

Quasilinear Utility and Two Market Monopoly

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Abstract

The use of quasilinear utility functions in economic analyses is widespread. This paper presents an overdue clarification on the implications of quasilinear utility for two market monopoly. The paper begins by deriving the demands facing a two market monopoly from a representative consumer with quasilinear utility. Expressions are derived for the profit margins expressed solely in terms of the own and cross-price elasticities of demand. The paper also analyzes the implications of quasilinear utility for other issues in two market monopoly: pricing below marginal cost in a market, third-degree price discrimination when the monopoly products are substitutes and pricing in the inelastic region of demands.

Keywords: quasilinear utility, two product monopoly, third-degree price discrimination

1. Introduction

Thanks in large part to Varian (1985, 1992) the assumption of quasilinear utility is ubiquitous in economic policy analyses because it allows one to simply measure social welfare as profit plus consumer surplus. This paper presents an overdue clarification of the implications of quasilinear utility for the behavior of two market monopoly and in the process corrects some errors in Varian (1989, 1992). After deriving the demands facing a two market monopoly from a representative consumer, the paper derives simple expressions for the monopoly profit margins expressed solely in terms of own and cross-price elasticities of demand. The paper also discusses three other implications of quasilinear utility for two market monopoly: (1) the possibility of pricing below marginal cost (2) the theory of third-degree price discrimination when the monopoly products are substitutes and (3) pricing in the inelastic region of demands.

2. Method

2.1 Quasilinear Utility, Monopoly Demands and Elasticities of Demand

Let the utility function of the representative consumer be given by $u(x_0, x_1, x_2) = x_0 + \varphi(x_1, x_2)$, where x_0 is a numeraire good and x_1 and x_2 are the two goods produced by the monopolist.¹ Assume that φ is a twice continuously differentiable, strictly concave function where $\varphi_1 \equiv \frac{\partial \varphi}{\partial x_1} > 0$, $\varphi_2 \equiv \frac{\partial \varphi}{\partial x_2} > 0$, $\varphi_{11} \equiv \frac{\partial^2 \varphi}{\partial x_1^2} <$

$$0, \varphi_{22} \equiv \frac{\partial^2 \varphi}{\partial x_2^2} < 0, \varphi_{12} \equiv \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \text{ and } \varphi_{11}\varphi_{22} > \varphi_{12}^2.$$

Let the price of the numeraire good be 1. Utility is maximized subject to the constraint: $x_0 + p_1x_1 + p_2x_2 \leq y$, where p_i is the price of good i and y is the consumer's exogenous income plus monopoly profit. Assuming an interior solution to the consumer's utility maximization problem, the consumer's inverse demand functions are: $p_1 = \varphi_1(x_1, x_2)$ and $p_2 = \varphi_2(x_1, x_2)$. Note that

$$\frac{\partial p_1}{\partial x_1} = \varphi_{11}, \frac{\partial p_2}{\partial x_2} = \varphi_{22} \text{ and, by Young's theorem, } \frac{\partial p_2}{\partial x_1} = \frac{\partial p_1}{\partial x_2} = \varphi_{12}.$$

Given the inverse demand functions,

¹The analysis can easily be extended to the case where a monopolist sells n goods. For the case where the monopolist sells n goods let the utility function be $u(x_0, x_1, \dots, x_n) = x_0 + \varphi(x_1, \dots, x_n)$.

$\varphi_1(x_1, x_2)$ and $\varphi_2(x_1, x_2)$, one might ask under what conditions do the demand functions $x_1(p_1, p_2)$ and $x_2(p_1, p_2)$ exist and how are the partial derivatives of these demand functions related to the partial derivatives of the inverse demand functions?

Write the inverse demand functions as

$$(1) F^i(p_i, x_1, x_2) = p_i - \varphi_i(x_1, x_2) = 0, \quad i = 1, 2.$$

Using the implicit function theorem we know that the demand functions $x_i = x_i(p_1, p_2)$, $i = 1, 2$, exist in a

neighborhood around a point that satisfies equation (1) as long as the Jacobian $|J| = \begin{vmatrix} \frac{\partial F^1}{\partial x_1} & \frac{\partial F^1}{\partial x_2} \\ \frac{\partial F^2}{\partial x_1} & \frac{\partial F^2}{\partial x_2} \end{vmatrix} = \varphi_{11}\varphi_{22} -$

$\varphi_{12}^2 \neq 0$. Because $\varphi(x_1, x_2)$ is by assumption strictly concave it follows that the Jacobian is always positive and that the demand functions $x_i = x_i(p_1, p_2)$ do exist at every point satisfying equation (1).

