Finite Time Synchronization between Two Different Chaotic Systems with Uncertain Parameters

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Abstract

This paper deals with the finite-time chaos synchronization between two different chaotic systems with uncertain parameters by using active control. Based on the finite-time stability theory, a control law is proposed to realize finite-time chaos synchronization for the uncertain systems Lorenz and Lü. The controller is simple and robust against the uncertainty in system parameters. Numerical results are presented to show the effectiveness of the proposed control technique.

Keywords: Finite-time synchronization, Active control, Finite-time stability theory, Uncertain parameters

1. Introduction

Synchronization of chaotic systems has been a hot topic since the pioneering work of Pecocra and Carroll (Pecora LM, Carroll TL. 1990). It can be applied in various fields such as chemical reactors, power converters, biological systems, information processing, secure communication(Colet P, Roy R. 1994)(Sugawara T, Tachikawa M, Tsukamoto T, Shimisu T. 1994)(Lu JA, Wu XQ, Lü JH. 2002), etc. A wide variety of approaches have been employed in the synchronization of chaotic systems which most of them are designed to synchronize two identical chaotic systems(Chen S, Lü J. 2002)(Agiza HN, Yassen MT. 2001)(Ho M-C, Hung Y-C. 2002)(Huang L, Feng R, Wang M. 2004)(Liao T-L. 1998)(Liao T-L, Lin S-H. 1999)(Yassen MT. 2003). The synchronization of two different chaotic systems, however, is not straightforward. Difference in the structure of the systems makes the synchronization a challenging problem in this case(M.-Ch. Ho, Y.-Ch. Hung, 2002)(H. Zhang, W. Huang, Z. Wang, T. Chai, 2006)(M.-T. Yassen, 2005). Some of the proposed approaches in the identical case are not applicable here or required different procedure to be designed. This problem becomes more difficult when two chaotic systems have some uncertain parameters.

In the literatures (Shahram Etemadi, Aria Alasty. 2007) used active sliding mode control to synchronize two different chaotic systems with uncertain parameters. However, the convergence of the synchronization procedure in (Shahram Etemadi, Aria Alasty. 2007) is exponential with infinite settling time. To attain fast convergence speed, many effective methods have been introduced and finite-time control is one of them. Finite-time synchronization means the optimality in convergence time. Moreover, the finite-time control techniques have demonstrated better robustness and disturbance rejection properties(Bhat S, Bernstein D. 1997)(Hua Wang, 2008).

In this paper, the goal is to force the two different chaotic systems with uncertain parameters to be synchronized in finite time. The method of active control is applied to control the chaos synchronization system. Based on finite-time stability theory, a controller is designed to achieve finite-time synchronization. Simulation results show that the proposed controller synchronizes the Lorenz and Lü chaotic systems in finite time.

2. Preliminary definitions and lemmas

Finite-time synchronization means that the state of the slave system can track the state of the master system after a finite-time. The precise definition of finite-time synchronization is given below.

Definition 1. Consider the following two chaotic systems:

$$\begin{aligned} \dot{x}_m &= f(x_m) \\ \dot{x}_s &= h(x_m, x_s) \end{aligned} \tag{1}$$

Where x_m, x_s are two $n-$dimensional state vectors. The subscripts 'm' and 's' stand for the master and slave systems, respectively. $f : R^n \to R^n$ and $h : R^n \to R^n$ are vector-valued functions. If there exists a constant $T > 0$, such that $\lim_{t\to T} \|x_m - x_s\| = 0$, and $\|x_m - x_s\| = 0$, if $t \geq T$, then synchronization of the system (1) is achieved in a finite-time.

Lemma 1. Suppose there exists a continuous function $V: D \to R$ such that the following hold :

1. $V(t)$ is positive definite.

2. There exists real numbers $c>0$ and $\alpha \in (0,1)$ and an open neighborhood $N \subseteq D$ of the origin such that

$$\dot{V}(x) \le -cV^{\eta}(x), x \in N \setminus \{0\} \tag{2}$$

Then the origin is a finite-time stable equilibrium.

