# Manifest Covariant Hamiltonian Theory of General Relativity

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## Abstract

The problem of formulating a manifest covariant Hamiltonian theory of General Relativity in the presence of source fields is addressed, by extending the so-called "DeDonder-Weyl" formalism to the treatment of classical fields in curved space-time. The theory is based on a synchronous variational principle for the Einstein equation, formulated in terms of superabundant variables. The technique permits one to determine the continuum covariant Hamiltonian structure associated with the Einstein equation. The corresponding continuum Poisson bracket representation is also determined. The theory relies on first-principles, in the sense that the conclusions are reached in the framework of a non-perturbative covariant approach, which allows one to preserve both the 4-scalar nature of Lagrangian and Hamiltonian densities as well as the gauge invariance property of the theory.

**Keywords:** General Relativity, Einstein equation, DeDonder-Weyl formalism, Synchronous variational principle, Covariant Hamiltonian theory, Poisson bracket representation

## 1. Introduction

The Hamiltonian description of classical mechanics, both for discrete and continuum systems, is of foremost importance and a mandatory prerequisite for the construction of quantum field theory (Dirac, 1964; Goldstein, 1980). The issue pertains necessarily also to General Relativity (GR), and specifically the so-called Standard Formulation to General Relativity (SF-GR) (Einstein, 2004; Hilbert, 1915; Kuchar, 1973; Misner et al., 1973), i.e., Einstein's original approach to his namesake field equation. The primary goal of this paper is to carry out the construction of a manifest covariant Hamiltonian description of GR in the context of SF-GR and such that it leaves unchanged the tensorial form of the Einstein field equations. This investigation is based on earlier work related to the establishment of a fully-covariant Lagrangian treatment of SF-GR (Cremaschini & Tessarotto, 2015). The latter is based in turn on the adoption of a new type of variational principle for continuum fields, i.e. the so-called *synchronous Lagrangian variational principle*, first introduced by Cremaschini and Tessarotto (2015) (please notice that below we shall refer to the same paper for all notations which are not explicitly indicated here). As we intend to show, this choice is also crucial in the present paper for the establishment of the axiomatic covariant Hamiltonian treatment.

The task indicated above, however, is by no means straightforward. Indeed, an axiomatic approach of this type should satisfy precise requisites. These include, in particular, both the Einstein general covariance principle and the principle of manifest covariance. It must be stressed that a Hamiltonian theory of GR fulfilling these requirements is still missing to date.

More precisely, the first requisite states that in the context of GR the Einstein field equation as well as its solution represented by the metric tensor $g(r) \equiv \left\{ g_{\mu\nu}(r) \right\}$ should be endowed with tensor transformation laws with respect to the group of transformations connecting arbitrary GR-reference frames, i.e., 4−dimensional curvilinear coordinate systems spanning the same prescribed space-time $D^4 \equiv \left( \mathbf{Q}^4, g(r) \right)$. Here the notation $r \equiv \{r^\mu\}$, to be used whenever necessary throughout the paper, actually identifies one of such possible coordinate parametrizations. In SF-GR such a setting is usually identified with the group of invertible local point transformations (i.e., suitably-smooth diffeomorphism).

Instead, the principle of manifest covariance is actually a particular realization of the general covariance principle (Einstein, 2004). It states that it should always be possible in the context of any relativistic theory, i.e., in particular SF-GR, to identify with 4-tensor quantities all continuum fields, such as the corresponding related Lagrangian and Hamiltonian continuum variables, as well as the relevant variational and/or extremal equations involved in the theory. The latter viewpoint is in agreement with the axiomatic construction originally developed by Einstein in which both requirements are, in fact, explicitly adopted. Indeed, according to the Einstein viewpoint in this reference, in order that physical laws have an objective physical character, they should not depend on the choice of the GR-reference frame. This requisite can only be met when all classical physical observables and corresponding physical laws and the mathematical relationships holding among them, are expressed in tensorial form with respect to the group of transformations connecting the said GR-frames.

The conjecture that a manifestly covariant Hamiltonian formulation, i.e., a theory satisfying simultaneously both of these principles, must be possible for continuum systems is also suggested by the analogous theory holding for discrete classical particle systems. Indeed, its validity is fundamentally implied by the state of the art theory for classical $N-$body systems subject to non-local electromagnetic (EM) interactions. The issue is exemplified by the Hamiltonian structure of the EM radiation-reaction problem in the case of classical extended particles as well as $N-$body EM interactions among particles of this type (Cremaschini & Tessarotto, 2011-2014), together with the corresponding non-local quantum theory (Cremaschini & Tessarotto, 2015).

On the other hand, it should be mentioned that in the case of continuum fields, the appropriate formalism is actually well-established, being provided by the DeDonder-Weyl Lagrangian and Hamiltonian treatments (De Donder, 1930; Echeverría-Enríquez et al., 1996; Forger et al., 2003; Günther, 1987; Kanatchikov, 1998; Kastrup, 1983; Kisil, 2004; Sardanashvily, 1995; Saunders, 1989; Struckmeier & Redelbach, 2008; Weyl, 1935). Such an approach is originally formulated for fields defined on the Minkowski space-time, while its extension to classical fields defined in curved space-time is well-known. Nevertheless, the inclusion of the gravitational field in this treatment, consistent with the full validity of the manifest covariance principle, is still missing. The need to adopt an analogous approach also in the context of classical GR, and in particular for the Einstein equation itself or its possible modifications, has been recognized before (Esposito et al., 1995; Reisenberger & Rovelli, 2002; Rovelli, 2003). Notice that this feature is of primary interest in particular in the case of non-perturbative classical and quantum approaches.

However, as far as the gravitational field is concerned, a common difficulty met by all previous non-perturbative Hamiltonian approaches of this type is that, from the start, they are based on the introduction of Lagrangian densities which have a non-covariant character, i.e., they are not 4-scalars. A further possible deficiency lies in the choice made by some authors of non-tensorial Lagrangian coordinates and/or momenta. Such a concept for example is intrinsic in the Dirac constrained dynamics approach (Castellani, 1982; Mukunda, 1980; Sudarshan & Mukunda, 1964). Both features actually prevent the possibility of establishing a theory in which the canonical variables are tensorial and the Euler-Lagrange equations, when represented in terms of them, are manifestly covariant. A possible way-out, first pointed out by De Witt (1967), might appear that based on the adoption of functional-derivative equations in implicit form, replacing the explicit Euler-Lagrange equations. This can be formally achieved provided the variations of the Lagrangian coordinates and momenta have a tensorial character (De Witt, 1967; Rovelli, 2003). Nevertheless, the issue of the possible violation of manifest covariance remains in place for the definition of the Lagrangian and Hamiltonian variables.

It must be stressed that a common underlying difficulty still characterizing these approaches is that the variational Lagrangian densities adopted there still lack, in a strict sense, the fundamental requirement needed for the application of the DeDonder-Weyl approach, namely their 4-scalar property. The physical consequence is that the action functional and consequently also the variational Lagrangian density do not exhibit the required gauge properties, which apply instead in the case of field Lagrangian formulations valid in Minkowski space-time. A solution to this issue has been pointed out by Cremaschini and Tessarotto (2015), based on the introduction of a new form of the variational principle, to be referred to as synchronous Lagrangian variational principle. The same type of problem can be posed in the context of continuum Hamiltonian theories. However, the achievement of such a goal may encounter potentially far more severe difficulties, unless $4-$tensor canonical variables are adopted. Such a choice may actually be inhibited (Arnowitt et al., 1962; Bergmann, 1950; Dirac, 1958 and 1959; Pirani & Schild, 1950; Pirani et al., 1952), for example when asynchronous variational principles are adopted (see definition given in Cremaschini and Tessarotto, 2015). Despite these difficulties, the possible solution of the problem appears of fundamental importance. In fact, it must be noted that the Hamiltonian formalism has a perspicuous and transparent

physical interpretation, even in the context of covariant formulation, an example of which is provided by Cremaschini and Tessarotto (2011). The symplectic structure of phase-space arises also for continuum systems, a feature which permits one to construct both local tensorial Poisson brackets and continuum canonical transformations.

In some respects to be later explained, the viewpoint adopted in this paper has analogies with the so-called *induced gravity* (or *emergent gravity*), namely the conjecture that the geometrical properties of space-time reveal themselves as a mean field description of microscopic stochastic or quantum degrees of freedom underlying the classical solution in the corresponding variational formulation. This is achieved by introducing a prescribed metric tensor $\widehat{g}_{\mu\nu}$ which is held constant in the variational principles and therefore also in the Euler-Lagrange equations. This tensor is to be distinguished from the variational one $g_{\mu\nu}$. More precisely, $\widehat{g}_{\mu\nu}$ acquires a geometrical interpretation, since by construction it raises/lowers tensorial indices and prescribes the covariant derivatives. In this picture, $\widehat{g}_{\mu\nu}$ arises as a macroscopic prescribed mean field emerging from a background of variational fields $g_{\mu\nu}$, all belonging to a suitable functional class. This permits to introduce a new representation for the action functional, depending both on $g_{\mu\nu}$ and $\widehat{g}_{\mu\nu}$. As shown here, such a feature is found to be instrumental for the identification of the covariant Hamiltonian structure associated with the classical gravitational field.

The aim of the paper is to provide a continuum Lagrangian and Hamiltonian variational formulations for the Einstein equation in the context of a non-perturbative treatment. Basic feature is the manifest covariance property of the theory at all levels, whereby all physical quantities including the canonical variables have a tensorial character. Two remarkable consequences follow. The first one is that the continuum Hamiltonian structure is displayed in terms of continuum tensor Poisson brackets. Second, as a result, the physical interpretation of the canonical variables and the extremal quantities (in particular in terms of $\widehat{g}_{\mu\nu}$ and of the extremal Hamiltonian function) clearly emerges.

In detail, the scheme of the paper is as follows. Section 2 deals with a historical background to the Lagrangian and Hamiltonian theories of GR given in the literature. Section 3 summarizes the physical motivations for the present study, in the framework of the DeDonder-Weyl covariant Hamiltonian formulation. Section 4 provides the definition of a constrained synchronous Lagrangian variational principle, which allows one to treat the contribution of the connection fields in the Lagrangian density, while warranting the consistency of the extremal Euler-Lagrange equations with the Einstein equation (see THM.1 and Corollary 1 to THM.1). In Section 5 the extension of the Lagrangian theory to the inclusion of a cosmological constant and source matter fields is illustrated (see Corollary 2 and 3 to THM.1). In Section 6 the axiomatic foundation to continuum Hamiltonian systems is given. In Section 7 the canonical theory of the Einstein equation is determined, providing a manifest covariant Hamiltonian theory (see THM.2). Section 8 deals with the classical 4-vector Poisson bracket representation of the same Hamiltonian theory. In Section 9 a discussion of the main results and their physical implications is presented. Finally, concluding remarks are reported in Section 10..

