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On Gravity Waves on the Surface of Tangential Discontinuity

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Abstract

On the basis of critical analysis of literature it is shown that the existing theory of surface gravity waves is incorrect and contradictory. Based on the new results published by the author dispersive equation for linear waves generated on the surface of tangential discontinuity between air and water was obtained. It is demonstrated that this equation is applicable only to capillary waves and effect of gravitational filed can be neglected in it. Thus, it is impossible to speak about capillary-gravitational waves in linear theory and consequently there is no condition restricting length of capillary wave. Contrastingly to the wide-spread opinion according to which capillary waves are generated only in deep water, it is demonstrated that they can be generated in shallow water as well where the phase speed of wave depends on the depth of water reservoir.

Keywords: tangential discontinuity, surface gravity waves, capillary-gravity waves, weakly inhomogeneous medium, strongly inhomogeneous medium, potential flow

1. Introduction

Problem of generation and propagation of waves on the surface of division of two media are solved with the help of the theory of hydrodynamic tangential discontinuity. The problem of gravitational waves on the surface of water reservoirs is one of the most vital among the problems of this type. As is known two limit cases are considered for gravity waves:

(1) Shortwave disturbances when the wave length λ is much smaller than the fluid depth h ($kh >> 1$, where $k = 2\pi/\lambda$ are called wave number). In this case, the water is considered infinitely deep and the influence of surface tension of fluid is taken into account. Waves generated in such conditions on the water surface are called deep water waves or capillary-gravity waves.

(2) Long wave disturbances when the wave length λ is much greater than the fluid depth h ($kh << 1$). In this case the surface tension influence is ignored and generated waves are called waves on shallow water.

According to the existing theory, the spectrum of frequencies of capillary-gravity waves is calculated by the formula

$$\omega = \sqrt{k\left(g + \frac{\alpha k^2}{\rho}\right) th(kh)} \tag{1.1}$$

where α is coefficient of surface tension of water, ρ is its density while g is acceleration of gravity. If the requirement is met

$$k << (\rho g / \alpha)^{1/2} \tag{1.2}$$

capillarity effect can be neglected. Taking into consideration that for water $\alpha = 0.073$ N/m and $\rho = 10^3$ kg/m^3, it is easy to calculate that the wave, length of which is $\lambda > 1.73$ cm, is purely gravitational, frequency of which in deep water ($th(kh) \approx 1$) is

$$\omega = \sqrt{kg} \tag{1.3}$$

In the second case, taking into consideration that $\alpha = 0$ and $th(kh) \approx kh$, we will have

$$\omega = k\sqrt{gh} \tag{1.4}$$

These formulae form the basis for a great deal of fundamental researches and are widely used in solution of applied problems. For example, phase speed of tsunami waves is calculated according to formula (1.4) (Kowalik, 2012; Ocean Waves, 2013; The shallow water wave …, 2011).

Correlation of (1.1), (1.3) and (1.4) follows from the Kelvin theory who was the first to solve the problem of surface capillary-gravity waves in 1871 (Landau & Lifshitz, 1988, See §62). His theory was based on the assumption of potentiality of fluid motion in the Earth gravity field. On the other hand it is known that fluid motion can be potential only in isentropic medium when entropy is equal in any of its points, i.e. $s = const$ (Landau & Lifshitz, 1988, See §§8,9). In the Earth gravity filed this condition is violated since entropy depends on vertical coordinate *z* and thus we assume that these correlations are doubtful and require to be revised.

To verify our doubts, in the second section we provide review of relevant paragraphs of the monograph (Landau & Lifshitz, 1988) and on the basis of analysis thereof demonstrate shortcomings of modern theory of hydrodynamic tangential discontinuity. In the third section, based on the generalized equation of gravity waves obtained in linear approximation in the paper (Kirtskhalia, 2012a) we derive an equation for surface capillary-gravity waves and demonstrate that the linear theory describes only capillary waves. Consequently, non-linear members must be taken into account in the system of hydrodynamic equations when considering gravity waves.

