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On The Improvement of Combinatorial Mathematics Teaching From Generating Function

Cheng Ji[1]

[1] College of Mathematic & Information, China West Normal University, Nan Chong, China

Correspondence: Cheng Ji, College of Mathematic & Information, China West Normal University, Nan Chong 637009, China. E-mail: chengjijx@163.com

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Abstract

Generating function is an effective method to solve combinatorial counting problem, but it most likely to be neglected in combinatorial mathematics teaching. In this paper, we provides a demonstration for combinatorial mathematics teaching improvement by using the generating function solving combination number and the sum of preceding n terms among sequence of numbers.

Keywords: combinatorial mathematics, generating function, teaching

1. Current Situation of Generating Function Teaching

Combinatorial mathematics is one of the fundamental courses for graduate students and advanced undergraduates in the department of mathematics, which mainly solves the problem of arranging multiple objects according to certain rules, and discuses its existence, enumeration and classification , construction and optimization. Because of the practical backgrounds and significance, combinatorial mathematics become one of the four pillars in competition mathematics together with algebra, geometry and theory of numbers.

In the course of combinatorial mathematics, the combinatorial counting problem is usually studied by means of permutation and combination, principle of inclusion and exclusion, recursive relation and generating function. Due to the impact of inadequate class hours but more contents, most of the teachers will focus on permutation and combination, principle of inclusion and exclusion, recursive relation, weakening on generating function. It induce the lack of knowledge, and incomplete of knowledge structure of combinatorial enumeration.

Generating function emerged in the 19th century, which is proposed by French mathematician Laplace P.S. as a simple and useful mathematical method. Generating function is an important general method for combinatorial enumeration.

In fact, generating function cleverly combined discrete mathematics and continuous mathematics together. If we can pay more attention to the application of generating function in teaching, it can play a better role in the integration of knowledge points in combinatorial mathematics.

2. Introduction of Generating Function

Generating function generally refers to power series generating function and exponential generating function, power series generating function is the most commonly used one.

Let $a_0, a_1, a_2, \cdots, a_n, \cdots$ as a sequence of numbers, its generating function is defined as

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{k=0}^{\infty} a_k x^k \text{ , and record as } f(x) = G\{a_n\}$$

The core of generating function is finite or infinite sequence of numbers, which integrates by power series. On the one hand, the generating function can be regarded as an algebraic object, and its formal processing makes it possible to realize combinatorial counting by algebraic method. On the other hand, the generating function is a power series expansion. Here, $x, x^2, \cdots, x^n, \cdots$ is formal symbols, omits its convergence field, that means the convergence of the series is not included in the definition. Thus, the generating function can be treat as formal power series with algebraic and analytical properties, can add, derivate, integrate one by one. Combinatorial counting problem could be solved by getting the sum function of formal power series, using one-one correspondence of generating functions and sequences of numbers, studying the properties of sequence.

As a formal power series, generating function satisfy the linear operation of addition and multiplication, and also derivation and integral calculus of rules.

Generally, there are three types of questions. Find the generation function from known sequence of numbers, especially sum function of power series. Find the sequence of numbers which corresponding to the function. Solving practical problems by generating function, by using the properties of the generating function, we can find out the sequence of numbers and the sum of preceding n terms among sequence of numbers.

3. Calculate Combinatorial Number Through Generating Function

There are three types of questions for combinatorial number in combinatorial mathematics:

1) Find the k combinatorial number of $\{a_1, a_2, \cdots, a_n\}$. This is a combinatorial problem with simple set, select k （$1 \le k \le n$） elements which are not allowed to repeat from n elements form a group. The total number of combination, here, we use combinatorial number for short, denoted as C_n^k, $C(n,k)$ or $\binom{n}{k}$.

2) Find the k combinatorial number of $\{\infty \cdot a_1, \infty \cdot a_2, , \infty \cdot a_n\}$. This is a combination problem with infinite sets. Select k （$1 \le k \le n$） elements which are allowed to repeat from n elements form a group. The the total number of combination, here, we name number of repetitions unlimited combinatorial number for short, denoted as $H_n^k = C_{n+k-1}^k$. In general, the combinatorial number and the number of non-negative integer solution of the indefinite equation $x_1 + x_2 + \cdots + x_n = k$ are corresponding.