Lemma 2. when a, b and $c<1$ are all positive numbers, the following inequality holds:

$$(a+b)^c \le a^c + b^c. \tag{3}$$

3. Systems description

Lorenz system is considered a paradigm, since it captures many of the feathers of chaotic dynamics. The Lorenz system is described by the following nonlinear equations:

$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = cx - xz - y \\ \dot{z} = xy - bz \end{cases} \tag{4}$$

which has a chaotic attractor when $a=10, b=8/3, c=28.$

Chen system is a typical chaos anti-control model, which has a more complicated topological structure than Lorenz attractor. The nonlinear differential equations that describe the Chen system are

$$\begin{cases} \dot{x} = \alpha(y-x) \\ \dot{y} = (\gamma-\alpha)x - xz + \gamma y \\ \dot{z} = xy - \beta z \end{cases} \tag{5}$$

which has a chaotic attractor when $\alpha=35, \beta=3, \gamma=28.$

Lü system is a typical transition system, which connects the Lorenz and Chen attractors and represents the transition from one to the other . The Lü system is described by

$$\begin{cases} \dot{x} = \rho(y-x) \\ \dot{y} = -xz + vy \\ \dot{z} = xy - \mu z \end{cases} \tag{6}$$

which has a chaotic attractor when $\rho=36, \mu=3.v=20.$

In the next sections, we will study the chaos synchronization between the chaotic dynamical systems Lorenz and Lü with uncertain parameters.

4. Finite-time synchronization between Lorenz and Lü systems with parameters uncertainty

This subsection deals with finite-time synchronization of uncertain Lorenz and Lü systems. It is valuable because practical systems are often disturbed by different factors. It is assumed that both the master system and the slave system hold uncertainties. Consider the following chaotic system with uncertain parameters:

$$\begin{cases} \dot{x}_1 = a(y_1-x_1) \\ \dot{y}_1 = (c+\Delta_1)x_1 - x_1z_1 - y_1 \\ \dot{z}_1 = x_1y_1 - (b+\Delta_2)z_1 \end{cases} \tag{7}$$

and

$$\begin{cases} \dot{x}_2 = \rho(y_2 - x_2) + u_1(t) \\ \dot{y}_2 = -x_2 z_2 + (v + \Delta_3) y_2 + u_2(t) \\ \dot{z}_2 = x_2 y_2 - (\mu + \Delta_4) z_2 + u_3(t) \end{cases} \tag{8}$$

where $\Delta_i, i = 1,2,3,4$ denote the bounded uncertain parameters, i.e. $|\Delta_i| \le \delta_i$, We have introduced three control functions $u_1(t), u_2(t)$, $u_2(t)$ in (8). Our goal is to determine the control functions $u_1(t), u_2(t)$, $u_2(t)$. In order to estimate the control functions, we subtract (7) from (8). We define the error system as the differences between the Lorenz system (7) and the controlled Lü system (8). Let us define the state errors as

$$e_1 = x_2 - x_1, e_2 = y_2 - y_1, e_3 = z_2 - z_1 \tag{9}$$

Use of the definition in (9), the error dynamics can be written as

$$\begin{cases} \dot{e}_1 = \rho(e_2 - e_1) + (\rho - a)(y_1 - x_1) + u_1(t) \\ \dot{e}_2 = ve_2 + vy_1 - x_1 e_3 - z_1 e_1 - e_1 e_3 - cx_1 + y_1 + \Delta_3 y_2 - \Delta_1 x_1 + u_2(t) \\ \dot{e}_3 = -\mu e_3 + e_1 y_1 + e_2 x_1 + e_1 e_2 - (\mu - b) z_1 - (\Delta_4 - \Delta_2) z_2 - \Delta_2 e_3 + u_3(t) \end{cases} \tag{10}$$

We define the active control functions $u_1(t), u_2(t)$, and $u_2(t)$ as follows

$$\begin{cases} u_1 = W_1(t) - (\rho - a)(y_1 - x_1) \\ u_2 = W_2(t) - vy_1 + z_1 e_1 + cx_1 - y_1 \\ u_3 = W_3(t) - e_1 y_1 + (\mu - b) z_1 \end{cases} \tag{11}$$

Hence the error system (10) becomes

$$\begin{cases} \dot{e}_1 = \rho(e_2 - e_1) + W_1(t) \\ \dot{e}_2 = -x_1 e_3 - e_1 e_3 + ve_2 + \Delta_3 y_2 - \Delta_1 x_1 + W_2(t) \\ \dot{e}_3 = -\mu e_3 + e_2 x_1 + e_1 e_2 - (\Delta_4 - \Delta_2) z_2 - \Delta_2 e_3 + W_3(t) \end{cases} \tag{12}$$

Our aim is to design a controller that can achieve the finite-time synchronization of uncertain Lorenz (7) and Lü systems (8). This problem can be converted to design a controller to attain finite-time stable of the error system (12). The design procedure consists of two steps as follows.