2. Modern Understanding of Surface Gravity Waves

Intensive studies of mechanisms of generation and propagation of gravity waves on the surface of water reservoirs began after L. Euler had formulated his fluid flow equation in 1755. Since then this issue has been treated in a great number of scientific works analysis of which is beyond the scope of this survey. We will refectory the reader mainly to the monograph of (Landau & Lifshitz, 1988) and bring only the most typical examples reflecting the shortcomings related to solution of this problem and showing that modern theory of gravity waves is incorrect and needs to be revised.

Dynamic processes occurring in fluids (gases) are described by the system of hydrodynamic equations which includes:

The flow equation (Euler's equation)

$$\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\nabla\right)\vec{V} = -\frac{\nabla P}{\rho} + \vec{g} \tag{2.1}$$

The continuity equation

$$\frac{\partial \rho}{\partial t} + div\left(\rho\vec{V}\right) = 0 \tag{2.2}$$

The adiabatic equation

$$\frac{\partial s}{\partial t} + \left(\vec{V}\nabla\right)s = 0, \tag{2.3}$$

where *p* is pressure, $\vec{V}$ is velocity and *s* is entropy. Exact solution of this system in general case is impossible, so scientists have to use approximate method of small disturbances when variables included in the system of Equations (2.1-2.3) have the form of the sum of their stationary and disturbed values $f(\vec{r},t) = f_0(\vec{r}) + f'(\vec{r},t)$, where $f'/f_0 < 1$ and

$$f'(\vec{r},t) = \tilde{f}(z)\cdot \exp[i(kx - \omega t)] \tag{2.4}$$

When using this method, the system (2.1-3) should be complemented with two equations:

The equation of equilibrium in the gravity field of the Earth

$$\nabla P_0 = \rho_0 \vec{g} \tag{2.5}$$

The medium state equation

$$\rho' = \frac{1}{C^2} p' \tag{2.6}$$

where *C* denotes the sound propagation velocity in a medium defined by means of its termodynamic characteristics.

This method for solution of the problem of stability of tangential discontinuity on the plane interface of two semi-bounded incompressible $C = \infty$ fluids (waves in deep water) without taking into account the influence of surface tension force and gravitation was first used by Helmholtz in 1868. He showed that the solutions of dispersion equation are two self-conjugate complex numbers (Landau & Lifshitz, 1988, See §29)

$$\omega = kV_0 \frac{\rho_2 \pm i\sqrt{\rho_1 \rho_2}}{\rho_1 + \rho_2} \tag{2.7}$$

where V_0-is speed of relative motion of fluids along plane surface of discontinuity. For the root with a positive imaginary part $(\mathrm{Im}\,\omega > 0)$, the wave amplitude increases with time, which means that the tangential discontinuity under consideration is absolutely unstable. Doubtful of solution of (2.7) is obvious, since from it follows that distribution of wave on the interface of two incompressible fluids at rest $(V_0 = 0)$ is impossible.

In 1944 Landau solved the problem of Helmholtz for compressible fluids with similar densities and discovered that such tangential discontinuity is stable when $V_0 > 2^{3/2} C$ (Landau & Lifshitz, 1988, See §84) and consequently, in compressible fluids tangential discontinuity is stable only in case of supersonic flows. This result contradicts to the Helmholtz result, according to which arbitrarily small speed of relative motion of fluids leads to instability of tangential discontinuity. Doubtfulness of its result was admitted by Landau himself (Landau, 1969). In the paper (Kirtskhalia, 1994) mistake was found in the method of solution of relevant dispersive equation and it was shown that such tangential discontinuity is also absolutely unstable.

