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Evolving Null Horizons Near an Isolated Black Hole

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Abstract

Totally geodesic null hypersurfaces have been widely used as models of time-independent event and isolated black hole horizons. However, in reality black hole being surrounded by a local mass distribution there is significant difference in the structure of the surrounding region of isolated black holes. In this paper, we use metric conformal symmetry which provides a class of a family of totally umbilical null hypersurfaces (Theorem 4), supported by a physical model and an example of time-dependent evolving null horizons (see Definition 6) conformally related to an isolated black hole. We establish an interrelation between the spacelike dynamical horizons (see Definition 7), isolated and evolving null horizons. Finally we propose further study on null geometry and physics of the surface closer to an isolated horizon.

Keywords: totally umbilical spacetime, null horizons, mean curvature, black hole

1. Introduction

Considerable work has been done on time-independent isolated black hole physics of asymptotically flat spacetimes which is still an active area of research. Such isolated black holes deal with the concept of an event and isolated horizons briefly explained as follows:

Event horizons. A boundary of a spacetime is called an event horizon beyond which events cannot affect the observer. An event horizon is intrinsically a global concept since its definition requires the knowledge of the whole spacetime to determine whether null geodesics can reach null infinity. For basic information on event horizons we refer (Hawking, 1972) and three papers of (Há́jičeke, 1973-74).

However, in practice an event horizon is generally not very useful since to actually locate a black hole one needs to know the full spacetime metric up to the infinite future. Moreover, even if one locates the event horizon, using it to calculate the physical parameters is extremely difficult. Therefore, attempts have been made to find a quasi-local concept of a horizon which requires only minimum number of conditions to detect a black hole and study its properties. For this purpose, (Asktekar et al., 1999) introduced following three notions of "Isolated Horizons".

Let (H, q) be a null hypersurface of a 4-dimensional spacetime (M, g) where q is the degenerate metric induced by the metric g of M. We assume that the null normal,say ℓ, is null geodesic future directed and is defined in some subset of M around H. This will permit to well-define the spacetime covarient derivative $\nabla\ell$ where ∇ denotes the Levi-Civita connection on M. The expansion $\theta_{(\ell)}$ is defined by $\theta_{(\ell)} = q^{ab}\nabla_a \ell_b$ and the vorticity-free Raychaudhuri equation is

$$\frac{d(\theta_{(\ell)})}{ds} = -R_{ab}\ell^a\ell^b - \sigma_{ab}\sigma^{ab} - \frac{\theta^2}{2},$$

where σ_{ab}, s and R_{ab} are shear tensor, a pseudo-arc parameter and Ricci tensor, respectively.

Definition 1. A null hypersurface (H, q) of a 4-dimensional spacetime (M, g) is called a *non-expanding horizon* (NEH) if

(1) H has a topology $R \times S^2$,

(2) Any null normal ℓ^a of H has vanishing expansion, $\theta_{(\ell)} = 0$.

(3) All equations of motion hold at H and the stress energy tensor T_{ab} is such that $-T^a_b\ell^b$ is future-causal for any future directed null normal ℓ^a,

The condition (1) implies that marginally trapped surfaces (Hayward, 1994) are related to a black hole spacetime. The conditions (2) and (3) imply from the Raychaudhuri equation that $\pounds_\ell q = 0$ on H, which further implies that the metric q is time-independent and H is totally geodesic in M. In general, there does not exist a unique induced connection on H due to degenerate q. However, on an NEH, the property $\pounds_\ell q = 0$ implies that the spacetime connection ∇ induces a unique (torsion-free) connection, say $\mathcal{D}$, on H which is compatible with q. We say that two null normals ℓ and $\bar{\ell}$ belong to the same equivalence class $[\ell]$ if $\ell = c\bar{\ell}$ for some positive constant c.

Definition 2. The pair $(H, [\ell])$ is called a weakly isolated horizon (WIH) if it is a NEH and each normal $\ell \in [\ell]$ satisfies $(\pounds_\ell \mathcal{D}_a - \mathcal{D}_a \pounds_\ell)\ell^a = 0$, i. e., $\mathcal{D}_a \ell^b$ is also time-independent.

Definition 3. A WIH $(H, q, [\ell])$ is called an *Isolated Horizon* (IH) if the full connection $\mathcal{D}$ is time-independent, that is, if $(\pounds_\ell \mathcal{D}_a - \mathcal{D}_a \pounds_\ell)V = 0$ for arbitrary vector fields V^a tangent to H.

