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How Can Non-Relativistic Projectile Motion Remain So in the Relativistic Limit?

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Abstract

Projectile motion in classical physics can be defined as a two-dimensional motion with constant acceleration in one direction and uniform motion in another direction. In classical mechanics this motion can be achieved if a force is applied in one direction and with some initial velocity in another direction. In relativistic dynamics, the equations of motion in both directions are coupled to each other and surprisingly the object experiences varying accelerations in both directions. We can say the non-relativistic projectile motion is not a projectile motion at all in relativistic physics. In this paper the conditions for the projectile motion in the relativistic regime are discussed. We discuss the two cases of constant force and constant acceleration. Another goal of this paper is that to show a computational example can be discussed in class for students to learn a topic in physics and improve their computational skills.

Keywords: special relativity, non-relativistic limit, projectile motion, constant force, constant acceleration

1. Introduction

The motion of an object subject to a constant force in one direction, for example $\hat{y}$, and some initial velocity in another direction, $\hat{x}$, is known as projectile motion in Newtonian mechanics. The trajectory of the motion is a parabola. The x component of the velocity remains constant and the y component increases as result of the force applied in that direction. This is a classical example of two-dimensional motion that is discussed in introductory physics. This is a basic and simple two-dimensional problem which can be discussed relativistic term. One-dimensional motion, such as that subject to constant applied force, sinusoidal time-dependent applied force, linear spring force and applied force as an inverse square law have been discussed by the authors (Asadi-Zeydabadi & Sadun, 2013). In the context of relativistic mechanics this problem is as simple as the force mentioned cases but what makes it interesting is the coupling nature of the two-dimensional problem in relativistic physics. The simplicity of this problem helps us to educate students about the nature of the relativistic physics and especially the coupling between x and y components of the equation of motion. In non-relativistic mechanics, the projectile motion is mathematically two uncoupled simple differential equations. In the relativistic case, the x component of the velocity does not remain constant even though there is no force in this direction. This is a direct consequence of the application of relativistic physics. The relativistic falling or projectile motion has been discussed by using general relativity (Lapidus, 1972; Adler & Robert, 1991; Strand, 1993). In this paper, we will consider this motion according to the special relativity only. The two separate cases of constant force and constant acceleration are discussed. Case one (applying constant force) also has been discussed elsewhere (Landaus & Lifshitz, 1975; Lapidus, 1972; Naddy, Dudley, & Haaland, 2002; Shahin, 2006; Price, 2005). For this case (applying constant force) analytical solutions are given elsewhere (Landaus & Lifshitz, 1975; Lapidus, 1972). Our interest is to integrate physics courses with computational method. This is one example of this style of teaching, so we shall solve the problem numerically. The question of "How can non-relativistic projectile motion remain so in the relativistic limit?" can be addressed in the modern physics classes to understand a fundamental transition from the classical to relativistic physics.

2. The Equation of Motion in Special Relativity

The relativistic equation of motion is given by,

$$\vec{p} = m\gamma\vec{u} \tag{1}$$

where $\gamma = 1/\sqrt{1-(\vec{u}.\vec{u}/c^2)}$.

For the two dimensional case, this equation can be written as the following

$$p_x = m\gamma u_x = m\frac{u_x}{\sqrt{1-\frac{u_x^2+u_y^2}{c^2}}} \tag{2-a}$$

$$p_y = m\gamma u_y = m\frac{u_y}{\sqrt{1-\frac{u_x^2+u_y^2}{c^2}}} \tag{2-b}$$

By taking the derivative of the above equations we have:

$$\frac{dp_x}{dt} = m\gamma\frac{du_x}{dt} + m\gamma^3 u_x\left(\frac{u_x\frac{du_x}{dt}+u_y\frac{du_y}{dt}}{c^2}\right) \tag{3-a}$$

$$\frac{dp_y}{dt} = m\gamma\frac{du_y}{dt} + m\gamma^3 u_y\left(\frac{u_x\frac{du_x}{dt}+u_y\frac{du_y}{dt}}{c^2}\right) \tag{3-b}$$

The above equations can be written with some arrangement in terms of the x and y components of the force

$$\left(\frac{dp_x}{dt} = F_x, \frac{dp_y}{dt} = F_y\right)$$

$$\frac{du_x}{dt} = \frac{(F_x/m)+(u_y/c)\left((F_x/m)(u_y/c)-(F_y/m)(u_x/c)\right)\gamma^2}{(1+((u_x^2+u_y^2)/c^2)\gamma^2)\gamma} \tag{4-a}$$

$$\frac{du_y}{dt} = \frac{(F_y/m)+(u_x/c)\left((F_y/m)(u_x/c)-(F_x/m)(u_y/c)\right)\gamma^2}{(1+((u_x^2+u_y^2)/c^2)\gamma^2)\gamma} \tag{4-b}$$

We discuss two cases: constant force and constant acceleration. In non-relativistic physics, both systems are equivalent and are known as projectile motion. In the relativistic case, they are different, and the second case is projectile motion.