The Helmholtz problem with account of forces of gravity and surface tension was solved by Kelvin in 1871 (Landau & Lifshitz, 1988, See §62). Assuming the flow to be potential, Kelvin introduced the velocity potential $\varphi(\vec{r}, t)$ satisfying the Laplace equation

$$\Delta \varphi(\vec{r}, t) = 0 \tag{2.8}$$

and related to the flow velocity of incompressible fluid by equation

$$\vec{V} = \nabla \varphi \tag{2.9}$$

The fluid pressure is defined by integration of the Euler's equation with application of (2.9) as well as condition of motion potentiality $rot\vec{V} = 0$ (Landau & Lifshitz, 1988, See §9):

$$P = -\rho \frac{\partial \varphi}{\partial t} - \rho g z - \rho \frac{V^2}{2} \tag{2.10}$$

where z is the coordinate along the axis normal to the surface of discontinuity. Assuming that the upper fluid moves with respect to the lower fluid along axis X with velocity V_0. The velocity potentials satisfying Equation (2.8) are written in the form

$$\varphi_1 = A_1 e^{-kz} \cos(kx - \omega t) + V_0 x, \qquad z > 0 \tag{2.11}$$

$$\varphi_2 = A_2 e^{+kz} \cos(kx - \omega t), \qquad z < 0 \tag{2.12}$$

Here it is accepted that the equation of discontinuity surface is $z = 0$. The difference in the exponent signs is explained by the requirement of decay of disturbances as the distance from discontinuity surface increases (surface waves). The constants A_1 and A_2 are assumed to be small values since they correspond to the disturbed values of the potentials. At the discontinuity surface the following conditions must be fulfilled:

$$(P_2 - P_1)\big|_{z=0} = -\alpha \frac{\partial^2 \xi}{\partial x^2} \tag{2.13}$$

$$v_{z1}\big|_{z=0} = \frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} + V_0 \frac{\partial \xi}{\partial x} \tag{2.14}$$

$$v_{z2}\big|_{z=0} = \frac{\partial \xi}{\partial t} \tag{2.15}$$

where ξ is displacement of discontinuity surface along the Z-axis; v_{z1} and v_{z2} are the z-components of the perturbed velocity values in areas $z > 0$ and $z < 0$ respectively. Condition (2.13) with consideration of (2.10) is written as follows

$$\rho_2 \frac{\partial \varphi_2}{\partial t} + \rho_2 g \xi - \alpha \frac{\partial^2 \xi}{\partial x^2} = \rho_1 \frac{\partial \varphi_1}{\partial t} + \rho_1 g \xi + \frac{\rho_1}{2}\left(V_1^2 - V_0^2\right) \tag{2.16}$$

where $V_1^2 = (\nabla\varphi_1)^2$. Representing ξ as

$$\xi = a\sin(kx - \omega t) \tag{2.17}$$

and substituting (2.11), (2.12) and (2.17) into the boundary conditions (2.14), (2.15) and (2.16) Kelvin obtained a system of three linear homogeneous algebraic equations with respect to the coefficients A_1, A_2 and a (the terms containing A_1^2 and A_2^2 are neglected). Equating the determinant of this system to zero, he finds the dispersion equation whose solution is

$$\omega = \frac{\rho_2 k V_0}{\rho_1+\rho_2} \pm \left[\frac{kg(\rho_2-\rho_1)}{\rho_1+\rho_2} - \frac{\rho_1\rho_2 k^2 V_0^2}{(\rho_1+\rho_2)^2} + \frac{\alpha k^3}{\rho_1+\rho_2}\right]^{1/2} \tag{2.18}$$

For $\alpha = 0$, $g = 0$, Kelvin's solution (2.18) transforms to Helmholtz' solution (2.7), and when $V_0 = 0$ and $\alpha = 0$ gives the dispersion Equation (1.3). From (2.18) it follows that the stability condition of capillary-gravity waves on water surface is the non-negativity of the expression in the square brackets, i.e.

$$\frac{\alpha k^2}{\rho_1+\rho_2} - \frac{\rho_1\rho_2 V^2}{(\rho_1+\rho_2)^2}k + \frac{g(\rho_2-\rho_1)}{\rho_1+\rho_2} \geq 0 \tag{2.19}$$