## 2. Historical Background

A common theoretical feature shared by General Relativity (GR) and classical field theory is the variational character of the fundamental dynamical laws which identify these disciplines. This concerns both the representation of the Einstein field equations and of the covariant dynamics of classical fields as well as of discrete (e.g., point-like or extended particles) or continuum systems in curved space-time. For definiteness, the non-vacuum Einstein equations for the metric tensor $g_{\mu\nu}$, assumed to have Lorentzian signature $(+, -, -, -)$, in the absence of cosmological constant are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1}$$

which are assumed to hold in the 4-dimensional space-time $D^4$, to be identified with a connected and time-oriented Lorentzian differentiable manifold $(\mathbf{Q}^4, g)$. Here $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ defines the symmetric Einstein tensor, while $R_{\mu\nu}$ is the Ricci curvature $4-$tensor

$$R_{\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}, \tag{2}$$

with $\Gamma^{\alpha}_{\mu\nu}$ denoting non-tensorial quantities which identify the so-called (standard) connection functions. Finally, the source term on the rhs is expressed in terms of the symmetric stress-energy tensor $T_{\mu\nu}$. Eq.(1) determines the dynamical equations of $g_{\mu\nu}(r)$ to be completed by the metric compatibility condition of the covariant derivative

$$\nabla_{\alpha}g^{\mu\nu} = 0, \tag{3}$$

which provides a unique relationship between $g_{\mu\nu}$ and the connection function contained in the covariant derivative operator $\nabla_\alpha$ (Landau & Lifschitz, 1957; Wald, 1984). In particular, this yields the definition of the Christoffel symbols as

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left( \frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right). \tag{4}$$

Fundamental issues related to possible alternative construction methods of Lagrangian variational formulations for the Einstein equation have been treated extensively in the work by Cremaschini and Tessarotto (2015). The analysis has permitted to uncover the existence of a new class of variational approaches for classical continuum fields, referred to as *synchronous Lagrangian variational principles*. These have been shown to depart from the customary approach adopted in the literature and the analogous one developed originally by Einstein himself for his namesake equation. The key feature of the new approach, which sets is apart from such literature approaches, lies in the prescription of the functional setting for the variational classical fields, whereby the 4−scalar 4−volume element $d\Omega \equiv d^4x \sqrt{-g}$ (with $g$ denoting the determinant of the metric tensor) is held fixed during arbitrary variations performed on the same fields. In contrast, traditional Lagrangian approaches to be found in the literature are based on so-called *asynchronous variations*, namely in which $d\Omega$ is actually suitably varied during the variations of classical fields. Such a property, as pointed out in Cremaschini and Tessarotto (2015), has important implications since it permits to cure deficiencies which include, in particular, the property of manifest covariance of the symbolic Euler-Lagrange equations and the gauge invariance property of the variational Lagrangians.

Based on these premises, in this paper we address the issue of constructing a manifestly covariant Hamiltonian theory of GR. Such a problem must be set in the proper historical perspective. Indeed, it is well-known that although in principle from the Lagrangian formulation it is possible to derive a Hamiltonian formulation, in the case of GR this is not an easy task. In fact, in the past such a Hamiltonian theory has been approached based on the principles of constrained dynamics and the related concepts of primary constraints (in which the constraints depend both on coordinates and momenta) and secondary constraints (i.e., those which are not primary). In case of Hamiltonian systems they coincide with the so-called first- and second-class constraints, originally introduced by Dirac in order to treat singular systems (Dirac, 1950; Sundermeyer, 1982). Based on this approach, some qualitative properties emerge which can be problematic. These are:

1) Lack of manifest covariance of the variational functions which appear both in the Lagrangian and Hamiltonian formulations. These include for example the same Lagrangian and Hamiltonian densities as well as the definition of the canonical momenta. A basic consequence is that the symbolic Euler-Lagrange equations are not manifestly covariant.

2) Lack of gauge invariance properties for the Hamiltonian theory. This property is actually inherited from the corresponding Lagrangian formulation, see Cremaschini and Tessarotto (2015).

3) Another possible issue concerns the definition of the variational functional class together with the boundary conditions to be satisfied by the varied fields.

4) A further deficiency which is in part connected to 3 is the fact that it is not possible to define boundary conditions for fields which do not have 4-tensor properties. In particular, such boundary conditions only apply in a given coordinate system. Therefore, they are mutually related by coordinate transformation. This means that, as a result, if the boundary condition is such that a 4-tensor is set to zero on the boundary $\partial \mathbb{D}^4$, then since its modulus is a 4-scalar which is identically zero, the same modulus condition must apply in any coordinate system.

In order to analyze systematically the previous issues, it is worth starting from the original approaches based on the Einstein-Hilbert Lagrangian. First attempts to develop a Hamiltonian theory are those due to Bergmann et al. (1950), Pirani and Schild (1950) and Pirani et al. (1952), in which the "natural" choice was made of using the covariant metric tensor as the canonical (coordinate) variable. Later, Dirac himself (1958-1959), adopting the same choice, proposed a Hamiltonian formulation in terms of a modified Einstein-Hilbert Lagrangian. All such approaches are based on the adoption of non-tensorial canonical momenta. In fact, in these methods one actually introduces a separation between "space" and "time" components of the field metric and its derivatives. On the other hand, it must be noted that this operation does not correspond to the introduction of preferred coordinate systems which can result in the violation of the intrinsic covariance property of the theory (Kiriushcheva et al., 2008 and 2011). As a consequence, the final set of variational equations are actually equivalent to the Einstein equations, so that in this sense the covariance of the Einstein equations is preserved, although the manifest covariance for such approaches is not extant.

The key principles of Dirac's Hamiltonian approach is that of singling out the "time" component of the 4−position vector in terms of which the generalized velocity is identified with $g_{\mu\nu,0}$. This lead him to identify the canonical momentum in terms of the manifest non-tensorial quantity

$$\pi^{\mu\nu}_{Dirac} = \frac{\partial \mathcal{L}_{EH}}{\partial g_{\mu\nu,0}}, \tag{5}$$

where $\mathcal{L}_{EH}$ is the Einstein-Hilbert variational Lagrangian density. One can even argue at this point on the correctness of such an identification of the canonical momentum. In particular, the definition of the momentum based on the partial derivative of a tensor can appear ambiguous in the framework of GR, where only covariant derivatives actually exhibit the proper covariance property.

Another well-known approach to the Hamiltonian formulation of the Einstein equations is the one developed by Arnowitt, Deser and Misner (Arnowitt et al., 1962, usually referred to as ADM theory). Detailed presentations of this issue can be found for example in Misner et al. (1973) and Wald (1984). There is a striking analogy with the Dirac approach, in that also in this case the role of the time coordinate is singled out. The theory provides a 3+1 decomposition of GR which is coordinate invariant but foliation dependent, in the sense that it relies on a choice of a family of observers according to which space-time is split into time and a 3-space (Alcubierre, 2008). Nevertheless, also in this case manifest covariance is lost, specifically because of the adoption, inherent in the same ADM approach, of non-4−tensor Lagrangian and Hamiltonian variables. This feature supports the objection raised by Hawking against the ADM theory, who stated that "the split into three spatial dimensions and one time dimension seems to be contrary to the whole spirit of Relativity" (Hawking, 1979; see also Hawking & Penrose, 1970, for additional critics on the 3+1 decomposition). This point of view seems physically well-founded. In fact, in the spirit of GR, "time" and "space" should be treated on equal footing as independent variables. The distinction among the entries of 4−tensors cannot be longer put on physical basis in GR, in contrast to what happens in flat space-time. As a result, the special role attributed to the "time" (or zero) component with respect to the "space" components does not appear consistently motivated.

The remaining critical points concern issues 2-4 indicated above. In fact both the Dirac and the ADM approaches rely on the original Einstein-Hilbert Lagrangian density. As pointed out in Cremaschini and Tessarotto (2015), this misses the correct gauge invariance properties required for a variational formulation in classical field theory. Therefore, also the corresponding Hamiltonian approaches share the same feature. In addition, it must be remarked that in both Hamiltonian approaches the variational functional class is not defined so it is not clear which boundary conditions are to be set on the varied fields. In this regard we also notice that it seems doubtful even the possibility of prescribing in a consistent unique way boundary conditions on non-tensorial fields.

Because of these reasons, it still remains to be ascertained whether alternative approaches can be envisaged which can ultimately overcome simultaneously these problems. The Lagrangian formulation of GR has been already discussed in Cremaschini and Tessarotto (2015). This leaves however the problem of the corresponding manifest covariant Hamiltonian formulation of GR still open.

### 3. Physical Motivations

In this section we introduce the physical motivations for the research program developed in this paper. In this regard a basic preliminary issue to be first addressed concerns the possible viewpoint denying the very possibility of a manifest covariant Hamiltonian field theory - in contrast to the corresponding Lagrangian one - (Wald, 1984). The goal of this paper is actually to prove that such a covariant theory can be constructed.

The first motivation arises from field theory. In fact in the literature the problem of the construction of manifest covariant Hamiltonian formulations for field theories has a long history. In this reference, the earliest contributions date back to the pioneering work by De Donder (1930) and Weyl (1935). More recently, the subject has been treated in Echeverría-Enríquez et al. (1996), Forger et al. (2003), Günther (1987), Kanatchikov (1998), Kastrup (1983), Kisil (2004), Sardanashvily (1995), Saunders (1989), Struckmeier and Redelbach (2008), where such an approach is usually referred to as "multi-symplectic" or "poly-symplectic field theory". The general method underlying this approach can be best illustrated by considering the example-case of a scalar field $\phi \equiv \phi(r^\mu)$ in flat space-time, being the treatment of tensorial fields analogous (Struckmeier & Redelbach, 2008). For definiteness let us consider a Minkowski space-time $(\mathbf{M}^4, \eta)$ with signature $(+, -, -, -)$. The corresponding variational Lagrangian density $\mathcal{L}_\phi$ becomes a 4−scalar, since $\sqrt{-g} = 1$ identically. The Lagrangian is assumed to exhibit a functional dependence of the form $\mathcal{L}_\phi = \mathcal{L}_\phi(\phi, \partial_\mu\phi, r^\mu)$, namely to depend on the field $\phi$, its first partial derivatives and possibly also

explicitly on the position 4−vector $r^\mu$. As a result, the action integral is written as

$$S_\phi = \int_{M^4} d^4 r \mathcal{L}_\phi \left( \phi, \partial_\mu \phi, r^\mu \right). \tag{6}$$

Variation of $S_\phi$ in a suitable functional class in which the value of $\phi$ is prescribed on a suitable fixed boundary of $M^4$ then provides the Euler-Lagrange equation

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}_\phi}{\partial \left( \partial_\mu \phi \right)} - \frac{\partial \mathcal{L}_\phi}{\partial \phi} = 0, \tag{7}$$

which determines the behavior of $\phi \left( r^\mu \right)$.