The implicit function rule of differentiation applied to equation (1) yields:

$$(2) \frac{\partial x_1}{\partial p_1} = \frac{\varphi_{22}}{\varphi_{11}\varphi_{22} - \varphi_{12}^2} < 0$$

$$(3) \frac{\partial x_2}{\partial p_2} = \frac{\varphi_{11}}{\varphi_{11}\varphi_{22} - \varphi_{12}^2} < 0$$

and

$$(4) \frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1} = \frac{-\varphi_{12}}{\varphi_{11}\varphi_{22} - \varphi_{12}^2}.$$

The sign of φ_{12} determines whether products 1 and 2 are substitutes, complements or independent. Products 1 and 2 are substitutes, that is $\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1} > 0$, if and only if $\varphi_{12} < 0$.² Products 1 and 2 are complements, that is

$\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1} < 0$, if and only if $\varphi_{12} > 0$. The intuition for these latter results is that when products 1 and 2 are

substitutes (complements) an increase in production of good 1 lowers (raises) the marginal utility of good 2 and lowers (raises) the maximum amount the representative consumer is willing to pay for good 2. Finally, if

$\varphi_{12} = 0$, $\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1} = 0$, the two demand functions are independent. Only in this special case will it be true that

$$\frac{\partial x_i}{\partial p_i} = \frac{1}{\varphi_{ii}} = \frac{1}{\frac{\partial p_i}{\partial x_i}}.$$
³

From the strict concavity of $\varphi(x_1, x_2)$ and equations (2) - (4) it also follows that $\frac{\partial x_1}{\partial p_1} \frac{\partial x_2}{\partial p_2} > \left(\frac{\partial x_1}{\partial p_2}\right) \left(\frac{\partial x_2}{\partial p_1}\right)$. Letting

$e_{ij} \equiv \frac{\partial x_j}{\partial p_i} \frac{p_i}{x_j}$ be the elasticity of product j with respect to the price of product i , the previous inequality expressed

²Varian (1989, p. 619, 1992, p. 249) mistakenly states that $\frac{\partial p_2}{\partial x_1} > 0$ when products 1 and 2 are substitutes. Note

also from equation (4) that $\frac{\partial x_i}{\partial p_j} \neq 1/\frac{\partial p_j}{\partial x_i}$ for $i \neq j$.

³Varian (1989, p. 618, 1992, p. 249) incorrectly assumes that $\frac{\partial x_i}{\partial p_i} = 1/\frac{\partial p_i}{\partial x_i}$ when the two products are substitutes.

in elasticity form is

$$(5) \quad e_{11}e_{22} - e_{12}e_{21} > 0$$

2.2 Profit maximization for a two product firm

Let the profit function for a two-market monopoly be $\pi(p_1, p_2) = (p_1 - c_1)x_1 + (p_2 - c_2)x_2$, where c_1 is constant marginal cost in market 1 and c_2 is constant marginal cost in market 2. The first order conditions for profit maximization are:

$$(6) \quad \pi_1 = \frac{\partial \pi}{\partial p_1} = x_1 + (p_1 - c_1) \frac{\partial x_1}{\partial p_1} + (p_2 - c_2) \frac{\partial x_2}{\partial p_1} = 0$$

$$(7) \quad \pi_2 = \frac{\partial \pi}{\partial p_2} = x_2 + (p_1 - c_1) \frac{\partial x_1}{\partial p_2} + (p_2 - c_2) \frac{\partial x_2}{\partial p_2} = 0$$

Let the profit margin in market i be $m_i = \frac{p_i - c_i}{p_i}$. Using the cross-price symmetry result $\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1}$,

equations (6) and (7) with some algebraic manipulation may be rewritten as:

$$(8) \quad m_1 e_{11} + m_2 e_{21} = -1$$

$$(9) \quad m_1 e_{12} + m_2 e_{22} = -1$$

Solving for the profit maximizing values of the profit margins yields:

$$(10) \quad m_1^* = \frac{-e_{22} + e_{21}}{e_{11}e_{22} - e_{12}e_{21}}$$

$$(11) \quad m_2^* = \frac{-e_{11} + e_{12}}{e_{11}e_{22} - e_{12}e_{21}}$$

It is assumed that the second order sufficient conditions for a maximum are satisfied at the profit margins given by equations (10) and (11). For $e_{12} = e_{21} = 0$, equations (10) and (11) simplify to $m_1^* = -\frac{1}{e_{11}}$

and $m_2^* = -\frac{1}{e_{22}}$.

The profit margin expressions given by equations (10) and (11) above are similar but somewhat simpler than expressions developed by Forbes (1988, 62). Forbes' expressions for the profit maximizing profit margins

are: $m_1^* = \frac{-e_{22} + e_{12} \frac{R_2}{R_1}}{e_{11}e_{22} - e_{12}e_{21}}$ and $m_2^* = \frac{-e_{11} + e_{21} \frac{R_1}{R_2}}{e_{11}e_{22} - e_{12}e_{21}}$, where $R_i = p_i x_i$. Assuming cross-price symmetry it is easy

to show that $e_{21} = e_{12} \frac{R_2}{R_1}$ and $e_{12} = e_{21} \frac{R_1}{R_2}$. Tirole's (1988,70) expressions for the profit maximizing profit

margins in the two good case are: $m_1^* = -\frac{1}{e_{11}} - m_2^* \frac{R_2 e_{12}}{R_1 e_{11}}$ and $m_2^* = -\frac{1}{e_{22}} - m_1^* \frac{R_1 e_{21}}{R_2 e_{22}}$, which with some algebra

can be shown to be equivalent to Forbes' expressions.