By solving inequality (2.19) with respect to k we find that the negativity of its discriminant gives condition of stability of tangential discontinuity for any k in the form

$$V_0^2 \leq \frac{2(\rho_1+\rho_2)}{\rho_1\rho_2}\sqrt{\alpha g(\rho_2-\rho_1)} \tag{2.20}$$

On the other hand, having solved (2.19) with respect to V_0, we find the stability condition in the form

$$V_0^2 \leq \frac{\left[g(\rho_2-\rho\ )+\alpha k^2\right](\rho_1+\rho_2)}{k\rho_1\rho_2} \tag{2.21}$$

It can be easily demonstrated that minimum value of the right part of inequality (2.21) is reached when $k = k_0 = \sqrt{g(\rho_2-\rho_1)/\alpha}$ and equals to the right part of inequality (2.20), however these conditions contradict to each other. Indeed, from (2.20) it follows that if $V_0 \neq 0$ the tangential discontinuity can be stable only in presence of both factors – the surface tension and the gravity field while according to (2.21) each of the parameters g and α makes its own contribution to the stability of the tangential discontinuity as it may be stable if one of them is absent. These contradictions which were first highlighted in the work (Kirtshalia & Rukhadze, 2008) are quite sufficient to get sure that the Kelvin theory is erroneous; however correlation (1.1) is considered classic up to now.

The basic reason of erroneousness of the Kelvin theory is violation of condition of potentiality of fluid motion, according to which motion can be potential only in isentropic medium. Indeed, in the Earth gravity filed z component of disturbed velocity must depend on gravity acceleration g, which must be taken into account in potentials of velocities (2.11) and (2.12). However, in this case the condition of potentiality of fluid motion $\partial v_x/\partial z = \partial v_z/\partial x$ is impossible to be met and consequently, the problem in such setting is unsolvable. That is why the constant term $-V_0^2$ is artificially introduced in the right side of the Equation (2.16).

The other threshold case—long gravity waves (waves on shallow water) is considered in monograph (Landau & Lifshitz, 1988, See §12). The authors solve the problem of wave propagation on water surface in gravity filed of Earth along the canal (along the axis X) with depth h and width b when $V_0 = 0$. Having applied the system of Equations (2.1) and (2.2) the authors obtain phase speed of wave on the shallow water in the form

$$U_p = \frac{\omega}{k} = \sqrt{gh} \tag{2.22}$$

which coincides with the expression (1.4). The Equation (2.3) is not used at all, assuming that it is satisfied identically. As is shown in the papers (Kirtskhalia, 2012a; 2012b). this is the fundamental error leading to incorrect solution of the problem on the whole. Indeed, the authors assume that v_z is so small, that $\partial v_z/\partial t = 0$. On this assumption, they write down the x and z components of Euler's Equation (2.1) in the form

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2.23}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g \tag{2.24}$$

Thereafter they are integrated the Equation (2.24) subject to

$$p\ \Big|_{z=\xi} = p_0 \tag{2.25}$$

and find

$$p = p_0 + \rho g(\xi - z) \tag{2.26}$$

Without going into further details, the said is quite sufficient to get sure that the ultimate result (2.22) here is also obtained by means of incorrect solution of the problem:

Firstly – the Equation (2.24) is nothing else but the fluid equilibrium condition in the Earth gravity field (2.5), which is fair only for stationary values of pressure and density and consequently, subject to fulfillment of this condition oscillations are impossible.

Secondly – from condition (2.25) it follows that pressure is constant on the disturbed liquid surface and thus it is unclear how the wave is propagated in this case.

Thirdly- the requirement $\partial v_z/\partial t = 0$ equals to the requirement $v_z = 0$ meaning that liquid surface is not displaced along the axis Z. Notwithstanding this the authors mark this displacement through ξ and obtain wave equation for it.