The corresponding Hamiltonian formulation is obtained by identifying the conjugate momentum $\pi_\phi^\mu$ in terms of the 4−vector

$$\pi_\phi^\mu = \frac{\partial \mathcal{L}_\phi}{\partial \left( \partial_\mu \phi \right)}, \tag{8}$$

so that, provided $\mathcal{L}_\phi$ is regular, the 4−scalar Hamiltonian density $\mathcal{H}_\phi = \mathcal{H}_\phi \left( \phi, \pi_\phi^\mu, r^\mu \right)$ is introduced by means of the Legendre transform

$$\mathcal{H}_\phi = \pi_\phi^\mu \partial_\mu \phi - \mathcal{L}_\phi, \tag{9}$$

which is usually referred to as the "De Donder-Weyl" Hamiltonian density for the scalar field $\phi$. The set of first-order Hamilton equations corresponding to the Euler-Lagrange equation (7) which define the continuum Hamiltonian system are provided by the covariant PDEs

$$\frac{\partial \mathcal{H}_\phi}{\partial \pi_\phi^\mu} = \frac{\partial \phi}{\partial r^\mu}, \tag{10}$$

$$\frac{\partial \mathcal{H}_\phi}{\partial \phi} = -\frac{\partial \pi_\phi^\mu}{\partial r^\mu}. \tag{11}$$

Remarkably, in such a framework the variational principle meets all the requirements indicated above. In particular, one notices that the canonical variables $\left( r^\mu, \pi_\phi^\mu \right)$ are both 4−vectors and the Hamiltonian density is a 4−scalar, so that the Hamilton equations (10) and (11) are manifestly covariant. Finally, the theory is manifestly gauge invariant.

The second motivation is provided by the analogy with the variational description of relativistic particle dynamics, which can be cast both in Lagrangian and Hamiltonian forms by preserving manifest covariance. The construction of the Hamiltonian formulation is a direct consequence of the synchronous Lagrangian variational principle (Cremaschini & Tessarotto, 2015). In fact, let us consider a point-like particle having rest mass $m_o$, charge $q_o$ and proper time $s$, so that the corresponding 4−velocity is $u^\mu (s) = \frac{dr^\mu(s)}{ds}$, while the line element is given by

$$ds^2 = g_{\mu\nu}(r) dr^\mu(s) dr^\nu(s). \tag{12}$$

Here the metric tensor $g_{\mu\nu}(r)$ and the Faraday tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ of the external EM fields are considered prescribed, with $A_\mu$ denoting the 4−vector potential. The Lagrangian functional for this system is given by

$$S_{pS} (r^\mu, u^\mu) = -\int_{s_1}^{s_2} ds L_{pS} \left( r^\mu (s), \dot{r}^\mu (s), u^\mu (s) \right), \tag{13}$$

where $\dot{r}^\mu \equiv \frac{dr^\mu}{ds}$ and $L_{pS}$ is the 4−scalar Lagrangian

$$L_{pS} \left( r^\mu (s), \dot{r}^\mu (s), u^\mu (s) \right) \equiv \left( u_\mu (s) + q A_\mu (r(s)) \right) \frac{dr^\mu (s)}{ds} - \frac{1}{2} u^\mu (s) u_\mu (s). \tag{14}$$

Here $q \equiv \frac{q_o}{m_o c^2}$ is the normalized charge. The virtue of this variational formulation is the adoption of superabundant variables, so that the kinematic constraint (mass-shell condition) $u^\mu (s) u_\mu (s) = 1$ acts only on the extremal function, while the line element $ds$ is by construction expressed by Eq.(12) in terms of the extremal curve $r^\mu (s)$.

Then, after introducing the customary Legendre transformation, one recovers the Hamiltonian variational formulation which is provided in terms of the functional

$$S_{pS}^H(r^\mu, p^\mu) = \int_{s_1}^{s_2} ds \left[ p_\mu(s) \frac{dr^\mu(s)}{ds} - H_{pS} \right], \tag{15}$$

where

$$H_{pS} = \frac{1}{2} \left( p_\mu - qA_\mu \right) \left( p^\mu - qA^\mu \right) \tag{16}$$

is the 4−scalar Hamiltonian function and $p_\mu \equiv u_\mu + qA_\mu$ is the canonical 4−momentum. The corresponding variational principle is realized in terms of the synchronous Hamilton variational principle. Following the work by Cremaschini and Tessarotto (2015), we introduce the synchronous functional class

$$\{r^\mu, p^\mu\}_S = \left\{ \begin{array}{c} r^\mu(s), p^\mu(s) \in C^2(\mathbb{R}) \\ \delta(ds) = 0 \\ r^\mu(s_k) = r_k^\mu, \; k = 1, 2 \\ p^\mu(s_k) = p_k^\mu, \; k = 1, 2 \end{array} \right\}, \tag{17}$$

where $\delta$ denotes the synchronous variation operator. This can be defined in particular with respect to the extremal quantities, namely

$$\delta r^\mu(s) = r^\mu(s) - r_{extr}^\mu(s), \tag{18}$$

$$\delta p^\mu(s) = p^\mu(s) - p_{extr}^\mu(s), \tag{19}$$

where $r_{extr}^\mu(s)$ and $p_{extr}^\mu(s)$ identify suitable extremal phase-space curves. Here, $r^\mu(s)$ and $p^\mu(s)$ are considered independent, so that $\delta r^\mu(s)$ and $\delta p^\mu(s)$ are independent too. In this setting, the variational principle is realized in terms of the synchronous variation of the functional $S_{pS}^H(r^\mu, p^\mu)$, namely the corresponding Frechet derivative

$$\delta S_{pS}^H(r^\mu, p^\mu) \equiv \left. \frac{d}{d\alpha} \Psi(\alpha) \right|_{\alpha=0} = 0, \tag{20}$$

to hold for arbitrary independent displacements $\delta r^\mu(s)$ and $\delta p^\mu(s)$. Notice that here $\Psi(\alpha)$ is the smooth real function $\Psi(\alpha) = S_{pS}(r^\mu + \alpha \delta r^\mu, p^\mu + \alpha \delta p^\mu)$, being $\alpha \in \,]-1, 1[$ to be considered independent of $r^\mu(s)$, $p^\mu(s)$ and $s$, so that the corresponding variational derivatives are

$$\frac{\delta S_{pS}^H(r^\mu, p^\mu)}{\delta r^\mu(s)} \equiv -\frac{D}{Ds} p_\mu - \frac{\partial H_{pS}}{\partial r^\mu} = 0, \tag{21}$$

$$\frac{\delta S_{pS}^H(r^\mu, p^\mu)}{\delta p_\mu(s)} \equiv \frac{dr^\mu(s)}{ds} - \frac{\partial H_{pS}}{\partial p_\mu} = 0, \tag{22}$$

which identify the corresponding 1-body Hamiltonian system. In Eq.(21) $\frac{D}{Ds}$ denotes the covariant derivative acting on $p_\mu$ (Landau & Lifschitz, 1957). In the literature (see for example Misner et al., 1973) Eq.(16) is usually referred to as "super-Hamiltonian", with Eqs.(21) and (22) as super-Hamiltonian equations. Nevertheless, we notice that the same equations exhibit the standard Hamiltonian structure (Cremaschini & Tessarotto, 2011). This means that the set $\left\{ H_{pS}, \mathbf{y} \equiv (r^\mu, p^\mu) \right\}$ defines in a proper sense a Hamiltonian system, even if it is represented in terms of a superabundant canonical state. Its remarkable property in this respect is that, once the constraint (12) on the line element to hold only for the extremal curves has been set, no further constraints are required on the extremal canonical state $\overline{\mathbf{y}}$. Indeed, such constraints are now identically fulfilled simply as a consequence of the Hamilton equations (21) and (22). These conclusions depart from the well-known Dirac generating-function formalism (DGF) where the canonical variables do not have a tensorial character, although the property of covariance (albeit not of manifest covariance) of the related Hamilton equations is still fulfilled (see related discussion in Cremaschini and Tessarotto, 2012).

Furthermore, it is interesting to notice the basic properties of the synchronous Hamiltonian variational principle (20), which involve: 1) coordinates and momenta which are 4−vectors; 2) the Hamiltonian function $H_{pS}$ and the action functional $S_{pS}^H(r^\mu, p^\mu)$ which are 4-scalars; 3) the manifestly-covariant Hamilton equations. In addition, it must be noticed that the Hamiltonian $H_{pS}$ has a special feature, namely its extremal value is a constant equal to

1/2. Such a theory is of general validity and relies on the adoption of Hamilton variational principle, which holds even for the treatment of the non-local interaction occurring in the EM radiation-reaction problem. A theory of this type has been recently established by Cremaschini and Tessarotto (2011-2015).

These considerations suggest the obvious conjecture that the same program can actually be worked out in curved space-time, and in particular for the gravitational field in such a way to determine a covariant Hamiltonian variational formulation for the Einstein equations.

### 4. Modified Synchronous Lagrangian Variational Principle

In view of the considerations outlined above, let us now pose the problem of constructing a modified version of the synchronous Lagrangian variational principle for the Einstein equation reported in Cremaschini and Tessarotto (2015). The issue, as will be clarified below, is actually propedeutic for the problem considered in this paper about the manifestly covariant Hamiltonian formulation of GR. Starting point is the synchronous Lagrangian variational principle presented in Cremaschini and Tessarotto (2015), recalled here for better clarity, using the same notation.