3) Find the k combinatorial number of $\{t_1 \cdot a_1, t_2 \cdot a_2, \cdots, \mathrm{t}_n \cdot a_n\}$, $t_i (1 \le i \le n)$ is finite positive integer. This is a combination problem with multiple sets. Select k elements which are allowed to repeat from n elements form a group. The total number of combination, here, we name number of repetitions limited combinatorial number for short. In general, the combinatorial number and the number of non-negative integer solution of the indefinite equation $x_1 + x_2 + \cdots + x_n = k$ are corresponding, and $0 \le x_i \le t_i (1 \le i \le n)$. In combinatorial mathematics teaching, principle of inclusion and exclusion is widely adopt, although the solution is cleat, but it's expensive in calculating.

If the generating function is introduced, the above three kinds of problems can be solved uniformly, which greatly simplifies the learning difficulty.

The combinatorial number in question 1) can be translate into coefficients of x^k in

$$(1+x)(1+x)\cdots(1+x) = (1+x)^n.$$

The combinatorial number in question 2) can be translate into coefficients of x^k in

$$(1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots) = (1+x+x^2+\cdots)^n = \frac{1}{(1-x)^n},$$

then, solved the coefficients by Newton binomial theorem.

The combinatorial number in question 3) can be translate into Coefficients of x^k in

$$(1+x+x^2+\cdots+x^{t_1})(1+x+x^2+\cdots+x^{t_2})\cdots(1+x+x^2+\cdots+x^{t_n}),$$

these also need Newton binomial theorem to solve the coefficients but more flexible.

The above three problems are distributed in multiple plates of combinatorial mathematics, spanned large. Thus, beginners will be difficult to clarity their thoughts. If generating function can be introduced, it will be easy to unite the three problems above. So we can get:

Theorem 1: Select k elements from $\{a_1, a_2, \cdots, a_n\}$, let the combinatorial number named b_k, the set of the number of times which the element a_i appeared defined as $s_i (1 \le i \le n)$. Then the generating function of combinatorial number sequence of numbers should be $G\{b_n\} = \prod_{i=1}^{n}\left(\sum_{m\in s_i} x^m\right)$.

When we use generating function for solving number of repetitions limited combinatorial number, the superiority is quite obvious, much better than principle of inclusion and exclusion.

Example1: If there are 3 weights of 1g, 4 weights of 2g, 2 weights of 4g, how many different weight can we get? And how many plans there are?

Solution: Weights of 1g corresponds generating function $1+x+x^2+x^3$, weights of 2g corresponds generating function $1+x^2+x^4+x^6+x^8$, weights of 4g corresponds generating function $1+x^4+x^8$. So the generating function should be

$$\begin{aligned} f(x) &= (1+x+x^2+x^3)(1+x^2+x^4+x^6+x^8)(1+x^4+x^8) \\ &= 1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7+5x^8+5x^9+5x^{10} \\ &\quad +5x^{11}+4x^{12}+4x^{13}+3x^{14}+3x^{15}+2x^{16}+2x^{17}+x^{18}+x^{19} \end{aligned}$$

Therefore, it can be weight of 0-19g, and the coefficients of x^k denote the number of plans for weight of k grams. The study found that the coefficients $(1,1,2,2,3,3,4,4,5,5,5,5,4,4,3,3,2,2,1,1)$ show high symmetry. It could be perfectly explained by the combination method. And we can get much more interesting finding in deeply research.

4. Find the Sum of Sequence of Numbers Through Generating Function

From build recursive relation to solve recursive relation, these process combined combinatorial mathematics and practical problem together. Generally, there's no universal method to solve every recursive relation (or we can say it's still not find). But for some special recursive relation, such as linear homogeneous recursive relation with constant coefficients or part of linear non-homogeneous recursive relation with constant coefficients, still could be solved. In spite of this, the calculation will be so complex.

Recursive relation is very useful when we meet combinatorial enumeration problem, and can solve the general formula of sequence of numbers in sequence of numbers field. When recursive relation is adopt, we need get the sum of preceding n terms among sequence of numbers, here, non-homogeneous recursive relation cannot be avoided.