3. Constant Force

Suppose a constant force, F_0, acts on an object only in the y direction ($F_x = \frac{dp_x}{dt} = 0$, and $F_y = \frac{dp_y}{dt} = F_0$) and the initial velocity of the object is in the $\hat{x}$ direction ($u_{0x} = u_0$, and $u_{0y} = 0$). Substituting $F_x = 0$, and $F_y = F_0$ into equations (4-a) and (4-b) one will get:

$$\frac{du_x}{dt} = -\frac{(F_0/m)(u_y u_x/c^2)\gamma}{(1+((u_x^2+u_y^2)/c^2)\gamma^2)} \tag{5-a}$$

$$\frac{du_y}{dt} = \frac{(F_0/m)\left(1+(u_x^2/c^2)\gamma^2\right)}{(1+((u_x^2+u_y^2)/c^2)\gamma^2)\gamma} \tag{5-b}$$

In the non-relativistic limit ($\gamma \to 1$, $u_y u_x/c^2 \to 0$, $u_x^2/c^2 \to 0$ and $\left(u_x^2+u_y^2\right)/c^2 \to 0$) the above equations are reduced to $\frac{du_x}{dt} = 0$ and $\frac{du_y}{dt} = \frac{F_0}{m}$, which is consistent with the Newtonian mechanics formulation.

4. Constant Acceleration

The condition for the projectile motion in special relativity can be found by substituting $a_x = du_x/dt = 0$ and $a_y = du_y/dt = a_0$, and therefore $u_x = u_{x0}$ and $u_y = u_{y0} + a_0 t$, into equations (3-a) and (3-b):

$$F_x(t) = m\gamma_t^3\frac{u_{x0}(u_{y0}+a_0 t)}{c^2}a_0 \tag{6-a}$$

$$F_y(t) = m\gamma_t\left(1+\gamma_t^2\frac{(u_{y0}+a_0 t)^2}{c^2}\right)a_0 \tag{6-b}$$

where $\gamma_t = \frac{1}{\sqrt{1-\frac{u_x^2+u_y^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{u_{x0}^2+(u_{y0}+a_0 t)^2}{c^2}}}$.

In the non-relativistic limit ($\gamma_t \to 1$, $\frac{u_{x0}(u_{y0}+a_0 t)}{c^2} \to 0$, and $\frac{(u_{y0}+a_0 t)^2}{c^2} \to 0$) the above equations are reduced to $F_x(t) = 0$ and $F_y(t) = ma_0$ (no force in $\hat{x}$ direction and constant force in $\hat{y}$ direction), which is again consistent with the Newtonian mechanics formulation.

If the x and y components of an applied force on an object satisfy the above equations then, the motion is projectile motion with uniform motion (no acceleration) in $\hat{x}$ direction and a constant acceleration of a_0 in $\hat{y}$ direction. Notice that, a_0, acceleration in $\hat{y}$ direction appears in both equations, and the x and y components of the force depend on the x and y components of the velocity. The requirements for projectile motion imply a constant x component of the velocity, $u_x = u_{0x}$, and linear time dependence of the y component of the velocity, $u_y = u_{y0} + a_0 t$ that obviously could violate $u^2 = u_x^2 + u_y^2 < c^2$ when time passes a certain value. These equations are valid only for $u_{x0}^2 + (u_{y0} + a_0 t)^2 - c^2 < 0$ which implies that the x and y components of an applied force which are given in 6-a and 6-b are valid for $t < \left(\sqrt{c^2 - u_{x0}^2} - u_{y0}\right)/a_0$. In summary, a projectile motion (zero acceleration in one direction and a constant acceleration in other direction) in the relativistic limit can occur if the x and y components of the applied force satisfy equations (6-a) and (6-b) and if time is less than $\left(\sqrt{c^2 - u_{x0}^2} - u_{y0}\right)/a_0$. The force that is described by equations (6-a) and (6-b) in the non-relativistic limit ($\gamma_t \to 1$) causes a linear acceleration in the $\hat{x}$ direction and a quadratic acceleration in time in the $\hat{y}$ direction.