To ease the presentation we consider the case of vacuum Einstein equations. The extension to the case of non-vacuum equations in the presence of EM and external matter sources is given in Cremaschini and Tessarotto (2015). As also shown in THM.2 of the same reference, a synchronous variational principle of the type

$$\delta S_1\left(Z, \widehat{Z}\right) = 0 \tag{23}$$

holds, which applies for arbitrary variations of the variational fields $Z$ and with the operator $\delta$ denoting the *synchronous variation operator*. By construction, this acts in such a way that both

$$\delta \widehat{Z} \equiv 0, \tag{24}$$
$$\delta Z_{extr} \equiv 0, \tag{25}$$

with $\widehat{Z}$ and $Z_{extr}$ identifying respectively the prescribed and the extremal tensor fields. In particular, $Z_{extr}$ is defined as the solution of the boundary-value problem associated with the Euler-Lagrange equations. Instead, when acting on any other arbitrary variational function $Z$ different from both $\widehat{Z}$ and $Z_{extr}$, the synchronous variation is defined as

$$\delta Z\left(r\right) = Z\left(r\right) - Z_{extr}\left(r\right). \tag{26}$$

This requires identifying $\Psi(\alpha) = S_1(Z + \alpha \delta Z, \widehat{Z})$, where the action functional is

$$S_1\left(Z, \widehat{Z}\right) = \int_{\widehat{D}^4} d\Omega L_1\left(Z, \widehat{Z}\right). \tag{27}$$

In this treatment, $Z$, $\delta Z$ and $\widehat{Z}$ denote respectively the variational, the variation and the prescribed fields, the latter being held fixed during synchronous variations, so that identically $\delta \widehat{Z}\left(r\right) = 0$ (see discussion in Cremaschini and Tessarotto, 2015). Thus, here $L_1$ is the 4−scalar variational Lagrangian density

$$L_1\left(Z, \widehat{Z}\right) \equiv -\frac{c^3}{16\pi G} g^{\mu\nu} \widehat{R}_{\mu\nu} h\left(Z, \widehat{Z}\right), \tag{28}$$

where $h\left(Z, \widehat{Z}\right)$ is the 4−scalar multiplicative factor

$$h\left(Z, \widehat{Z}\right) \equiv \left(2 - \frac{1}{4} g^{\alpha\beta} g_{\alpha\beta}\right), \tag{29}$$

and $\widehat{R}_{\mu\nu}$ is defined according to Eq.(2) and is evaluated for $g_{\mu\nu}\left(r\right) = \widehat{g}_{\mu\nu}\left(r\right)$, while here $g^{\alpha\beta} g_{\alpha\beta} \neq \delta_\alpha^\alpha$ for variational

curves (see below). Here, both $Z$ and $\widehat{Z}$ belong to the functional class

$$\{Z\}_{E-S} \equiv \left\{ \begin{array}{c} Z_1(r) \equiv g_{\mu\nu}(r) \\ \widehat{Z}_1(r) \equiv \widehat{g}_{\mu\nu}(r) \\ \widehat{Z}_2(r) \equiv \widehat{R}_{\mu\nu}(r) \\ Z(r), \widehat{Z}(r) \in C^k\left(\mathbb{D}^4\right) \\ g_{\mu\nu}(r)|_{\partial\mathbb{D}^4} = g_{\mu\nu\mathbb{D}}(r) \\ \delta\widehat{Z}(r) = \delta Z_{extr}(r) = 0 \\ \delta(d\Omega) = 0 \\ g_{\mu\nu} = g^{\alpha\beta}\widehat{g}_{\alpha\mu}\widehat{g}_{\beta\nu} \\ g^{\alpha\beta} = \widehat{g}^{\alpha\mu}\widehat{g}^{\beta\nu}g_{\mu\nu} \\ \nabla_\alpha \equiv \widehat{\nabla}_\alpha \end{array} \right\}, \tag{30}$$

where $k \geq 3$ and all the fields $\left\{ g_{\mu\nu}(r), \widehat{g}_{\mu\nu}(r), \widehat{R}_{\mu\nu}(r) \right\}$ are by assumption symmetric in the indices $\mu, \nu$. We stress that in $\{Z\}_{E-S}$ it is assumed that the variational metric tensor $Z_1(r) \equiv g_{\mu\nu}(r)$ does not raise/lower indices. Instead, the covariant varied function $g_{\alpha\beta}(r)$ must be transformed into the corresponding contravariant representation $g^{\alpha\beta}(r)$ by means of the fixed metric tensor $\widehat{g}_{\alpha\beta}$ only, namely $g_{\alpha\beta} = \widehat{g}_{\alpha\mu}\widehat{g}_{\beta\nu}g^{\mu\nu}$. Furthermore, $\widehat{\nabla}_\alpha$ denotes the covariant derivative expressed in terms of $\widehat{g}_{\alpha\beta}$.

Based on these premises, the following modified form of the Lagrangian action principle can be established.

*THM.1 - Extended form of the synchronous Lagrangian variational principle*

*Given validity of THM.2 reported in Cremaschini and Tessarotto (2015), let us introduce in the action functional the 4-scalar contribution to the Lagrangian variational density*

$$\Delta L_1\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right) \equiv \frac{1}{2}\kappa\widehat{\nabla}^k g_{\mu\nu}\widehat{\nabla}_k g^{\mu\nu}h\left(Z, \widehat{Z}\right), \tag{31}$$

*where $\widehat{\nabla}_k$ denotes the covariant derivative operator expressed by the Christoffel symbols evaluated in terms of the prescribed metric tensor $\widehat{g}_{\mu\nu}$ which is held constant during synchronous variations. Moreover, the same tensor is used to raise/lower indices in $\Delta L_1\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right)$, so that in particular $\widehat{\nabla}^k \equiv \widehat{g}^{ik}\widehat{\nabla}_i$. Finally, $\kappa$ is a suitable 4-scalar constant. Then, the following propositions hold:*

T1$_1$) *Denoting by $\Delta S_1\left(Z, \widehat{Z}\right)$ the corresponding action functional, one finds that the synchronous variation of $\Delta S_1\left(Z, \widehat{Z}\right)$ performed in the class $\{Z\}_{E-S}$ and evaluated for $g_{\mu\nu} = \widehat{g}_{\mu\nu}$ is identically satisfied for arbitrary variations $\delta g^{\mu\nu}$ in the same functional class.*

T1$_2$) *The dimensional constant $\kappa$ can always be expressed as*

$$\kappa = \frac{c^3}{16\pi G}, \tag{32}$$

*namely the dimensional factor which appears in the Einstein-Hilbert functional.*

T1$_3$) *The variational Lagrangian density in the action functional can always be identified with*

$$\begin{aligned} L_2\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right) &\equiv L_1\left(Z, \widehat{Z}\right) + \Delta L_1\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right) \\ &= -\kappa\left[ g^{\mu\nu}\widehat{R}_{\mu\nu} - \frac{1}{2}\widehat{\nabla}^k g_{\mu\nu}\widehat{\nabla}_k g^{\mu\nu} \right]h\left(Z, \widehat{Z}\right), \end{aligned} \tag{33}$$

*so that the Lagrangian functional becomes finally*

$$S_2\left(Z, \widehat{Z}\right) = \int_{\widehat{D}^4} d\Omega L_2\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right). \tag{34}$$

T1$_4$) *The symbolic Euler-Lagrangian equations corresponding to the variational Lagrangian density $L_2\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right)$ take the form*

$$\widehat{\nabla}_i \frac{\partial L_2\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right)}{\partial\left(\widehat{\nabla}_i g^{\mu\nu}\right)} - \frac{\partial L_2\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right)}{\partial g^{\mu\nu}} = 0, \tag{35}$$

*and are therefore manifestly covariant. Explicit evaluation gives*

$$\widehat{\nabla}_i \left[ -h \widehat{\nabla}^i g_{\mu\nu} \right] - \frac{\partial}{\partial g^{\mu\nu}} \left[ g^{\mu\nu} \widehat{R}_{\mu\nu} h \right] + \frac{1}{2} \widehat{\nabla}^k g_{\mu\nu} \widehat{\nabla}_k g^{\mu\nu} \frac{\partial}{\partial g^{\mu\nu}} h = 0. \tag{36}$$

T1$_5$) *If the solution $\overline{g}^{\mu\nu}(r)$ of Eq.(35), to be denoted as extremal curve of the Lagrangian action $S_2(Z, \widehat{Z})$, is identified with the prescribed metric tensor $\widehat{g}^{\mu\nu}(r)$, then Eq.(36) reduces to the vacuum Einstein equation.*

*Proof* – To prove T1$_1$, let us evaluate the synchronous variational derivative

$$\frac{\delta \Delta S_1 (Z, \widehat{Z})}{\delta g^{\mu\nu}} = \frac{1}{4} g_{\mu\nu} \kappa \widehat{\nabla}^k g_{\alpha\beta} \widehat{\nabla}_k g^{\alpha\beta} - \kappa \widehat{\nabla}_k \left[ h \widehat{\nabla}^k g_{\alpha\beta} \right] = 0. \tag{37}$$

If $g_{\alpha\beta}$ is identified with the prescribed metric tensor $\widehat{g}_{\alpha\beta}$, by construction the lhs of such an equation vanishes identically. To prove the statement T1$_2$ it is sufficient to notice that, from a dimensional analysis it follows that $[\kappa] = \left[ \frac{c^3}{G} \right]$, so that one can choose the customary coefficient of the Einstein Lagrangian density. This permits one to represent the total Lagrangian in terms of the same coefficient $\kappa$. In addition, the normalization coefficient $\frac{1}{2}$ is chosen in analogy with the usual coefficient carried by kinetic terms in field/particle Lagrangians and for convenience with the subsequent Hamiltonian formulation (see for example Wald, 1984). Finally, proposition T1$_3$ is an immediate consequence of T1$_1$ and T1$_2$. To prove T1$_4$ one simply needs to evaluate the variational derivative of the functional $S_2(Z, \widehat{Z})$. It follows that

$$\frac{\delta S_2 (Z, \widehat{Z})}{\delta g^{\mu\nu}} = -\widehat{\nabla}_i \frac{\partial L_2 (Z, \widehat{\nabla}_\mu Z, \widehat{Z})}{\partial \left( \widehat{\nabla}_i g^{\mu\nu} \right)} + \frac{\partial L_2 (Z, \widehat{\nabla}_\mu Z, \widehat{Z})}{\partial g^{\mu\nu}} = 0, \tag{38}$$

which completes the proof. Finally, the proof of proposition T1$_5$ follows immediately by identifying $g^{\mu\nu}(r)$ with $\widehat{g}^{\mu\nu}(r)$ and recalling that by construction $h(\widehat{g}^{\mu\nu}(r)) = 1$, while $\widehat{\nabla}_\alpha \widehat{g}^{\mu\nu}(r) = 0$.

*Q.E.D.*

The following remarks must be added:

1) Based on the results reported in Cremaschini and Tessarotto (2015), one can show that THM.1 above can be readily extended to treat the case of non-vacuum Einstein equations. In particular, this includes the Maxwell equations and the treatment of classical source matter.

2) We stress that the adoption in the variational Lagrangian of the covariant derivative $\widehat{\nabla}_i$ expressed in terms of the same $\widehat{g}_{\alpha\beta}$ is actually consistent with the adoption of the constrained synchronous variational principle. This warrants for example that, since $d\Omega$ is held fixed in terms of $\widehat{g}_{\alpha\beta}$, the integration by part in terms of $\widehat{\nabla}_i$ works in the customary way as is permitted by the Gauss theorem.

3) The variational principle in terms of $S_2(Z, \widehat{Z})$ exhibits the property of manifest covariance.

4) The property of gauge invariance, in the sense pointed out in Cremaschini and Tessarotto (2015), is also warranted.

5) The contribution $\Delta L_1 (Z, \widehat{\nabla}_\mu Z, \widehat{Z})$ considered above can be viewed as representing a new type of gauge invariance, since its extremal value vanishes together with its functional derivatives when evaluated for the extremal fields. As a consequence its introduction does not affect, by construction, the validity of GR and the Einstein equations in particular.