Step1: Let $W_1 = -\rho e_2 - |e_1|^m sign(e_1), m \in (0,1)$,substituting this control input W_1 into the first equation of (12) yields

$$\dot{e}_1 = -\rho e_1 - |e_1|^m sign(e_1) \tag{13}$$

Choose a candidate Lyaupunov function

$$V_1 = \frac{1}{2} e_1^2$$

The derivative of V_1 along the trajectory of (13) is

$$\dot{V}_1 = e_1 \dot{e}_1 = -\rho e_1^2 - |e_1|^{1+m} \le -|e_1|^{1+m} = -2^{\frac{1+m}{2}} V_1^{\frac{1+m}{2}}$$

From Lemma 1, the system (13) is finite-time stable. That means that is a $T_1 > 0$ such that $e_1 = 0$ provided that $t \ge T_1$.

Step 2: When $t > T_1, e_1 = 0$. The last two equation of system (12) become

$$\begin{cases} \dot{e}_2 = -x_1e_3 + ve_2 + \Delta_3 y_2 - \Delta_1 x_1 + W_2(t) \\ \dot{e}_3 = -\mu e_3 + e_2x_1 - (\Delta_4 - \Delta_2)z_2 - \Delta_2 e_3 + W_3(t) \end{cases} \tag{14}$$

Select $W_2(t) = -ve_2 - |e_2|^m sign(e_2) - \gamma_1 |x_1| sign(e_2) - \gamma_2 |y_2| sign(e_2)$, $\gamma_1 \geq \delta_1$, $\gamma_2 \geq \delta_3$,and

$W_3(t) = -|e_3|^m sign(e_3) - \gamma_3 |z_2| sign(e_3)$,where $\gamma_3 \geq \delta_2 + \delta_4$.

Substitute W_2 and W_3 into the system (14) and consider the following candidate Lyapunov function:

$$V_2 = \frac{1}{2}({e_2}^2 + {e_3}^2)$$

The derivative of V_2 along the trajectory of (14) is

$$\begin{aligned} \dot{V}_2 &= e_2(-x_1e_3 + \Delta_3 y_2 - \Delta_1 x_1 - |e_2|^m sign(e_2) - \gamma_1 |x_1| sign(e_2) - \gamma_2 |y_2| sign(e_2)) \\ &\quad + e_3(-\mu e_3 + e_2x_1 - (\Delta_4 - \Delta_2)z_2 - \Delta_2 e_3 - |e_3|^m sign(e_3) - \gamma_3 |z_2| sign(e_3)) \\ &= \Delta_3 y_2 e_2 - \Delta_1 x_1 e_2 - |e_2|^{1+m} - \gamma_1 |x_1e_2| - \gamma_2 |y_2e_2| - (\mu + \Delta_2){e_3}^2 - (\Delta_4 - \Delta_2)z_2e_3 - \gamma_3 |z_2e_3| - |e_3|^{1+m} \\ &\leq -\gamma_1 |x_1e_2| - \gamma_2 |y_2e_2| - \gamma_3 |z_2e_3| - |e_2|^{1+m} - |e_3|^{1+m} + \Delta_3 y_2e_2 - \Delta_1 x_1e_2 - (\Delta_4 - \Delta_2)z_2e_3 \\ &= -(\gamma_1 + \Delta_1 sign(x_1e_2))|x_1e_2| - (\gamma_2 - \Delta_3 sign(y_2e_2))|y_2e_2| - (\gamma_3 + (\Delta_4 - \Delta_2) sign(z_2e_3))|z_2e_3| \\ &\quad - |e_2|^{1+m} - |e_3|^{1+m} \\ &\leq -|e_2|^{1+m} - |e_3|^{1+m} = -2^{\frac{1+m}{2}} V_2^{\frac{1+m}{2}} \end{aligned}$$

From Lemma 1, it follows that (14) is finite-time stabilized. Thus, the uncertain slave system (8) can synchronize the uncertain master system (7) in finite time.

5. Simulations results

In this simulation, the 4th order Runge–Kutta algorithm was used to solve the sets of differential equations related to the master and slave systems. We select the parameters of Lorenz system as $a = 10, b = 8/3, c = 28$ and the parameters of Lü system as $\rho = 36, v = 20, \mu = 3$. The initial values of Lorenz system and Lü system are $[x_1(0)\ \ y_1(0)\ \ z_1(0)] = [5\ \ 6\ \ 9], [x_2(0)\ \ y_2(0)\ \ z_2(0)] = [15\ \ 17\ \ 10]$.