5. Results

The equations of motion, (5-a) and (5-b), have been solved numerically. We have used, $c = 1, \frac{F_0}{m} = a_{Newton} = 1$, and with the initial conditions of $u_{0x} = u_0 = 0.1$, and $u_{0y} = 0$. The results are showed in figure 1.

Since the magnitude of the velocity cannot exceed the speed of light, $u^2 = u_x^2 + u_y^2 < c^2$. Therefore as the magnitude of the y component of the velocity increases as a result of applying force, the x component must decrease to satisfy $u_x^2 + u_y^2 < c^2$ according to equations (5-a) and (5-b). In the long term, as the y component of velocity approaches the speed of light, c, the x component approaches zero. In the non-relativistic limit, the x component of velocity remains constant, and there is no limit on the magnitude of the y component of the velocity, and on the speed of the object.

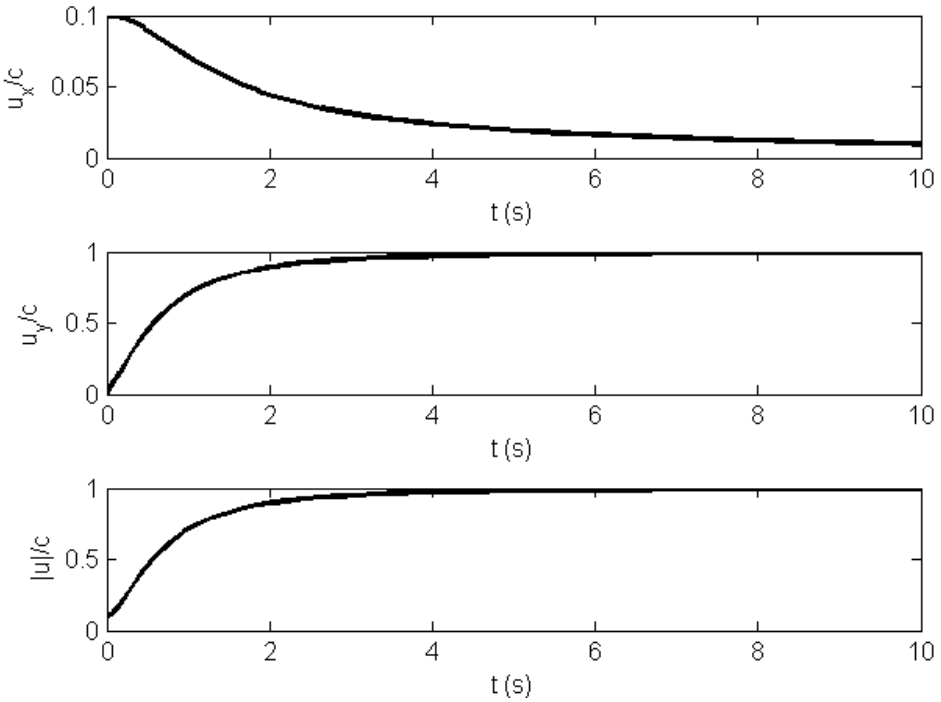

Figure 1. The x and y component and the absolute value of the velocity

Since there is no force in the $\hat{x}$ direction, the x component of the acceleration is initially zero. Then, as a result of the change on the x component of the velocity, the object experiences acceleration (deceleration) in $\hat{x}$ direction. This acceleration depends on both x and y components of the velocity (equations 5-a and 5-b), and consequently depends on time as well. Figure 2 shows the results. We know the ultimate speed of the object approaches the speed of light, and the x component of the velocity finally approaches a constant value of zero. Therefore there must be a minimum in the x component of the acceleration. This result can be seen easily from (5-a). Since the y component of the velocity is initially zero, ($u_y = 0$ at $t = 0$), the x component of the acceleration, $\left(a_x = \frac{du_x}{dt}\right)$, is initially zero. As $t \to \infty$ the x component of the velocity approaches zero, $u_x \to 0$, therefore in accordance with equation (5-a) $a_x = \frac{du_x}{dt} \to 0$. The y component of the acceleration decreases and finally approaches zero as the y component of the velocity approaches the constant value of the speed of light. This result also can be found from equation (4-b). As $t \to \infty$, the x and y components of the velocity approach zero and the speed of light, respectively, and $a_y = \frac{du_y}{dt} = \frac{(F_0/m)\gamma^2}{(1+\gamma^2)\gamma}$. At this limit $|u| \to c$, and $\gamma \to \infty$ therfore $a_y = \frac{du_y}{dt} \to (F_0/m)(1/\gamma)$ goes to zero.