For information and examples on isolated horizons we refer (Lewandowski, 2000), (Asktekar-Krishnan, 2002-3) , (Gourgoulhon-Jaramillo, 2006) and several others listed in these papers. However, in reality the present day research indicates that black hole has a cosmological background or it is surrounded by a local mass distribution. Therefore, there is significant difference in the structure and properties of the surrounding dynamical region of isolated black holes. The purpose of this paper is to use a conformal symmetry on an isolated black hole spacetime (M, g) which brings in a family of totally umbilical null hypersurfaces representing time-dependent null horizons near an isolated black hole. We also explain how this family of null horizons may evolve (for some cases) into a black hole isolated horizon.

2. Method

Let (M, g) be a spacetime with a conformal symmetry defined by a map $\phi : M \to M$ such that the

$$g(\phi_\star X, \phi_\star Y) = G_s(X, Y) = e^{\Omega_s} g(X, Y), \qquad \forall X, Y \in TM \tag{1}$$

where $\phi_\star$ is the differential (tangent) map of ϕ and Ω_s is a scalar function on M for some parameter value of s. The set of all conformal maps, satisfying (1), form a group of conformal motions under composition of mappings. Let each V_s be a smooth vector field on M and $\mathcal{U}$ denote a neighborhood of each $p \in M$ with local coordinate system. Let the integral curve of each V_s, through any point p in $\mathcal{U}$, be defined on an open interval $(-\epsilon, \epsilon)$ for $\epsilon > 0$. For each t in this interval we define a map ϕ_t on $\mathcal{U}$ such that for p in $\mathcal{U}$, $\phi_t(p)$ is that point with parameter value t on the integral curve of V_s through p. Then, each V_s generates a local 1-parameter group of transformations $\phi_t : x^a \to x^a + tV_s^a$. If ϕ_t satisfies the conformal symmetry equation (1), then, we say that V_s is a conformal vector field, briefly denoted by CKV. In local coordinates, V_s conformal implies that

$$\partial_c\,(x^a + tV_s^a)\,\partial_d\,(x^b + tV_s^b)\,G_{s_{ab}}(x + tV_s) = e^{\Omega_s} g_{cd}(x).$$

Expanding $G_{s_{ab}}(x + tV_s)$ up to first order in t, and then using the Lie derivative operator $\pounds_{V_s}$, we get

$$t(\pounds_{V_s} G_s) = (e^{\Omega_s} - 1)g.$$

As t is small, so is Ω_s. Setting $\Omega_s = t\,\sigma_s$ and expanding $e^{t\sigma_s}$ up to first order in t, we get

$$\pounds_{V_s} G_s = \sigma_s G_s, \quad \Omega_s = t\,\sigma_s. \tag{2}$$

Above equations are well-known as conformal Killing equations. In particular, V_s is homothetic or a Killing vector field according as σ_s is a no-zero constant or zero. Let $\mathcal{M}$ be the space of all smooth Lorentzian metrics on M. Consider a family $C = (G_s) \subset \mathcal{M}$ whose each member is conformally related to the metric g of (M, g) with conformal symmetry defined by (1). Denote by

$$\mathcal{F} = \{(M, (G_s)) : G_s = e^{\Omega_s} g \in C\} \tag{3}$$

a family of spacetimes conformally related to (M, g) and $G_s \in (G_s)$ for some parameter value of s.

3 Results

Let $\mathcal{S} = ((H_u), (h_u), (\ell_u))$ be a family of null hypersurfaces of $(M, (G_s))$ where u is corresponding parameter induced by the parameter s. For simplicity, we consider (H, h, ℓ) a member of the family $\mathcal{S}$ for some parameter value of u. The "bending" of H in M is described by the *Weingarten map*:

$$\mathcal{W}_\ell : T_pH \to T_pH, \quad X \to \nabla_X \ell.$$

$\mathcal{W}_\ell$ associates each X of H the variation of ℓ along X, with respect to the spacetime connection ∇. The second fundamental form, say B, of H is the symmetric bilinear form and is related to the $\mathcal{W}_\ell$ by

$$B(X, Y) = h(\mathcal{W}_\ell X, Y) = h(\nabla_X \ell, Y). \tag{4}$$

(H, h) is totally umbilical in M if and only if there is a smooth function f on H such that

$$B(X, Y) = f h(X, Y), \quad \forall X, Y \in TH. \tag{5}$$

In particular, H or a portion of H is called totally geodesic if and only if B vanishes, i.e., if and only if f vanishes on H or some portion of H.