Solution of the task No.2 (Landau & Lifshitz, 1988, See §12) seems to be the most correct. Assuming that $\alpha = 0$ and $V_0 = 0$, and considering motion to be potential, the authors find the dispersive equation for gravity waves on interface of two liquids, density and depth of which equal to ρ_1, h_1 and ρ_2, h_2 respectively. Dispersive equation is obtained in the form

$$\omega^2 = \frac{kg(\rho_2 - \rho_1)}{\rho_2 cth(kh_2) + \rho_1 cth(kh_1)} \tag{2.27}$$

Three cases are considered:

1. Both liquids are infinitesimally deep ($kh_1 >> 1$ and $kh_2 >> 1$) and then the Equation (2.27) gives

$$\omega^2 = kg\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \tag{2.28}$$

2. Depth of both liquids is small ($kh_1 << 1$ and $kh_2 << 1$), and then we have

$$\omega^2 = k^2\frac{g(\rho_2 - \rho_1)h_1 h_2}{\rho_1 h_2 + \rho_2 h_1} \tag{2.29}$$

3. The lower liquid is shallow while the upper liquid is deep ($kh_2 << 1$ and $kh_1 >> 1$) which results in

$$\omega^2 = k^2 g h_2 \frac{\rho_2 - \rho_1}{\rho_2} \tag{2.30}$$

For $\rho_2 >> \rho_1$, (2.28) and (2.30) give the results (1.3) and (1.4), respectively. Mathematical correctness of solution of this task is conditioned by absence of relative motion of liquids ($V_0 = 0$). However, incorrectness related to physics remains in force and consequently it can be claimed that solution (2.27) is erroneous.

As was mentioned above, this problem is dealt with in a great number of scientific works which give the analogous results and therefore it is senseless to consider them here.

3. Capillary Waves

In the paper (Kirtskhalia, 2012a) it was demonstrated that the known system of hydrodynamic (gas dynamic) equations is fair only for homogeneous media which do not exist in the nature. This system needs to be generalized for real inhomogeneous media which qualitatively changes a number of existing notions on dynamic processes occurring in them (Kirtskhalia, 2013).

In the paper (Kirtskhalia, 2012a) a generalized equation of gravity wave in inhomogeneous medium is obtained in linear approximation, which is given by

$$\Delta p' - \vec{g}\left(\frac{1}{C^2} - \frac{1}{C_p^{\ 2}}\right)\nabla p' + \frac{\vec{g}}{C^2}\left(\frac{2\nabla C}{C} - \frac{\vec{g}}{C_p^2}\right)p' - \frac{1}{C^2}\frac{\partial^2 p'}{\partial t^2} = 0 \tag{3.1}$$

Here

$$C^2 = \frac{C_s^2 C_p^2}{C_s^2 + C_p^2} \tag{3.2}$$

is squared speed of sound in the medium which is reduced from squares of adiabatic C_s and isobaric C_p of sound speeds (Kirtskhalia, 2012a, 2012b). Existence of isobaric speed of sound associated with propagation of isobaric perturbation of density leads to necessity of generalization of the equation of mass discontinuity for inhomogeneous medium. Detailed presentation of the procedure which provides this equation in the form of

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + (\vec{v}\nabla)\rho = -\rho\nabla\vec{v} - \frac{\vec{v}\nabla p}{C_p^2} \tag{3.3}$$

is set out in the paper (Kirtskhalia, 2012a). From (3.3) it is obvious that the equation of mass discontinuity accepted in the existing theory is fair only for homogeneous medium when $C_p = \infty$. Consequently, homogeneous medium cannot be considered incompressible $(\nabla\vec{v} = 0)$ since in such case ρ = *const* and generation of mechanic wave in it is impossible. Thus, the Equation (3.3) provides qualitatively new definition of criteria of compressibility and incompressibility. This issue is extensively considered in the paper (Kirtskhalia, 2013), where it is proved that weakly inhomogeneous medium is always compressible while strongly inhomogeneous medium is always incompressible. Absence of such understanding of criteria of compressibility and incompressibility is one of the main reasons of incorrectness of wave theory in gas and hydrodynamic in whole (e.g. internal gravity waves (Kirtskhalia, 2013)), and in the theory of surface gravity waves in particular.