Figure 3 shows the deviation of the trajectory of the non-relativistic from the relativistic case. The x and y components of the acceleration approach zero as $t \to \infty$, and the x and y components of the velocity approach zero and the speed of light respectively. The displacement in $\hat{x}$ direction goes to zero and object moves only asymptotically in $\hat{y}$ direction as $t \to \infty$ in the relativistic case.

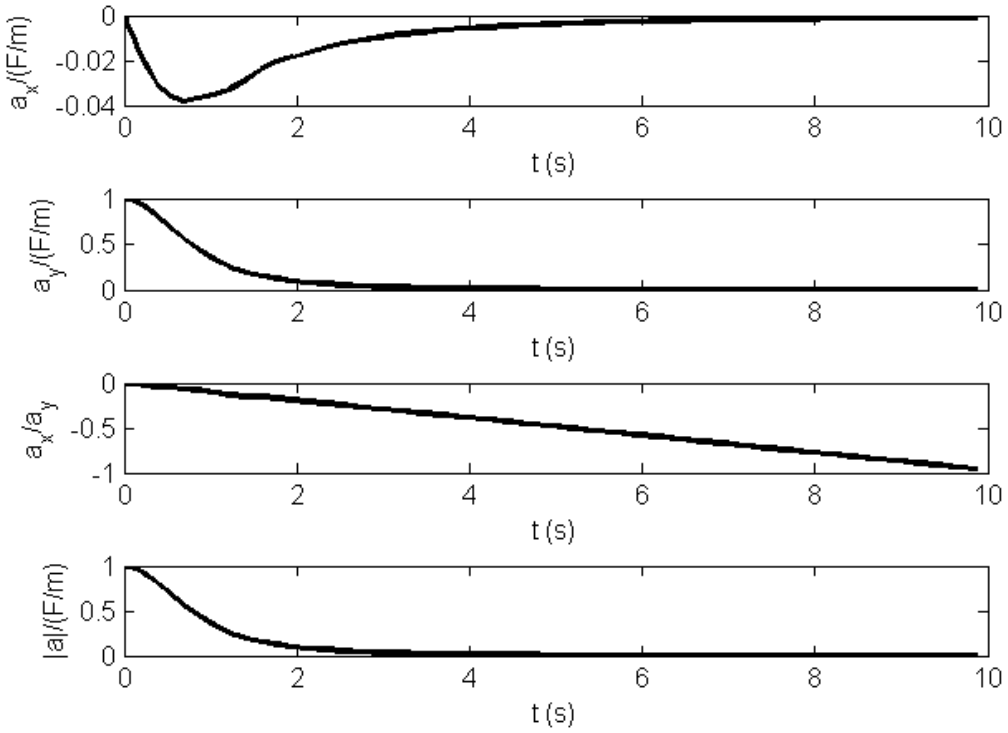

Figure 2. The x and y components of acceleration, and their ratio, and the absolute value of total acceleration