Theorem 4. Let $\mathcal{S} = ((H_u), (h_u), (\ell_u))$ be a family of null hypersurfaces of a family $\mathcal{F} = (M, (G_s))$ of spacetimes defined by (3) whose each memeber is conformally related to an isolated black hole spacetime (M, g), with an isolated horizon (H, q) as defined by (1), which lies to the future of each (H_u, h_u). Then,

(a) there exists a family of maps $\phi_u : ((H_u), (h_u), (\ell_u)) \to (H, q)$ such that each metric h_u transforms to $h_u = e^{\Omega_u} q$, where Ω_u is the induced conformal factor of Ω_s.

(b) Each (H_u, h_u) is totally umbilical in $(M, (G_s))$.

(c) (H_u, h_u) may coincide with (H, q) on a portion of H_u only if Ω_u vanishes on that portion and, then, this common null hypersurface (H, q) has null mean curvature $\theta = 0$.

Proof. Part (a) of the theorem follows easily since conformal transformations preserve causal structure and, therefore, each member of $\mathcal{S}$ is conformal to (H, q), with the induced conformal factor Ω_u for some value of u. To prove (b) we observe that, as explained in Section 2, the conformal structure in (a) will induce a family of null conformal Killing vector (CKV) fields, say (ℓ_u) of the family (H_u) such that for each value of u

$$\pounds_{\ell_u} h_u(X, Y) == \sigma_u h_u(X, Y), \quad \Omega_u = t\sigma_u, \quad \forall X, Y \in TH_u. \tag{6}$$

Express the left side of above equation in the form $\pounds_{\ell_u} h_u(X, Y) = h_u(\nabla_X \ell_u, Y) + h_u(\nabla_Y \ell_u, X)$. Then, using (4) and (5) with $B(X, Y)$ symmetric it follows that

$$B(X, Y) = \frac{1}{2}\pounds_\ell h_u(X, Y) = \frac{1}{2}\sigma_u h_u(X, Y), \quad \forall X, Y \in TH,$$

which is well defined up to conformal rescaling (related to the choice of ℓ_u). Thus, each (H_u, h_u) is totally umbilical , which proves (b). For the case (c) observe that (H_u, h_u) approaches (H, q) for some value of u only if $h_u = q$ for that value of u, which further means that only if Ω_u vanishes for that value of u. This proves the first part of (c). Now, as per definition of isolated horizons, $(H_u, h_u) = (H, q)$ is totally geodesic. Moreover, It is well-known that any null hypersurface of a semi-Riemannian manifold has zero mean curvature if and only if it is totally geodesic, which proves (c). □

Now we address the question of how the Theorem 4 can be used to show the existence of a family of time-dependent null horizons near an isolated black hole. For this purpose we recall that (Perlick, 2005) proved following general result for a totally umbilical submanifold (also holds for totally geodesic case) H of a semi-Riemannian manifold M.

"A null geodesic vector field of M that starts tangential to H remains within H for some parameter interval around the starting point".

Above result satisfies a requirement for the existence of a null horizon in relativity. Since we assume that each null normal ℓ_u of the family of hypersurfaces $\mathcal{S}$ is null geodesic, using above result of (Perlick, 2005) we state the following corollary as a physical consequence of the Theorem 4 (proof is easy).

Corollary 5. Let (M, g) be a null geodesically complete spacetime obeying the null energy condition $Ric(X, X) \geq 0$ for all null vectors X and the hypothesis of Theorem 4 holds. Then, each null geodesic vector ℓ_u of $\mathcal{S}$ is contained in its respective smooth totally umbilical null hypersurface (H_u, ℓ_u) of (M, G_s) In particular, this property will also hold for the totally geodesic hypersurface (H, q) of (M, g) .

In support of Theorem 4, we now present following physical model and an example.

Physical Model. In two recent papers of (Duggal, 2012, 2014) a new class of null hypersurfaces of a spacetime was studied using the following definition:

Definition 6. A null hypersurface (H, h, ℓ) of a spacetime (M, g) is called an *Evolving Null Horizon*, briefly denoted by (ENH), if

(i) H is totally umbilical in (M, g) and may include a totally geodesic portion.

