$$\eta_t^{i_1 \dots i_t, j} = h_t(S_t^{i_1 \dots i_t}) \quad \text{if } h_t(S_t^{i_1 \dots i_t}) >= [1/(b-1)] \textstyle\sum_{i=1, i\neq j}^{b} exp(-R_{t+1}^{i_1 \dots i_t, i})\theta_{t+1}^{i_1 \dots i_t, i}$$

$$= exp(-R_{t+1}^{j})\theta_{t+1}^{i_1 \dots i_t, j} \quad \text{if } h_t(S_t^{i_1 \dots i_t}) < [1/(b-1)] \textstyle\sum_{i=1, i\neq j}^{b} exp(-R_{t+1}^{i_1 \dots i_t, i})\theta_{t+1}^{i_1 \dots i_t, i} \quad (4)$$

for $j = 1, \dots, b$

Then let

$$\theta_t^{i_1 \dots i_t} = (1/b) \quad \textstyle\sum_{j=1}^{b} \eta_t^{i_1 \dots i_t, j} \quad \text{for } t = 0, \dots, T\text{-}1 \quad (5)$$

3. Implementation

The implementation of our study is undertaken using Matlab program. Our Matlab codes are available upon request. Follow the *depth-first* procedure suggestion by Broadie and Glasserman (1997), the storage requirements for this algorithm are minimal, especially with the much increased computer capacity these days.

3.1 Monte Carlo Simulation

Assume the stock price follows a geometric Brownian motion process. Specifically, assume that the risk neutralized price of the stock, S_t, follows the stochastic differential equation

$$dS_t = S_t\,[(r-\delta)dt + \sigma dZ_t] \quad (6)$$

where Z_t is a standard Brownian motion process. Under the risk neutral measure, and $\ln(S_i/S_{i-1})$ is normally distributed with mean $(r-\delta-\sigma^2/2)*(t_i-t_{i-1})$ and variance $\sigma^2(t_i-t_{i-1})$.

Given S_{i-1}, a discrete time approximation to S_i can be simulated using

$$S_i = S_{i-1}\, exp[\,(r-\delta-\sigma^2/2)*(t_i-t_{i-1}) + \sigma\sqrt{t_i - t_{i-1}}\; z] \quad (7)$$

where Z is a standard normal random variable.

3.2 Variance Reduction Technique

Any Monte Carlo simulation involves variation in the estimates due to sampling error. The goal of variance reduction is to improve the computational efficiency of Monte Carlo simulations. There are a number of variance reduction techniques, such as antithetic variates, control variates, moment matching methods, stratified and Latin hypercube sampling, important sampling, conditional Monte Carlo, and low-discrepancy sequences (quasi-random sequences). In this paper, we apply the control variate technique to our Monte Carlo simulations. In general, the control variate technique can lead to very substantial error reductions, but its effectiveness hinges on finding a good control for each problem. More specifically, a good control variate is the one that is highly correlated with the original estimator. In our case of pricing an American put option, a naturally good candidate for the control variate is the European put option under the same terms. Note that the higher the correlation between the control variate and the original estimator, the more effective is this technique. In fact, the correlation must be high enough to offset the variance of the additional estimator introduced by the control variate technique. This theoretical prediction can be seen in the numerical results shown in Table 1 shown in the next section.

4. Results and Conclusion

Table 1 shows the numerical results from our analysis. The benchmark "true" value for comparison is the estimate from binomial model with large time steps (N), as it is known that the binomial estimate converges to true value in the limit. The relative errors are used to measure the overall accuracy of the across time steps and/or options. The importance and effectiveness of variance reduction is clearly demonstrated in our results. Moreover, alternative variance reduction techniques, besides the control variates method examined in this paper, could be incorporated to the Monte Carlo method to further improve the efficiency of the pricing method. Notice that the control variates technique does not work as well for deep-in-the-money options. This is because deep-in-the-money American options are more likely to be exercised early, thus the value of the American options are less in line (or less correlated) with those of their European counterparts.

Table 1. Relative pricing error with and without control variate

Model Specification and Parameters

American Put Option on a Single Asset

K=100 r=0.08 q=0.12 sigma=0.2 T=3

Without Control Variate

Stock Price	low p	sd	high p	sd	lower CB	upper CB	point P	TRUE	**Relative Error**
80	24.5941	1.4272	25.1106	1.4518	24.1985	25.5130	24.8524	25.6577	3.1%
90	18.6027	1.2826	19.0078	1.3382	18.2472	19.3787	18.8053	20.0832	6.4%
100	13.2266	1.2992	13.7632	1.3792	12.8665	14.1455	13.4949	15.4981	12.9%
110	9.6895	1.0119	9.8983	1.0814	9.4090	10.1980	9.7939	11.8032	17.0%
120	6.8045	1.0044	6.9433	0.8524	6.5261	7.1796	6.8739	8.8856	22.6%

With Control Variate

Stock Price	low p	sd	high p	sd	lower CB	upper CB	point P	TRUE	**Relative Error**
80	24.5941	1.4272	26.9087	0.1714	24.1985	26.9562	25.7514	25.6577	0.4%
90	18.6027	1.2826	21.0846	0.1311	18.2472	21.1209	19.8437	20.0832	1.2%
100	13.2266	1.2992	16.238	0.1175	12.8665	16.2706	14.7323	15.4981	4.9%
110	9.6895	1.0119	12.3508	0.0877	9.4090	12.3751	11.0202	11.8032	6.6%
120	6.8045	1.0044	9.2695	0.0629	6.5261	9.2869	8.0370	8.8856	9.6%

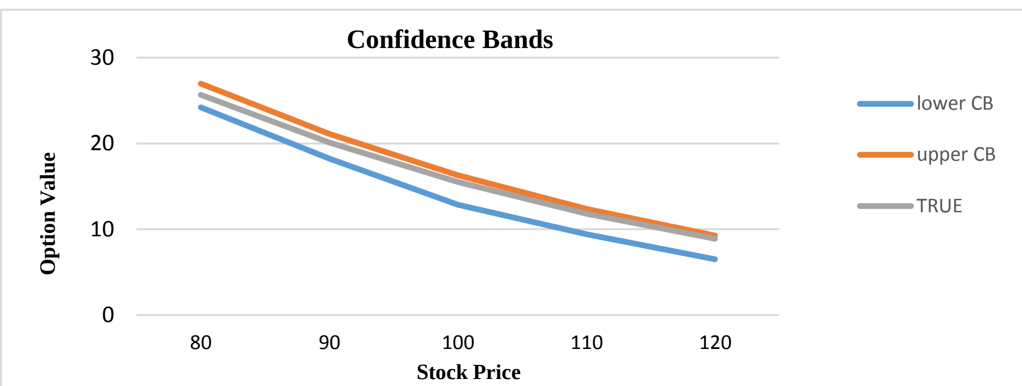

Figure 1. Confidence bands for the option value from Table 1 with control variate

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