6) Finally, it follows that the Lagrangian differential form and the corresponding action functional can be defined up to an arbitrary gauge contribution of the form

$$\Delta S (\widehat{Z}) = \int_{\widehat{D}^4} d\Omega K(\widehat{Z}, r), \tag{39}$$

where $K$ denotes an arbitrary $4-$scalar which depends on prescribed fields and possibly also on the position $4-$vector. This implies that the Lagrangian $L_2(Z, \widehat{Z})$ can be equivalently represented as

$$L_{2-G} (Z, \widehat{\nabla}_\mu Z, \widehat{Z}) = L_2 (Z, \overline{\nabla}_\mu Z, \overline{Z}) + G(\widehat{Z}, r), \tag{40}$$

where for example the gauge $G(\widehat{Z}, r)$ can be set so that identically

$$L_{2-G}\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right) = 0. \tag{41}$$

As a final comment, we stress that, despite the validity of proposition $T1_5$, $\widehat{g}_{\alpha\beta}$ and the extremal value of $\overline{g}_{\alpha\beta}$, namely the solution of the corresponding Euler-Lagrange equations given above, remain in principle independent. For the same reason, the boundary-value functions $g_{\mu\nu\mathbb{D}}(r)$ should be regarded as independent from the corresponding ones holding for $\widehat{g}_{\alpha\beta}$. The previous considerations notwithstanding, the identification $\widehat{g}_{\alpha\beta} = \overline{g}_{\alpha\beta}$ remains mandatory for recovering the classical form of the Einstein equation.

These conclusions lead us to the following basic result.

*Corollary 1 to THM.1 - Necessary condition for the solution of the Einstein equation*

*Given validity of THM.1, it follows that $\widehat{g}_{\mu\nu}(r)$ is a particular solution of the vacuum Einstein equation subject to the boundary conditions*

$$\widehat{g}_{\mu\nu}(r)|_{\partial\mathbb{D}^4} = g_{\mu\nu\mathbb{D}}(r) \tag{42}$$

*iff it satisfies the boundary-value problem represented by Eq.(36) together with the same boundary conditions. Then, necessarily the extremal curve $\overline{g}_{\mu\nu}(r)$ solution of the same equation in the functional class $\{Z\}_{E-S}$ coincides identically with $\widehat{g}_{\mu\nu}(r)$.*

*Proof* – If $\overline{g}_{\mu\nu}(r)$ is a particular solution of the vacuum Einstein equation subject to the boundary conditions (42), then it follows by construction that, since the covariant derivative $\widehat{\nabla}_k \widehat{g}^{\mu\nu} = 0$ identically, $\widehat{g}_{\mu\nu}(r)$ satisfies also Eq.(36) and this coincides with the Einstein equation. Viceversa, if $\widehat{g}_{\mu\nu}(r)$ satisfies the boundary-value problem represented by Eqs.(36) and (42), then again due to the vanishing of $\widehat{\nabla}_k \widehat{g}^{\mu\nu}$, it obeys also to the Einstein equation. Finally, due to the uniqueness of the solutions of the boundary-value problem associated with the Einstein equation, if $\widehat{g}_{\mu\nu}(r)$ satisfies the same problem, necessarily the extremal curve $\overline{g}_{\mu\nu}(r)$ coincides with $\widehat{g}_{\mu\nu}(r)$.

*Q.E.D.*

## 5. Cosmological Constant and Non-Vacuum Lagrangians

In this section we analyze how THM.1 can be extended to the treatment of the following two cases:

1) The presence of a non-vanishing cosmological constant $\Lambda$.

2) The presence of non-vanishing source matter fields, which determine the non-vacuum Einstein equation in terms of a stress-energy tensor $T_{\mu\nu}$.

The treatment is based on the scheme developed in Cremaschini and Tessarotto (2015) for synchronous Lagrangian variational principles. The solution to points 1 and 2 is provided by the following propositions. The straightforward proofs are left to the reader.

*Corollary 2 to THM.1 - Cosmological constant*

*Given validity of THM.1, in the presence of a non-vanishing cosmological constant $\Lambda$ the variational Lagrangian density becomes*

$$L_{2\Lambda}\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right) = L_2\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right) + L_\Lambda\left(Z, \widehat{Z}\right), \tag{43}$$

*where $L_2\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right)$ is given by Eq.(33), while $L_\Lambda\left(Z, \widehat{Z}\right)$ is defined as*

$$L_\Lambda\left(Z, \widehat{Z}\right) = -2k\Lambda h\left(Z, \widehat{Z}\right). \tag{44}$$

*Then THM.1 holds also for the modified Lagrangian $L_{2\Lambda}\left(Z, \overline{Z}\right)$ in the functional class $\{Z\}_{E-S}$. The term $L_\Lambda\left(Z, \widehat{Z}\right)$ gives the contribution*

$$\frac{\partial L_\Lambda\left(Z, \widehat{Z}\right)}{\partial g^{\mu\nu}} = k\Lambda g_{\mu\nu}. \tag{45}$$

*As a consequence, the Euler-Lagrange equation evaluated for $\overline{g}_{\mu\nu} = g_{\mu\nu}$ is*

$$\overline{R}_{\mu\nu} - \frac{1}{2}\left(\overline{g}^{ik}\overline{R}_{ik}\right)\overline{g}_{\mu\nu} + \Lambda\overline{g}_{\mu\nu} = 0. \tag{46}$$

*Corollary 3 to THM.1 - Non-vacuum Einstein equations*

*Given validity of THM.1, in the presence of source matter fields, the variational Lagrangian density becomes*

$$L_{2F}\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right) = L_2\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right) + L_{1F}\left(Z, \widehat{Z}\right), \tag{47}$$

*where $L_2\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right)$ is given by Eq.(33), while $L_{1F}\left(Z, \widehat{Z}\right)$ is defined according to THM.3 in Cremaschini and Tessarotto (2015) as*

$$L_{1F}\left(Z, \widehat{Z}\right) = L_F\left(Z, \widehat{Z}\right) h\left(Z, \widehat{Z}\right), \tag{48}$$

*with $L_F\left(Z, \widehat{Z}\right)$ to be suitably identified with the appropriate EM and matter sources (see Cremaschini & Tessarotto, 2015). As a consequence, the stress-energy tensor, to be evaluated for $\widehat{g}_{\mu\nu} = \overline{g}_{\mu\nu}$, which enters the extremal non-vacuum Einstein equation is defined as*

$$T_{\mu\nu}(r) = -2\frac{\partial L_{1F}\left(Z, \overline{Z}\right)}{\partial g^{\mu\nu}} + g_{\mu\nu}L_{1F}\left(Z, \overline{Z}\right). \tag{49}$$

*The corresponding synchronous Lagrangian variational principle defined in THM.1 and evaluated in terms of $L_{2F}\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right)$ yields as extremal equation, when letting $\overline{g}_{\mu\nu} = g_{\mu\nu}$, the non-vacuum Einstein equation (1).*

The validity of the two corollaries permits to retain the full form of the non-vacuum Einstein equation, including also the cosmological constant in the subsequent developments. Here we stress that in view of Eq.(41), also in the case of non-vacuum theories, the total Lagrangian density can be prescribed in such a way that its extremal value (obtained letting $Z = \overline{Z}$) vanishes identically.

## 6. Axiomatic Formulation to Continuum Hamiltonian Systems

In view of the considerations outlined above, let us now pose the problem of formulating an axiomatic approach in which all the physical requirements pointed out are included by means of suitable axioms. In particular, this concerns the extension of the theory outlined by De Donder and Weyl to the setting of curved space-time.

Let us now introduce the Axioms required for the construction of a manifest covariant Hamiltonian theory of GR, to be based on such a synchronous Lagrangian formulation. This is achieved by imposing the following physical requirements:

1. Axiom #1: there exists a 4−scalar functional $S_H\left(Z, \widehat{Z}\right)$ defined as

$$S_H\left(Z, \widehat{Z}\right) \equiv \int d\Omega L\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right), \tag{50}$$

   where $L$ is the 4−scalar variational Lagrangian density, which carries at most first-order derivatives $\widehat{\nabla}_\mu Z$ (generalized velocities). The latter is assumed to be a quadratic dependence only. Notice that the covariant derivative in the functional dependence of $L$ is taken to be fixed, namely expressed by the Christoffel symbols represented in terms of the prescribed metric tensor $\widehat{g}_{\alpha\beta}$. By assumption, the corresponding Euler-Lagrange equations associated with the synchronous variational principle must recover the Einstein equation for the metric tensor $g_{\mu\nu}$ when the identification $g_{\mu\nu} = \widehat{g}_{\mu\nu}$ is made everywhere in the curved space-time $\mathbb{D}^4$.

2. Axiom #2: the functional $S_H\left(Z, \widehat{Z}\right)$ is defined in a suitable functional class of variations $\{Z\}$. For this purpose the ensemble field $Z$ is identified with the canonical set $Z = x$, where

$$\{x\} = \left\{q^{\mu\nu}, p^\alpha_{\mu\nu}\right\}, \tag{51}$$

   which belongs to the 50-dimensional phase-space spanned by the state vector $x$. In particular, $p^\alpha_{\mu\nu}$ denotes the canonical momentum conjugate to the coordinate field $q^{\mu\nu}$, to be defined in terms of the Lagrangian density $L$ as

$$p^\alpha_{\mu\nu} = \frac{\partial L}{\partial\left(\widehat{\nabla}_\alpha q^{\mu\nu}\right)}. \tag{52}$$

For completeness we introduce also the corresponding prescribed state $\{\widehat{x}\} = \left\{\widehat{q}^{\mu\nu}, \widehat{p}^{\alpha}_{\mu\nu}\right\}$ in which $\widehat{q}^{\mu\nu}$ identifies $\widehat{Z}$, while $\widehat{p}^{\alpha}_{\mu\nu}$ follows by evaluating the rhs of Eq.(52) for $Z = \widehat{Z}$. The functional class is therefore identified with the synchronous canonical class

$$\{x\} \equiv \left\{ \begin{array}{c} x(r) : x(r) \in C^2(\mathbb{R}^4); \\ x(r)|_{\partial D^4} = x_D(r) \\ \delta(d\Omega) = 0 \end{array} \right\}. \tag{53}$$

Notice that no additional constraint is placed, in analogy with the functional class $\{Z\}_{E-S}$ defined above. Here, however, the ensemble of fields $x$ identifies a symplectic structure.

3. Axiom #3: the continuum Hamiltonian system. The canonical fields $\left\{q^{\mu\nu}, p^{\alpha}_{\mu\nu}\right\}$ belonging to the functional class $\{x\}$ must obey the covariant Hamiltonian equations

$$\frac{\partial H}{\partial p^{\alpha}_{\mu\nu}} = \widehat{\nabla}_\alpha q^{\mu\nu}, \tag{54}$$

$$\frac{\partial H}{\partial q^{\mu\nu}} = -\widehat{\nabla}_\alpha p^{\alpha}_{\mu\nu}, \tag{55}$$

where $H = H\left(q^{\mu\nu}, p^{\alpha}_{\mu\nu}\right)$ is the Hamiltonian density

$$H = p^{\alpha}_{\mu\nu} \widehat{\nabla}_\alpha q^{\mu\nu} - L. \tag{56}$$

The set $\{H, x\}$, with $x$ belonging to the functional class $\{x\}$ and the fields satisfying Eqs.(54) and (55), prescribes a so-called continuum Hamiltonian system.