(ii) All equations of motion hold at H and energy tensor T_{ij} is such that $T^a_b \ell^b$ is future-causal for any future directed null normal ℓ.

Comparing the two conditions of above definition with Theorem 4, we notice that the first part of condition (i) is same as the conclusion (2) of Theoren 4 and for the second part we first observe that the energy condition of (ii) requires $R_{ab}\ell^a\ell^b$ non-negative for any ℓ, which implies from page 95 of (Hawking & Ellis, 1973) that $\theta_{(\ell)}$ monotonically decreases in time along ℓ, that is, M obeys the null convergence condition, which further means that the null hypersurface (H, h) is time-dependent in the region where $\theta_{(\ell)}$ is non-zero and may evolve into a time-independent totally geodesic hypersurface as a model of an isolated horizon. Thus, above two conditions of the Definition 6 clearly show that there exists a Physical Model of a class $\mathcal{S} = ((H_u), (h_u), (\ell_u))$ of a family of totally umbilical null hypersurfaces of a family $\mathcal{F} = (M, (G_s))$, satisfying the hypothesis and three conclusions of Theorem 4, such that its each member is an evolving null horizon(ENH) which may evolve into an isolated horizon. Simple example is a family of null cones non of which evolves into an isolated horizon. We refer (Duggal, 2012, 2014, 2015) for this and some more examples with details on the geometry and physics of ENHs.

Physical example. To construct an example we first recall that (Asktekar-Krishnan, 2003) studied the following quasi-local concept of dynamical horizons (briefly denoted by DH) which model the present day evolving black holes and their asymptotic states are isolated horizons.

Definition 7. A smooth, 3-dimensional spacelike submanifold (possibly with boundary) Σ of a spacetime is said to be a dynamical horizon (DH) if it can be foliated by a family of closed 2-manifolds such that

1. on each leaf L its future directed null normal ℓ has zero expansion, $\theta_{(\ell)} = 0$,
2. and the other null normal, $\mathbf{k}$, has negative expansion $\theta_{(\mathbf{k})} < 0$.

Above definition requires that Σ be spacelike except for a special case in which portions of marginally trapped surfaces lie on a spacelike horizon and the remainder on a null horizon. Recall that the concept of marginally trapped surfaces was first introduced by (Hayward, 1994) as an attempt to describe the surface of an evolving black hole. In the null case, Σ reaches equilibrium for which the shear and the matter flux vanish and this portion is represented by a weakly isolated horizon. Since in this paper we only focus on null horizons, we refer(Asktekar-Krishnan, 2003) for details on DHs and their properties.
Here, in order to construct a new physical example of an ENH satisfying the three conclusions of Theorem 4, we use the following Vaidya metric of a spacetime (M, g) which is an explicit example of dynamical horizons with their equilibrium states-the weakly isolated horizons (WIS).
Let (v, r, θ, ϕ) be the Eddington-Frinkelstein coordinates (Hawking-Ellis, 1973) of the metric g given by

$$\begin{aligned} g_{ab} &= -\left(1 - \frac{2Gm(v)}{r}\right)\nabla_a v\nabla_b v + 2\nabla_{(a}\nabla_{b)}r \\ &\quad + r^2\left(\nabla_a\theta\nabla_b\theta + \sin^2\theta\nabla_a\phi\nabla_b\phi\right). \end{aligned} \tag{7}$$

Using the notations of (Asktekar-Krishnan, 2003), we assume the Einstein field equations

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G T_{ab},$$

where $m(v)$ is any smooth non-decreasing function of v and the stress energy tensor is

$$T_{ab} = \frac{\dot{m}}{4\pi r^2}\nabla_a v\nabla_b v, \quad \dot{m} = dm/dv.$$

T_{ab} satisfies the dominant energy condition if $\dot{m} \geq 0$ and vanishes if $\dot{m} = 0$. There exist metric 2-spheres defined by v = constant and r = constant. One can calculate the outgoing and ingoing null normals to these 2-spheres and, in particular, the expansion of the outgoing null normal ℓ is given by

$$\theta_{(\ell)} = \frac{r - 2Gm(v)}{r}.$$

Thus, it is easy to see that the only spherically symmetric marginally trapped surfaces are the 2-spheres v = constant and $r = 2Gm(v)$. For this example (Asktekar-Krishnan, 2003) have shown that the hypersurface Σ given by $r = 2Gm(v)$ with $dm/dv > 0$ is a dynamical horizon (DH) and if $dm/dv = 0$, then, part of Σ evolves into a weakly isolated horizon which (using our notations) we denote by (H, q) such that its induced degenerate metric q is given by

$$q_{ab} = 4\,(Gm(v))^2 \left(\nabla_a\theta\nabla_b\theta + \sin^2\theta\nabla_a\phi\nabla_b\phi\right), \quad dm/dv = 0. \tag{8}$$