Let us now consider the correct linear theory of surface waves with consideration of surface tension force and gravity. Assuming that the surface of tangential discontinuity between water and air is plane z = 0. Having ignored the third summand in Equation (3.1) due to its obvious smallness and taking into consideration that $1/C^2 - 1/C_p^2 = 1/C_s^2$, it can be rewritten as

$$\Delta p' - \vec{g}\frac{1}{C_s^2}\nabla p' - \frac{1}{C^2}\frac{\partial^2 p'}{\partial t^2} = 0 \tag{3.4}$$

Since in the Earth atmosphere at sea level $C_p = \infty$ (Kirtskhalia, 2012a; 2012b), air should be considered compressible ($C = C_s$) and for it the Equation (3.4) will be as follows:

$$\Delta p_1' - \vec{g}\frac{1}{C_{s1}^2}\nabla p_1' - \frac{1}{{C_{s1}}^2}\frac{\partial^2 p_1'}{\partial t^2} = 0 \tag{3.5}$$

Let us present p_1 in the form of

$$p_1'(x,z,t) = \tilde{p}_1(z)\exp[i(kx - \omega t)] \tag{3.6}$$

following which the Equation (3.5) will be as follows:

$$\frac{d^2\tilde{p}_1(z)}{dt^2} + \frac{g}{C_{s1}^2}\frac{d\tilde{p}_1(z)}{dz} - \left(k^2 - \frac{\omega^2}{C_{s1}^2}\right)\tilde{p}_1(z) = 0 \tag{3.7}$$

Solution of the Equation (3.7) with consideration of decay of waves when $z \to \infty$ (surface wave) is

$$\tilde{p}_1(z) = A\exp(-\gamma kz) \tag{3.8}$$

where

$$\gamma = -\frac{1}{\theta_1}\left[1 + \sqrt{1 + \theta_1^2\left(1 - \frac{U_p^2}{C_{s1}^2}\right)}\right] < 0, \tag{3.9}$$

$$\theta_1 = \frac{2kC_{s1}^2}{g} \tag{3.10}$$

$U_p = \omega/k$ - is the phase speed of wave and C_{s1} is sound speed in air at sea level.

For water the Equation (3.5) will be as follows:

$$\Delta p_2' - \vec{g}\frac{1}{C_{s2}^2}\nabla p_2' - \frac{1}{C_2^2}\frac{\partial^2 p_2'}{\partial t^2} = 0 \tag{3.11}$$

From determination of isobaric speed of sound (Kirtskhalia, 2012b) we can easily find

$$C_p = \left| \rho_0 \left(\frac{\partial \rho_0}{\partial T} \right)_p^{-1} \right| \left(\frac{c_p}{T} \right)^{1/2} = \frac{1}{\beta} \left(\frac{c_p}{T} \right)^{1/2} \tag{3.12}$$

where $\beta = (1/V)(\partial V/\partial T)_p$ – is the coefficient of thermal dilatation and c_p – is thermal capacity of unit mass of substance at constant pressure. For water $\beta = 1.5\times10^{-40}$ K^{-1}, $c_p = 4.19\times10^3$ J/kg·K and then from formula (3.13) at temperature $T = 288$ K we will get $C_{p2} = 25210$ m/sec. On the other hand, by of experiment sound speed in water at the same temperature to a high accuracy equals to $C_2 = 1480$ m/sec. Then from formula (3.2) we have $C_{s2} = C_2 C_{p2} / \sqrt{C_{p2}^2 - C_2^2} = 1482.6$ m/sec. As we see, in water sound speed practically equals to adiabatic sound speed i.e. $C_2 = C_{s2}$. Similar calculations for example for iron ($\beta = 33.9\times10^{-60}$ K^{-1}, $c_p = 476.4$ J/kg·K, $T = 288$ K, $C_p = 38968$ m/sec, $C = 5130$ m/sec) give $C_s = 5175$ m/sec, and sound speed in iron is also adiabatic ($C = C_s$). Consequently, according to the new determination of criteria of compressibility and incompressibility (Kirtskhalia, 2013), water and iron should be considered compressible media in terms of thermodynamics and incompressibility condition $\nabla \vec{v} = 0$ must not be applied to them. Moreover, paradoxical as it may seem, water and iron are more compressible than air in upper layers of the atmosphere like, for example at a height of 11 km, $C_s \approx C_p \approx 300$ m/sec (Kirtskhalia, 2012b). Thus, the terms "compressibility" and "incompressibility" in thermodynamics characterize not the state of aggregation of matter, but physical process of sound propagation in it.