4. Axiom #4: principle of general covariance. The manifest covariance property of the theory must hold both for all the variational and extremal quantities. This means that the canonical fields $\left\{q^{\mu\nu}, p^{\alpha}_{\mu\nu}\right\}$ must be tensorial fields, so that the variational Hamiltonian density $H$ is a $4-$scalar.

Let us briefly comment the physical motivations behind the Axioms. First we notice that the Axioms generalize the approach by De Donder and Weyl to the curved space-time. In particular, the choice of the synchronous variational principle is a natural one corresponding to their approach in flat-space time, because the $4-$volume element is treated as an invariant in the variational principle and at the same time is a $4-$scalar. As a basic consequence, necessarily the variational Lagrangian density is a $4-$scalar. The feature regarding the tensorial property of the canonical fields is maintained, by replacing the partial derivatives with the covariant ones. Finally, the previous features warrant the manifest covariance property of the theory.

## 7. The Canonical Theory of Einstein Equation

Based on the axiomatic formulation given above, in this section we proceed with the formulation of a manifest covariant Hamiltonian (or canonical) theory for the Einstein field equations. The starting point is the identification of the functional setting, in particular the definitions of the canonical coordinates. In fact the latter prescribe automatically the momenta in terms of the variational Lagrangian density. A possible (non-unique) choice of the Lagrangian density $L$ is provided by THM.1 and its Corollaries. In the case of the vacuum Einstein equation this is prescribed by

$$L\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right) \equiv L_2\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right), \tag{57}$$

where $L_2\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right)$ is given by Eq.(33). Instead, in the case of non-vacuum sources and possibly non-vanishing cosmological constant the Lagrangian density must be defined as

$$L\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right) \equiv L_2\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right) + L_\Lambda\left(Z, \widehat{Z}\right) + L_{1F}\left(Z, \widehat{Z}\right), \tag{58}$$

as given by Corollaries 2 and 3 to THM.1. It is important to remark at this point that the form of the Lagrangian is actually gauge-dependent. In fact, the gauge term can be identified either with an arbitrary real constant, an exact differential, an arbitrary real function of the extremal curves or more generally by arbitrary functions such that the same functions as well as their variations vanish identically for the extremal fields when also the replacement $\overline{g}_{\mu\nu} = g_{\mu\nu}$ is made.

As a consequence, by identifying $Z \equiv q^{\mu\nu} \equiv g^{\mu\nu}$, the canonical momenta are identified with $p_{\mu\nu}^{\alpha} \equiv \Pi_{\mu\nu}^{\alpha}$, where

$$\Pi_{\mu\nu}^{\alpha} = \frac{\partial L\left(Z, \widehat{\nabla}_{\mu}Z, \widehat{Z}\right)}{\partial \left(\widehat{\nabla}_{\alpha}g^{\mu\nu}\right)} = \kappa h\left(Z, \widehat{Z}\right)\widehat{\nabla}^{\alpha}g_{\mu\nu}. \tag{59}$$

Therefore, it follows that the canonical state can be represented as $\{x\} = \left\{Z, \Pi_{\mu\nu}^{\alpha}\right\}$. By construction $\Pi_{\mu\nu}^{\alpha}$ is symmetric in the lower indices, in the sense that $\Pi_{\mu\nu}^{\alpha} = \Pi_{\nu\mu}^{\alpha}$. It must be stressed that in this context, the canonical momentum $\Pi_{\mu\nu}^{\alpha}$ is non-vanishing since $g_{\mu\nu} \neq \overline{g}_{\mu\nu}$ and must be considered as non-extremal. On the other hand, the extremal value of $\Pi_{\mu\nu}^{\alpha}$, namely $\overline{\Pi}_{\mu\nu}^{\alpha} = \kappa \overline{\nabla}^{\alpha}\overline{g}_{\mu\nu}$ vanishes identically (the same property holds also for the prescribed fields). We notice that, thanks to the quadratic dependence required by Axiom #1 for the Lagrangian with respect to the generalized velocity, Eq.(59) is invertible. As a consequence, it recovers a form analogous to the customary relationship between momenta and generalized velocities occurring in relativistic particle dynamics, where the factor $\kappa h\left(Z, \widehat{Z}\right)$ plays the role of the rest mass. This similarity justifies the introduction of the normalization coefficient $\frac{1}{2}$ in the corresponding Lagrangian term.

In view of the previous considerations, the following proposition holds.

*THM.2 - Manifest covariant Hamiltonian theory*

*Given validity of THM.1 and invoking the definition of conjugate canonical momentum given by Eq.(59), it follows that:*

T2$_1$) *The Hamiltonian density $H = H\left(x, \widehat{x}\right)$ associated with the Lagrangian $L\left(Z, \widehat{\nabla}_{\mu}Z, \widehat{Z}\right)$ is provided by the Legendre transform*

$$L\left(Z, \widehat{\nabla}_{\mu}Z, \widehat{Z}\right) \equiv \Pi_{\mu\nu}^{\alpha}\widehat{\nabla}_{\alpha}g^{\mu\nu} - H\left(x, \widehat{x}\right), \tag{60}$$

*where*

$$L\left(Z, \widehat{\nabla}_{\mu}Z, \widehat{Z}\right) = -\kappa\left[g^{\mu\nu}\widehat{R}_{\mu\nu} - \frac{1}{2}\widehat{\nabla}^{\alpha}g_{\mu\nu}\widehat{\nabla}_{\alpha}g^{\mu\nu}\right]h\left(g^{\mu\nu}, \widehat{g}^{\mu\nu}\right). \tag{61}$$

*Here all the hatted quantities are evaluated with respect to the prescribed metric tensor $\widehat{g}^{\mu\nu}$ and the function $h\left(g^{\mu\nu}, \widehat{g}^{\mu\nu}\right)$ is defined above by Eq.(29). Then, written in canonical variables $L\left(Z, \widehat{\nabla}_{\mu}Z, \widehat{Z}\right) = L\left(x, \widehat{x}\right)$, it follows that*

$$L\left(x, \widehat{x}\right) = -\kappa h g^{\mu\nu}\widehat{R}_{\mu\nu} + \frac{1}{2\kappa h}\Pi_{\mu\nu}^{\alpha}\Pi_{\alpha}^{\mu\nu}. \tag{62}$$

T2$_2$) *The Hamiltonian density $H\left(x, \widehat{x}\right)$ is given by*

$$H\left(x, \widehat{x}\right) = \frac{1}{2}\frac{1}{\kappa h}\Pi_{\mu\nu}^{\alpha}\Pi_{\alpha}^{\mu\nu} + \kappa h g^{\mu\nu}\widehat{R}_{\mu\nu}. \tag{63}$$

T2$_3$) *Let us introduce the Hamiltonian action functional*

$$\begin{aligned} S_H\left(x, \widehat{x}\right) &= \int d\Omega L\left(x, \widehat{x}\right) \\ &= \int d\Omega\left[\Pi_{\mu\nu}^{\alpha}\widehat{\nabla}_{\alpha}g^{\mu\nu} - H\left(x, \widehat{x}\right)\right], \end{aligned} \tag{64}$$

*where $L\left(x, \widehat{x}\right)$ is given by Eq.(62) and $\{x\} = \left\{Z, \Pi_{\mu\nu}^{\alpha}\right\}$ is prescribed so that $Z$ belongs to the functional class $\{Z\}_{E-S}$ defined by Eq.(30), while $\Pi_{\mu\nu}^{\alpha}$ belongs to the set*

$$\{\Pi\}_{E-S} \equiv \left\{ \begin{array}{c} \Pi_{\mu\nu}^{\alpha}(r) \in C^k\left(\mathbb{D}^4\right) \\ \widehat{\Pi}_{\mu\nu}^{\alpha}(r) = 0 \\ \Pi_{\mu\nu}^{\alpha}(r)|_{\partial\mathbb{D}^4} = \Pi_{\mu\nu\mathbb{D}}^{\alpha}(r) \\ \delta\widehat{\Pi}_{\mu\nu}^{\alpha}(r) = \delta\overline{\Pi}_{\mu\nu}^{\alpha}(r) = 0 \end{array} \right\}, \tag{65}$$

*where $\overline{\Pi}_{\mu\nu}^{\alpha}(r)$ is the extremal curve solution of the corresponding continuum Hamilton equation with prescribed boundary conditions. Then, the synchronous Hamiltonian variational principle*

$$\delta S_H\left(x, \widehat{x}\right) \equiv \left. \frac{d}{d\alpha}\Psi(\alpha)\right|_{\alpha=0} = 0 \tag{66}$$

*is required to hold for arbitrary independent variations $\delta g^{\mu\nu}(r)$ and $\delta\Pi^\alpha_{\mu\nu}(r)$ in the respective functional classes. Here $\Psi(\alpha)$ is the smooth real function $\Psi(\alpha) = S_H(\widehat{Z + \alpha\delta Z}, \widehat{Z})$, being $\alpha \in ]-1, 1[$ to be considered independent of $g^{\mu\nu}$, $\Pi^\alpha_{\mu\nu}$ and $r^\mu$. Furthermore, $\delta g^{\mu\nu}(r)$ and $\delta\Pi^\alpha_{\mu\nu}(r)$ are defined respectively as in Eq.(26) and*

$$\delta\Pi^\alpha_{\mu\nu}(r) = \Pi^\alpha_{\mu\nu}(r) - \Pi^\alpha_{\mu\nu,extr}(r), \tag{67}$$

*where here $\Pi^\alpha_{\mu\nu,extr}(r) \equiv \overline{\Pi}^\alpha_{\mu\nu}(r)$.*

*The corresponding variational derivatives yield the continuum Hamilton equations*

$$\frac{\delta S_H(x,\widehat{x})}{\delta g^{\mu\nu}(r)} \quad \equiv \quad -\frac{\partial H(x,\widehat{x})}{\partial g^{\mu\nu}} - \widehat{\nabla}_\alpha\Pi^\alpha_{\mu\nu} = 0, \tag{68}$$

$$\frac{\delta S_H(x,\widehat{x})}{\delta\Pi^\alpha_{\mu\nu}(r)} \quad \equiv \quad \widehat{\nabla}_\alpha g^{\mu\nu} - \frac{\partial H(x,\widehat{x})}{\partial\Pi^\alpha_{\mu\nu}(r)} = 0. \tag{69}$$

*Written explicitly, these become*

$$\widehat{\nabla}_\alpha\Pi^\alpha_{\mu\nu} \quad = \quad -\frac{\partial H(x,\widehat{x})}{\partial g^{\mu\nu}}, \tag{70}$$

$$\widehat{\nabla}_\alpha g^{\mu\nu} \quad = \quad \frac{1}{\kappa h}\Pi^{\mu\nu}_\alpha, \tag{71}$$

*so that the second one recovers as usual the definition of the canonical momentum. These equations coincide with the Euler-Lagrange equation (36) given in THM.1.*

T2$_4$) *In validity of the* collapse condition, *for which $\widehat{g}_{\alpha\beta}$ is identified with the extremal solution, the previous equations reduce respectively to the Einstein equation and the condition $\overline{\Pi}^\alpha_{\mu\nu} = 0$.*

T2$_5$) *The Hamiltonian density $H(x,\widehat{x})$ given by Eq.(63) can always be replaced by the equivalent 4-scalar density*

$$H_G(x,\widehat{x}) = H(x,\widehat{x}) + G(\widehat{x}, r), \tag{72}$$

*where the gauge $G(\widehat{x}, r)$ can be set so that identically*

$$H_G(\overline{x},\overline{x}) = 0. \tag{73}$$

*Proof* - The proof of proposition T2$_1$) follows directly from Axioms #2-#3 once the Lagrangian is identified according to Eq.(57), i.e. based on THM.1. Then the representation of $L(x,\widehat{x})$ in terms of canonical variables in Eq.(62) follows by the validity of Eq.(59). Similarly, the proof of T2$_2$) follows from elementary algebra by combining Eqs.(59), (60) and (62).