Take $1^3+2^3+3^3+\cdots+n^3$ for example, it can be interpreted n terms among sequence of numbers $\{n^3\}$. This kind of problem is impossible for middle school.

In the combinatorial mathematics teaching, many solutions can be given for the above example. For permutation and combination part, iterating through combinatorial number property $n^3 = 6C_n^3 + 6C_n^2 + C_n^1, C_{n+1}^k = C_n^k + C_n^{k-1}$. For recursive relation part, linear non-homogeneous recursive relation with constant coefficients $f(n) = f(n-1) + n^3$ could be constructed. There are two methods to solve the problem. One is translate into homogeneous recursive relation and solved, another one is construct special solution and solved. Of cause, we can also solve this problem through mathematical induction which based on induction and supposition.

The following property of generating function could take more efficient solution.

Property: Let $A(x) = \sum_{k=0}^{\infty} a_k x^k$, $B(x) = \sum_{k=0}^{\infty} b_k x^k$, if $b_k = \sum_{i=0}^{k} a_i$, then $B(x) = \dfrac{A(x)}{1-x}$.

Example 2. Find $1^3 + 2^3 + 3^3 + \cdots + n^3$.

Solution: The generating function of $\{n^2\}$ is $f(x) = \dfrac{x(1+x)}{(1-x)^3} = \sum_{k=0}^{\infty} k^2 x^k$ (2.1)

For (2.1) take the derivative of both sides with respect to x, then multiply both sides by x, then get the generating function of $\{n^3\}$: $g(x) = \dfrac{x(x^2+4x+1)}{(1-x)^4} = \sum_{k=0}^{\infty} k^3 x^k$ (2.2)

Let $b_n = 1^3 + 2^3 + 3^3 + \cdots + n^3$, since the above property, then,

$$h(x) = \frac{g(x)}{1-x} = \frac{x(x^2+4x+1)}{(1-x)^5} = (x^3+4x^2+x)\cdot\frac{1}{(1-x)^5} = (x^3+4x^2+x)\cdot\sum_{k=0}^{\infty} C_{k+4}^{k} x^k \quad (2.3)$$

On the other side, $h(x) = \sum_{k=0}^{\infty} b_k x^k$, compare the x^n coefficient in both sides of the equation.

Get $b_n = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$.

Find the sum of sequence of numbers $\{n^3\}$ through generating function might not the simplest way, but the property of generating function above take a general mothed for solving the sum of general sequence of numbers.

5. The Suggestion for Combinatorial Mathematics Teaching

In the teaching process of combinatorial mathematics, we should strengthen the generating function part. On the one hand it can unify together all the combinatorial number solution. On the other hand, it can also provide a general method to solve the sum of general sequence of numbers.

For combinatorial mathematics teaching, more time could be saved from permutation and combination part. We can focus on generating function and other enumeration method. On the other hand, only concerning the knowledge points such as permutation and combination, principle of inclusion and exclusion, recursive relation, generating function horizontally is not enough, we should connect all the knowledge vertically, make the knowledge system three-dimensional.

References

Richard, A. B. (2001). Combinatorial Mathematics, Bei Jing, *China Machine Press*.

Xu, Y., & Sun, S. (2010). Combinatorial Mathematics Introduction, He Fei, *Press of University of Science and*

Technology of China.

Chen, J.(2011), Generating Function and Application, *Science Technology and Engineering, 11*(19), 4547-4549.

Feng, W., & Wu, J. (2012), The Geneal Formula of a Linear Homogeneous Recursive Sequence with Constant Coefficients and Generating[J], *Mathematics in Practice and Theory,42*(6), 257-260.

Sharma, K. (2012). On partial bilateral and improper partial bilateral genarating functions involving some basic classical poplynomials. *Int. Electron. J. Pure Appl. Math* (2), 79-82.

Nakagawa, & Toshio (2005). The universal generating function in reliability analysis and optimization. *Springer*.

Banderier, C., Bousquet-M, Lou, M., Denise, A., Flajolet, P., & Gardy, D., et al. (2002). Generating functions for generating trees. *Discrete Mathematics, 246*(246), 29-55. https://doi.org/10.1016/S0012-365X(01)00250-3