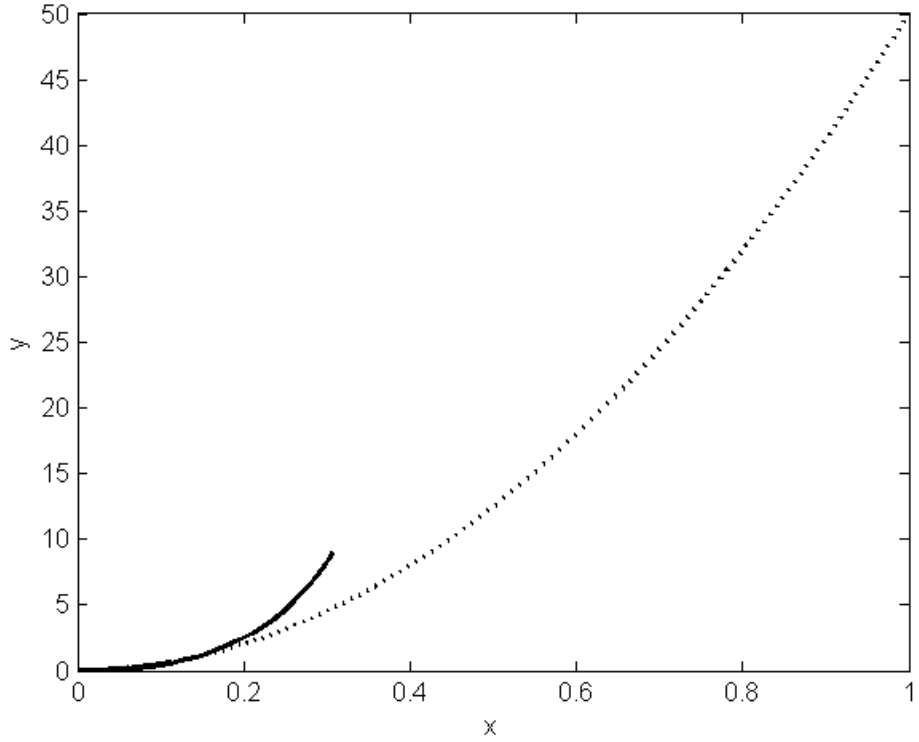

Figure 3. The trajectory of the motion in the non-relativistic and the relativistic cases

The x and y components of the applied force for projectile motion (constant acceleration) are given by equations (6-a) and (6-b), and these equations are valid only for $t < \left(\sqrt{c^2 - u_{x0}^2} - u_{y0}\right)/a_0$. Figure 4 shows the x and y components of the force for this case. We used $u_{x0} = 0.1, u_{y0} = 0$ and $a_0 = 1$ for this result.

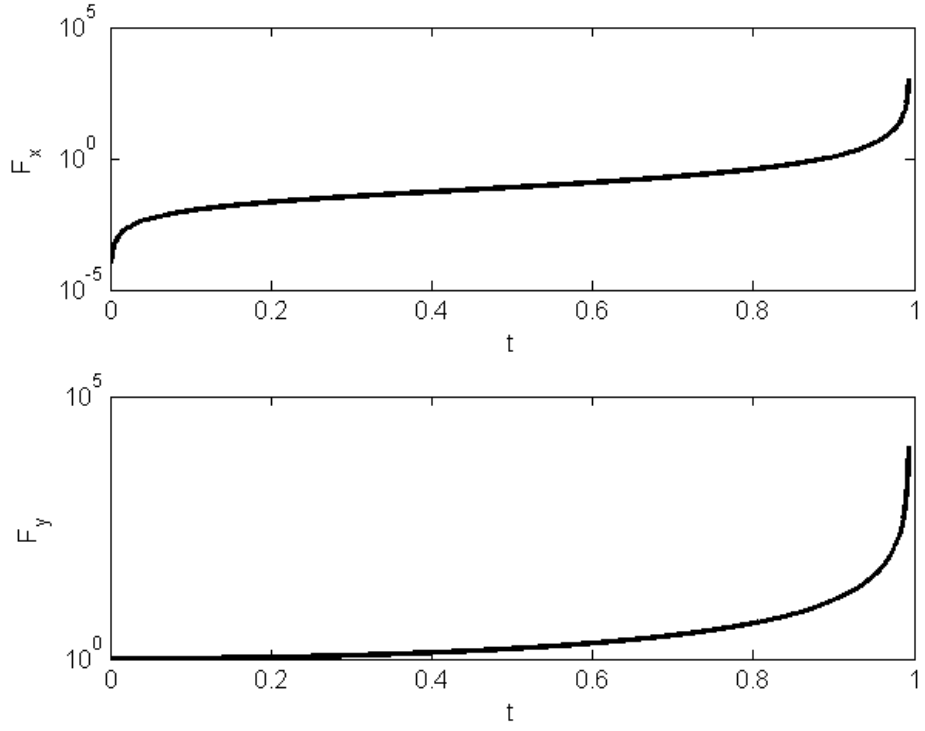

Figure 4. The x and y components of the applied force for projectile motion (constant acceleration) as semi-log plot. Here $t < \left(\sqrt{c^2 - u_{x0}^2} - u_{y0}\right)/a_0 = 0.9950$

6. Conclusion

In this paper we have investigated the motion of an object under a force in one direction with some initial velocity in another direction. This motion is known as projectile motion in classical mechanics. The motion of the object in the direction without the applied force remains uniform and in the other direction moves with constant acceleration. Mathematically, the motion in each perpendicular direction can be considered as two independent motions. In the relativistic mechanics the motion in each direction is coupled to each other, and the components of the velocity of the object in each direction changes. The object experiences acceleration not only in the direction with applied force, but also in the direction without applied force. It gains acceleration as a result of the change in the velocity in that direction. The projectile motion is possible in relativistic physics for only a limited time under a time dependent applied force on the object in both $\hat{x}$ and $\hat{y}$ directions. In this paper these conditions have been discussed.

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