Now as explained in section 2, we set up a conformal map $\phi : M \to M$ satisfying (1) to get a family $\mathcal{F} = \{(M, (G_s)) : G_s = e^{\Omega_s} g \in C\}$ of spacetimes conformally related to the spacetime (M, g) with Vaidya metric (7) along with a weakly isolated horizon (H, q). Thus, the hypothesis of Theorem 4 holds and, therefore, its three conclusions are satisfied. Consequently, as explained before, there exists a family of evolving null horizons with their equilibrium states-the weakly isolated horizons of the spacetime (M, g) with Vadiya metric (7), which completes this physical example.

4. Discussion

We first recall that for more than 50 years the research on black hole physics was limited to time-independent event and isolated horizons until first attempt on time-dependent horizons was made by (Hayward, 1994) describing the geometry of the surface of dynamical black hole by using the following definition of *future, outer, trapping horizons*.

Definition 8.A future, outer, trapped horizon (FOTH) is a three manifold Σ, foliated by family of closed 2-surfaces such that (i) one of its future directed null normal, say ℓ, has zero expansion, $\theta_{(\ell)} = 0$; (ii) the other null normal, $\mathbf{k}$, has negative expansion $\theta_{(\mathbf{k})} < 0$ and (iii) the directional derivative of $\theta_{(\ell)}$ along $\mathbf{k}$ is negative; $\pounds_{\mathbf{n}}\theta_{(\ell)} < 0$.

Σ is either spacelike or null (at the equilibrium state of spacetime) for which $\theta_{(\mathbf{k})} = 0$ and $\pounds_{\mathbf{k}}\theta_{(\ell)} = 0$ He derived following general laws of black hole dynamics:

(a) Zeroth law. *The total trapping gravity of a compact outer marginal surface has an upper bound, attained if and only if the trapping gravity is constant.*

(b) First law. *The variation of the area form along an outer trapping horizon is determined by the trapping gravity and an energy flux.*

Then (Ashtekar-Krishnan, 2003) introduced the concept of dynamical horizons(DH)(see Definition 7) and obtained the expressions of fluxes of energy and angular momentum carried by gravitational waves across DHs and a generalization of the first and the second laws of mechanics. Overall their work has provided a new perspective covering quantum gravity, numerical relativity and gravitational wave phenomenology and much more. However, their definition requires that DH is spacelike and is time-independent null horizon only at the equilibrium state of a spacetime. Since then some researchers have published papers on time-dependent horizons which are always null geodesic hypersurfaces of a spacetime. For example, we refer two papers each of(Sultana-Dyer, 2004, 2005) and (Duggal, 2012, 2014) and more referred therein.

We highlight that our approach in this paper of using conformal symmetry and proving Theorem 4 is an important step forward towards the existence of time-dependent null horizons near an isolated horizon, supported by a physical example. We also mention that the physical use of research on time-dependent null horizons, including our approach in this paper, may have connection (though it is early to be sure) with the latest LIGO [Laser Interferometer Gavitational-Waves Observatory] experiment (as per Press Release of February 11, 2016) confirming the presence of gravitational waves produced during the final fraction of a second of the merger of two black holes into a single massive spinning black hole. Thus, this latest LIGO experiment strengthens the ongoing research on time-dependent null horizons, in particular, near such spinning black holes surrounded with gravitational waves. For information on LIGO via video one may try: http://mediaassets.caltech.edu/gwave

Finally, our paper has opened the possibility of working on interrelated geometries of dynamical, isolated and evolving null horizons. Precisely, we have the following relationship:

$$(M, g) \Rightarrow \text{ Spacelike}(DH) \Rightarrow \text{ Null}(IH) \Leftarrow \text{ Null}(ENH) \Leftarrow \text{ conformal symmetry}((M, (G_s)) .$$

Based on above relation we propose further work on null version of spacelike results proved using DHs.

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