To detail the proof of proposition T2$_3$) it is sufficient to consider the explicit evaluation of the Euler-Lagrange equations. In particular, from the synchronous variational principle, the variational derivative with respect to $\delta g^{\mu\nu}(r)$ gives identically Eq.(68), where

$$\frac{\partial H(x,\widehat{x})}{\partial g^{\mu\nu}} = \frac{1}{4}\frac{1}{\kappa h^2}\Pi^\alpha_{\gamma\beta}\Pi^{\gamma\beta}_\alpha g_{\mu\nu} + \kappa h\widehat{R}_{\mu\nu} - \frac{1}{2}\kappa g^{\alpha\beta}\widehat{R}_{\alpha\beta}g_{\mu\nu}. \tag{74}$$

This provides the explicit representation for the rhs of Eq.(70). To prove the identity with Eq.(36) one has first to substitute Eq.(71) in the first term on the lhs of Eq.(70) and then evaluate explicitly the rhs of the same equation in terms of the Hamiltonian density (to be regarded as a function of $g^{\mu\nu}$ and $\widehat{\nabla}_\alpha g^{\mu\nu}$). This gives

$$\widehat{\nabla}_i\left[h\widehat{\nabla}^i g_{\mu\nu}\right] = -\frac{\partial}{\partial g^{\mu\nu}}\left[g^{\alpha\beta}\widehat{R}_{\alpha\beta}h\right] + \frac{1}{2}\widehat{\nabla}^k g_{\alpha\beta}\widehat{\nabla}_k g^{\alpha\beta}\frac{\partial}{\partial g^{\mu\nu}}h, \tag{75}$$

which completes the proof of T2$_3$). The proof of proposition T2$_4$) is achieved by noting first that, by definition $\overline{\nabla}^\alpha\overline{g}_{\mu\nu} = 0$, so that Eq.(71) implies $\overline{\Pi}^\alpha_{\mu\nu} = 0$. As a consequence, the lhs of Eq.(70) vanishes identically, while, upon identifying $\widehat{g}_{\alpha\beta}$ with the extremal solution and recalling that $h\left(\widehat{g}_{\alpha\beta}\right) = 1$, the rhs gives

$$\frac{\partial H(x,\widehat{x})}{\partial g^{\mu\nu}}\bigg|_{\overline{g}_{\mu\nu},\overline{\Pi}^\alpha_{\mu\nu}} = \kappa\overline{R}_{\mu\nu} - \frac{1}{2}\kappa\overline{g}^{\alpha\beta}\overline{R}_{\alpha\beta}\overline{g}_{\mu\nu} = 0, \tag{76}$$

which coincides with the correct Einstein vacuum equation. Finally, the proof of T2$_5$) follows invoking the gauge invariance property of the corresponding Lagrangian density (see Eq.(41)).

*Q.E.D.*

For completeness, we notice that the Lagrangian density $L$ given in Eq.(60) can obviously be considered as a function of the canonical state once the exchange term (namely the first term on the rhs of the same equation) is expressed in terms of the canonical variables.

Let us now consider the case of the non-vacuum Einstein equation in order to determine the corresponding form of the Hamiltonian, required for the validity of THM.2 in such a case. The recipe is elementary, by noting that by assumption the source terms do not carry contributions to the canonical momenta, as they depend only on $\widehat{g}_{\alpha\beta}$, $g_{\alpha\beta}$ and $r$. For completeness, we include the expression due to the possible non-vanishing cosmological constant and the source fields. Invoking the Corollaries 2 and 3 to THM.1 given above, the resulting Hamiltonian is found to be

$$H_{tot}\left(x,\widehat{x}\right) = H\left(x,\widehat{x}\right) - L_{\Lambda}\left(Z,\widehat{Z}\right) - L_{1F}\left(Z,\widehat{Z}\right), \tag{77}$$

where $L_{\Lambda}\left(Z,\widehat{Z}\right)$ is given by Eq.(44) and $L_{1F}\left(Z,\widehat{Z}\right)$ by Eq.(48). See also the work given in Cremaschini and Tessarotto (2015) for further related discussion concerning the specific realization of the Lagrangian density $L_{1F}\left(Z,\widehat{Z}\right)$.

## 8. Classical 4−Vector Poisson Brackets

In this section we develop the formalism of Poisson brackets appropriate for the treatment of classical fields in the context of GR according to the DeDonder-Weyl formalism. In this respect, the peculiarity of the covariant canonical formalism developed here must be stressed. In fact, our goal is to reach a description of the canonical equations (70) and (71) in terms of local Poisson brackets. This choice departs from the traditional approach for continuum fields used in part of the previous literature (Goldstein, 1980).

Let us consider first two 4−scalars $A$ and $B$ and then two arbitrary tensor fields $\mathbf{A} \equiv A^{\alpha_1..\alpha_n}$ and $\mathbf{B} \equiv B_{\beta_1..\beta_m}$, generally of different order, so that $n \neq m$, and all to be considered smoothly dependent only on the canonical set $\{Z\}$ defined by Eq.(51). Then, respectively for $(A, B)$ and $(\mathbf{A}, \mathbf{B})$ the canonical Poisson brackets in terms of $\{Z\}$ are defined in terms of the 4−vector operators

$$[A, B]_{x,j} \equiv \frac{\partial A}{\partial g^{\mu\nu}}\frac{\partial B}{\partial \Pi_{\mu\nu}^{j}} - \frac{\partial A}{\partial \Pi_{\mu\nu}^{j}}\frac{\partial B}{\partial g^{\mu\nu}}, \tag{78}$$

$$[\mathbf{A}, \mathbf{B}]_{x,j} \equiv \frac{\partial \mathbf{A}}{\partial g^{\mu\nu}}\frac{\partial \mathbf{B}}{\partial \Pi_{\mu\nu}^{j}} - \frac{\partial \mathbf{A}}{\partial \Pi_{\mu\nu}^{j}}\frac{\partial \mathbf{B}}{\partial g^{\mu\nu}}, \tag{79}$$

where in the second equation the two indices $\mu$ and $\nu$ saturate in both terms. *On the contrary, the index $j$ does not saturate, a feature which is characteristic of the DeDonder-Weyl covariant theory, in which coordinates and momenta have different tensorial ranks.* Hence, in the first case the rhs is a 4−vector, while in the second one the rhs is a tensor of dimension $n + m + 1$. The fundamental Poisson brackets holding for the canonical set are therefore

$$\left[g^{\beta\gamma}, \Pi_{\beta\gamma}^{\alpha}\right]_{x,j} \equiv \frac{\partial g^{\beta\gamma}}{\partial g^{\mu\nu}}\frac{\partial \Pi_{\beta\gamma}^{\alpha}}{\partial \Pi_{\mu\nu}^{j}} - \frac{\partial g^{\beta\gamma}}{\partial \Pi_{\mu\nu}^{j}}\frac{\partial \Pi_{\beta\gamma}^{\alpha}}{\partial g^{\mu\nu}} = \delta_{\mu}^{\beta}\delta_{\nu}^{\gamma}\delta_{j}^{\alpha}, \tag{80}$$

$$\left[g^{\beta\gamma}, g_{\beta\gamma}\right]_{x,j} \equiv \frac{\partial g^{\beta\gamma}}{\partial g^{\mu\nu}}\frac{\partial g_{\beta\gamma}}{\partial \Pi_{\mu\nu}^{j}} - \frac{\partial g^{\beta\gamma}}{\partial \Pi_{\mu\nu}^{j}}\frac{\partial g_{\beta\gamma}}{\partial g^{\mu\nu}} = 0, \tag{81}$$

$$\left[\Pi_{\alpha}^{\beta\gamma}, \Pi_{\beta\gamma}^{\alpha}\right]_{x,j} \equiv \frac{\partial \Pi_{\alpha}^{\beta\gamma}}{\partial g^{\mu\nu}}\frac{\partial \Pi_{\beta\gamma}^{\alpha}}{\partial \Pi_{\mu\nu}^{j}} - \frac{\partial \Pi_{\alpha}^{\beta\gamma}}{\partial \Pi_{\mu\nu}^{j}}\frac{\partial \Pi_{\beta\gamma}^{\alpha}}{\partial g^{\mu\nu}} = 0. \tag{82}$$

In terms of the fundamental brackets, the Poisson brackets with the non-vacuum Hamiltonian density $H_{tot} \equiv H_{tot}\left(g^{\mu\nu}, \Pi_{\mu\nu}^{\alpha}\right)$ defined by Eq.(77) are

$$\left[g^{\beta\gamma}, H_{tot}\right]_{x,j} \equiv \frac{\partial g^{\beta\gamma}}{\partial g^{\mu\nu}}\frac{\partial H_{tot}}{\partial \Pi_{\mu\nu}^{j}} = \frac{\partial H_{tot}}{\partial \Pi_{\beta\gamma}^{j}}, \tag{83}$$

$$\left[\Pi_{\beta\gamma}^{\alpha}, H_{tot}\right]_{x,j} \equiv -\frac{\partial \Pi_{\beta\gamma}^{\alpha}}{\partial \Pi_{\mu\nu}^{j}}\frac{\partial H_{tot}}{\partial g^{\mu\nu}} = -\delta_{j}^{\alpha}\frac{\partial H_{tot}}{\partial g^{\beta\gamma}}. \tag{84}$$

As a consequence, the continuum canonical equations (70) and (71) can be extended to the non-vacuum case and equivalently represented in terms of the 4-vector Poisson brackets as

$$\widehat{\nabla}_\alpha \Pi^\alpha_{\mu\nu} = \left[ \Pi^\alpha_{\mu\nu}, H_{tot} \right]_{x,\alpha}, \tag{85}$$

$$\widehat{\nabla}_\alpha g^{\mu\nu} = \left[ g^{\mu\nu}, H_{tot} \right]_{x,\alpha}. \tag{86}$$

The fundamental Poisson brackets Eqs.(82)-(82) together with the continuum Hamilton equations (85) and (86) display the Hamiltonian structure characteristic of the synchronous variational principle given in THM.2.