3. Results

3.1 Pricing below Marginal Cost under Two Market Monopoly

It has been well known for some time that if the two products are complements then it's possible that price may be below marginal cost in one market.⁴ See Allen (1938, 359-62) and the references cited therein. Without loss

⁴If price were below marginal cost in both markets the monopolist's profit would be negative.

of generality let $m_1^* > m_2^*$. From equation (10) the condition for price to be below marginal cost in market 2 in the present model is $e_{11} - e_{12} > 0$. Because e_{11} is assumed to be negative, for the profit margin to be negative in market 2 we must have $e_{12} < 0$ and $|e_{12}| > |e_{11}|$. For example, a baseball team that sells beer (product 2) and tickets (product 1) would price beer below its marginal cost if lowering the price of tickets by 1% led to a greater % increase in beer sold than the % increase in tickets sold.

3.2 Third-degree Price Discrimination When Demands Are Interdependent

Suppose a monopolist sells a single product with the equal marginal costs in two markets at different prices but the two markets are not perfectly sealed off from each other so that there is some substitution between the markets. See for example Varian's discussion (1989, 1992) of this subject. An example of this type of price discrimination is a movie theatre that shows afternoon movies at a lower price than the same movie shown at night. Because movies shown at night and during the day are substitutes, $e_{12}, e_{21} > 0$. Profit maximization for the price discriminating monopolist requires satisfying equations (10) and (11). The condition for $m_1^* > m_2^*$ is $-e_{22} + e_{21} > -e_{11} + e_{12}$. Note if $e_{12} = e_{21}$ then the previous condition reduces to $|e_{22}| > |e_{11}|$, which is the well known result that a price discriminating monopoly has a lower profit margin in the more elastic market.

3.3 Pricing in the Inelastic Regions of Demands

It is well known that an unregulated single-product monopolist will never operate in the price inelastic region of demand because that implies marginal revenue is negative. We now investigate whether this result extends to the case of a two-market monopoly. Because $m_1^* \leq 1$ and $m_2^* \leq 1$, we can derive the following inequalities from equations (10) and (11):

$$(12) \quad e_{21}(e_{12} + 1) \leq e_{22}(e_{11} + 1) \quad (\text{holds with equality if } m_1^* = 1)$$

$$(13) \quad e_{12}(e_{21} + 1) \leq e_{11}(e_{22} + 1) \quad (\text{holds with equality if } m_2^* = 1)$$

If $|e_{11}| < 1$ then from inequality (12) we must have $e_{12} < 0, e_{21} < 0$ and $|e_{12}| < 1$. In words, if demand is inelastic in market 1, then the two products must be complements and the absolute value of e_{12} must be less than 1. Similarly, if demand is price inelastic in market 2 inequality (13) implies that the two products must be complements and that the absolute value of e_{21} must be less than 1. To verify that both markets can have price inelastic demands, let $e_{11} = -.8, e_{22} = -.9, e_{12} = e_{21} = .5$. From equations (10) and (11) we find $m_1^* = .851$ and $m_2^* = .638$.

If marginal cost is zero in a market, the two products are complements and the absolute values of the cross-price elasticities are less than 1, then it must be the case that demand is inelastic in that market. Consider the case where $c_1 = 0$ and hence $m_1^* = 1$. If $e_{12}, e_{21} < 0$, and $|e_{12}| < 1$ then it follows from equation (12) that $|e_{11}| < 1$. Similarly if $c_2 = 0$ and hence $m_2^* = 1$, then $e_{12}, e_{21} < 0$, and $|e_{21}| < 1$ implies $|e_{22}| < 1$. Finally, consider the case where $c_1 = c_2 = 0$ which implies $m_1^* = m_2^* = 1$, then $e_{12}, e_{21} < 0$, $|e_{12}| < 1$ and $|e_{21}| < 1$ implies $|e_{11}| < 1$ and $|e_{22}| < 1$.

Major league sports teams provide a good application of ticket pricing in the inelastic region of demand. All major league sports teams can be regarded as multiproduct monopolies selling complementary goods: tickets and concessions. The marginal cost of allowing another person to see a game is nearly zero as long as there are empty seats. It follows from the discussion above that ticket prices should be set in the inelastic region of demand as long as the absolute value of the cross-price elasticity of demand for concessions with respect to ticket prices is less than 1. Fort (2004) and Krautmann and Berri (2007) both argue forcefully that ticket pricing in the inelastic region of demand is commonplace in sporting events.

4. Discussion

This paper has clarified several implications of the assumption of quasilinear utility for two market monopoly. After deriving the demands facing a two market monopoly from a representative consumer, this paper derives

expressions for profit maximizing profit margins expressed solely in terms of own and cross-price elasticities of demand. The paper also derives and discusses the condition for a two market monopoly to price one of its products below marginal cost and clarifies the theory of third-degree price discrimination when the two monopoly products are substitutes. The paper ends with a discussion of pricing in the inelastic region of demand under two market monopoly. It is shown that a two market monopolist may (and in some conditions must) operate in the price inelastic region of demand in one or even both markets.

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