The initial errors are $e_1(0) = 10, e_2(0) = 11, e_3(0) = 1$.

The uncertain parameters of Lorenz system and Lü system are adopted as $\Delta_1 = 0.5\sin t$, $\Delta_2 = 0.5\cos t$, $\Delta_3 = 0.1$, $\Delta_4 = \cos t, \gamma_1 = 0.8, \gamma_2 = 0.5, \gamma_3 = 2$ .The controller parameter m is selected as 1/3 to satisfy given condition. The simulation results are given in Fig. 1 for the case that the Lorenz system drives the Lü system.Fig.2 shows the errors responses of the uncertain Lorenz and Lü systems. As we expect, the slave system synchronizes with the master system and

the system have strong robustness to the uncertainties.

6. Conclusion

In this paper, an effective control method for synchronizing different chaotic systems with uncertain parameters has been proposed using active control. Based on the finite-time stability theory, the proposed controller enables stabilization of synchronization error dynamics to zeros in finite time. Finite time synchronization between the pairs of the Lorenz and Lü systems is achieved. Numerical simulations are also given to validate the synchronization approach.

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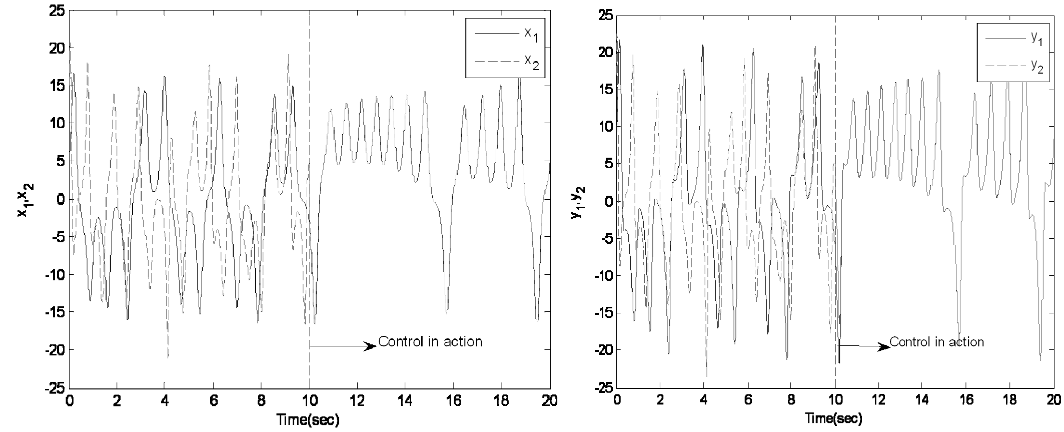

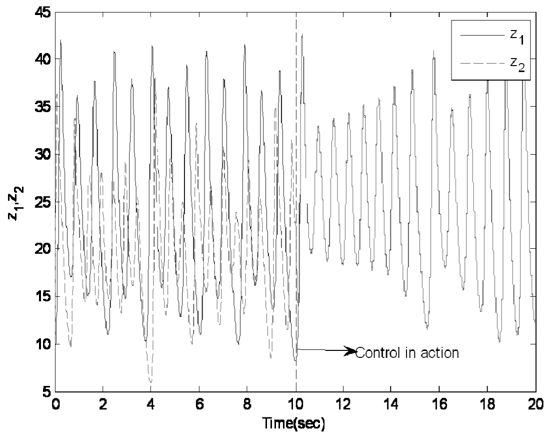

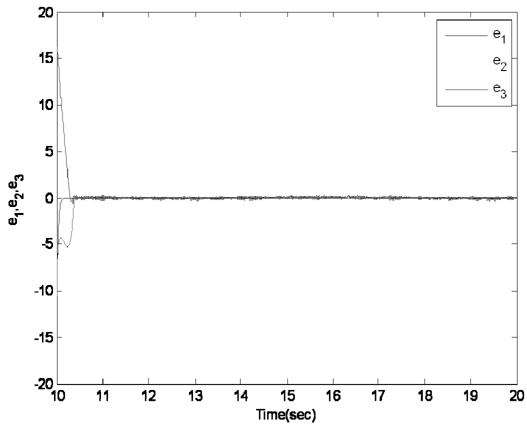

Figure 1. Results of synchronizing Lorenz and Lü systems.

Figure 2. The synchronization errors at $t \in [10,20]$ of controlled Lü