### 9. Discussion and Physical Implications

Let us now analyze the main physical aspects and the interpretation of the theory developed here.

The first basic feature is that the boundary-value problem associated with the Einstein equation (both in vacuum and non-vacuum) has been shown to be characterized by an intrinsic Hamiltonian structure. Remarkably, such a property arises when a manifestly-covariant variational approach is adopted based on the synchronous variational principles given in THMs.1 and 2. In particular, in this framework both the Lagrangian and the Hamiltonian structures of the theory naturally emerge when appropriate superabundant tensorial variables are adopted. In the case of the Lagrangian treatment, this involves the adoption of a 10-dimensional configuration space spanned by the symmetric variational tensor field $Z \equiv g^{\mu\nu}(r)$. Instead, the corresponding phase-space is determined by the Lagrangian state, namely the 50-dimensional tensor field $\left( Z, \widehat{\nabla}_\mu Z \right)$. In fact, based on the continuum Lagrangian representation determined according to THM.1, the canonical formalism is obtained following the DeDonder-Weyl approach, namely introducing canonical momenta and the Hamiltonian density. This means that also the canonical state $\{x\} = \left\{ Z, \Pi^\alpha_{\mu\nu} \right\}$ is 50-dimensional.

A notable feature of the variational approach, both for the Lagrangian and Hamiltonian treatments, is the use of a synchronous variational principle. This is realized by introducing in the action functional a prescribed metric tensor $\widehat{g}^{\beta\gamma}$, which remains unaffected by the synchronous variations and is considered independent of the corresponding variational tensor field $g^{\beta\gamma}$ as well its extremal value. The latter is determined by the solution of the boundary-value problem of the Euler-Lagrange equations determined by the variational principle.

Starting point is the physical interpretation of the Lagrangian variational approach given in THM.1. In fact, the form of the modified action functional adopted here, which differs from that given in Cremaschini and Tessarotto (2015), has been introduced specifically to permit the corresponding Hamiltonian formulation given in THM.2. In particular, in the Lagrangian density, the new term $\Delta L_1 \left( Z, \widehat{\nabla}_\mu Z, \widehat{Z} \right)$ given by Eq.(31) plays the role analogous to the kinetic energy in particle dynamics, being quadratic in the generalized velocity $\widehat{\nabla}_k g^{\mu\nu}$. As shown by Corollary 1 to THM.1, the Einstein equation is recovered for the extremal $g^{\mu\nu}(r)$ when the prescribed metric tensor $\widehat{g}^{\mu\nu}(r)$ is suitably identified. Remarkably, however, the Lagrangian structure holds for arbitrary smooth metric tensor fields $\widehat{g}^{\mu\nu}(r)$. In such a case $g^{\mu\nu}(r)$ obeys an Euler-Lagrange equation which generally departs from the standard Einstein form.

Let us now consider the Hamiltonian formulation given by THM.2. Basic feature of the treatment is that the Lagrangian density is regular, in analogy with the Lagrangian treatment well-known in particle dynamics. Indeed, the transformation from the Lagrangian state to the corresponding Hamiltonian one is non-singular, being realized by a smooth bijection. As for the Lagrangian treatment, also the Hamiltonian structure emerges only when tensorial superabundant variables are adopted in the framework of the synchronous principle. The feature is essential because it permits to display the contribution of the canonical momenta when they are non-vanishing. This property occurs when $g_{\mu\nu} \neq \widehat{g}_{\mu\nu}$, namely the variational curves differ from the prescribed metric tensor. In fact, from the definition (59), in such a case one has that $\Pi^\alpha_{\mu\nu} \neq 0$, since generally $\widehat{\nabla}^\alpha g_{\mu\nu} \neq 0$. This feature turns out to be instrumental for the construction of the continuum canonical equations given in THM.2 as well as the subsequent formulation in terms of Poisson brackets.

On the other hand, once the constraint $g_{\mu\nu} \equiv \widehat{g}_{\mu\nu}$ is set on the extremal equations, the Hamiltonian structure $\{H_{tot}, x\}$ is only apparently lost, because of the vanishing of the canonical momenta in such a case. In fact, in this limit the state $x$ collapses to the 10-dimensional hypersurface of the phase-space spanned by the canonical state $x = \left[ g_{\mu\nu}, 0 \right] \equiv x_0$ having identically-vanishing momenta. Nevertheless, inspection of the Hamilton equations shows that they actually hold also when $x = x_0$ is set, namely the partial derivatives of the Hamiltonian density are evaluated at such state. Hence, the Hamiltonian structure indeed is preserved also in this case. This conclusion is of fundamental importance, for its physical implication. In fact it identifies uniquely the covariant continuum

Hamiltonian system associated with the Einstein equation. We remark that this result is ultimately due to the validity of the synchronous variational principle formulated in THM.2, which is based on the introduction of the constrained functional class . Such a feature is not unique in classical physics. In fact, in this regard, one may view the metric-compatibility condition $\overline{\nabla}_\alpha \overline{g}^{\mu\nu} = 0$ as playing a role analogous to the mass-shell condition $u^\mu u_\mu = 1$ in the so-called super-Hamiltonian approaches to classical particle dynamics. In both cases, the constraints do not affect the validity of the corresponding Hamiltonian structure, since they hold only for the extremal curves.

It must be remarked that the adoption of the superabundant field variables $\left\{ g_{\mu\nu}, \widehat{g}_{\mu\nu} \right\}$ permits one to avoid the introduction of extra-dimensions in the space-time. The physical interpretation of such a representation is as follows: 1) $g_{\mu\nu}$ is a *tensor field*, whose dynamical equations are provided by the Euler-Lagrange equations determined by the synchronous variational principle. In this sense, in the action functional $g_{\mu\nu}$ has initially no geometrical interpretation, since it does not raise or lower indices nor it appears in the covariant derivatives or the Ricci tensor $\widehat{R}_{\mu\nu}$. 2) Instead, $\widehat{g}_{\mu\nu}$ is the *geometrical field*. In fact, it determines a number of relevant geometric properties: the invariant $4-$volume element, the covariant derivatives and the Ricci tensor $\widehat{R}_{\mu\nu}$ and finally it raises/lowers tensor indices. 3) The contribution $g^{\mu\nu}\widehat{R}_{\mu\nu}$ appearing in both the Lagrangian and Hamiltonian densities can be interpreted as an effective coupling term between the physical field $g^{\mu\nu}$ and the geometrical quantity $\widehat{R}_{\mu\nu}$. In this sense, this term is similar to the well-known coupling term $A_\mu J^\mu$ occurring in the corresponding Lagrangian formulation for the EM field.

Finally, we notice that there is another possible interpretation of the formalism developed here, and in particular the distinction between the metric tensors $g_{\mu\nu}$ and $\widehat{g}_{\mu\nu}$. This is provided by the so-called *induced gravity* (or *emergent gravity*), namely the idea that the space-time geometrical properties emerge as a mean field approximation of an underlying microscopic stochastic or quantum degrees of freedom. Indeed, it is tempting to view $\widehat{g}_{\mu\nu}$ as a macroscopic mean field emerging on a (fluctuating) background represented by the variational field $g_{\mu\nu}$. Such an idea is naturally connected with the process of quantization. As shown in the present paper, this gives rise to a new type of action which depends both on $g_{\mu\nu}$ and $\widehat{g}_{\mu\nu}$ and permits the identification of a covariant Hamiltonian structure associated with the classical gravitational field. Remarkably, the customary non-vacuum Einstein equation *emerges* when the Hamiltonian structure is collapsed on the 10-dimensional subset of phase-space on which canonical momenta vanish identically.

## 10. Conclusions

Historically, the problem of the identification of the manifestly covariant Hamiltonian structure associated with the Einstein equation of General Relativity (both in vacuum and non-vacuum) and which retains also the correct gauge transformation properties, has remained apparently unsolved to date. Actual reasons remain substantially a guesswork. For example one such possible conjecture lies in the strong influence set by some of the most-distinguished authors dealing with the subject. One example among them can probably be ascribed to Dirac himself. This refers, in particular, to his famous 1948 treatment of relativistic Hamiltonian systems based on constrained dynamics. In his approach, in fact, although overall covariance of the Hamilton equations is still warranted, explicit non-tensor Hamiltonian variables were adopted. On the other hand, there is good evidence that Dirac approach had actually a deep influence in the subsequent literature. This might possibly explain how and why the opposite view took stand, namely the concept that only manifestly non-covariant Hamiltonian treatments of GR are actually possible.

Contrary to such a viewpoint, in this paper the route of the manifestly covariant Hamiltonian theory has been pursued to a full extent, enabling us to establish a manifestly covariant Hamiltonian formulation for Einstein equation. The goal has been reached by generalizing the so-called "DeDonder-Weyl" formalism in the context of curved space-time and the introduction of a suitable kind of synchronous Hamiltonian variational principles extending analogous Lagrangian formulations earlier recently developed in the literature.

A number of notable new features emerge. The first one is the adoption of a constrained variational formulation, the constraints being intrinsically related to the notion of synchronous variation and of synchronous variational principle.

The second one is the manifest covariance property of the theory at all levels, i.e., beginning from the definition of the action functional itself, its Lagrangian density, the prescription of the Hamiltonian variables as well as, finally, the construction of the Hamiltonian Euler-Lagrange equations.

The third one involves the adoption of a superabundant canonical-variable approach. In this reference, a critical element turns out to be the adoption of a synchronous variational principle. This feature, in fact, permits one to

determine the Einstein equation via a Lagrangian formulation expressed in superabundant variables. As a consequence, it has been shown that the Lagrangian equation can be cast also in the equivalent manifestly-covariant Hamiltonian form and in terms of continuum Poisson brackets.

The fundamental key feature which arises is that the Hamiltonian structure of the Einstein equations emerges in a natural way when the manifestly covariant approach is realized.

These features suggest the theory presented here as a promising research topic, being susceptible of a plethora of potential applications, besides its natural framework, i.e., GR. In particular, possible subsequent developments range from the investigation of the Hamiltonian and Hamilton-Jacobi structure of GR, to cosmology as well as relativistic quantum field theory and quantum gravity.

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