Taking into consideration that $C_2 = C_{s2}$, solution of Equation (3.5) for amplitude of pressure disturbance in water gives

$$\tilde{p}_2(z) = B_1 \exp(\delta_1 kz) + B_2 \exp(\delta_2 kz) \tag{3.13}$$

where

$$\delta_1 = -\frac{1}{\theta_2}\left[1 - \sqrt{1 + \theta_2^2\left(1 - \frac{U_p^2}{C_{s2}^2}\right)}\right] > 0, \tag{3.14}$$

$$\delta_2 = -\frac{1}{\theta_2}\left[1 + \sqrt{1 + \theta_2^2\left(1 - \frac{U_p^2}{C_{s2}^2}\right)}\right] < 0 \tag{3.15}$$

$$\theta_2 = \frac{2kC_{s2}^2}{g} \tag{3.16}$$

Thus, for disturbed values of pressure in air and water we will have

$$p_1'(x,z,t) = A \exp(\gamma kz) \exp[i(kx - \omega t)] \tag{3.17}$$

$$p_2'(x,z,t) = [\ B_1 \exp(\delta_1 kz) + B_2 \exp(\delta_2 kz)\ ] \exp[i(kx - \omega t)] \tag{3.18}$$

After use of equilibrium condition (2.5) and medium state Equation (2.6) when $C = C_s$ the linearized Euler's equation will be as follows:

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho_0}\nabla p' + \frac{p'}{\rho_0 C_s^2}\vec{g} \tag{3.19}$$

Having represented in (3.19) all variables in the form (2.4), for amplitude of the z component of disturbance velocity we will get

$$\tilde{v}_z(z) = -\frac{i}{k\rho_0 U_p}\left[\frac{d\tilde{p}(z)}{dz} + \frac{\tilde{p}(z)}{C_s^2} g\right] \tag{3.20}$$

From (3.20) for air and water respectively we will have

$$v_{1z}(x,z,t) = -\frac{i}{k\rho_1 U_p}\left[\frac{d\tilde{p}_1(z)}{dz} + \frac{\tilde{p}_1(z)}{C_{s1}^{\ 2}} g\right]\exp[i(kx - \omega t)] \tag{3.21}$$

$$v_{2z}(x,z,t) = -\frac{i}{k\rho_2 U_p}\left[\frac{d\tilde{p}_2(z)}{dz} + \frac{\tilde{p}_2(z)}{C_{s2}^{\ 2}} g\right]\exp[i(kx - \omega t)] \tag{3.22}$$

As we see disturbance velocity in area $z > 0$ as well as in area $z < 0$ obviously depends on g, which does not occur in Kelvin theory (Landau & Lifshitz, 1988). Now, to boundary conditions (2.13), (2.14), and (2.15) when $V_0 = 0$, should be added the condition on the bottom of water reservoir

$$v_{2z}\big|_{z=-h} = 0 \tag{3.23}$$

and substituting values of relevant quantities from formulae $\xi = a\exp[i(kx - \omega t)]$ (2.17), (3.17), (3.18), (3.21), and (3.22) into these conditions, we will get the system of linear homogeneous algebraic equations regarding coefficients A, B_1, B_2, and a. Equating the determinant of this equation to zero, we shall find dispersive equation of linear surface capillary-gravity waves in the form of

$$\begin{gathered}(\rho_1 U_p^2 - \gamma^* \alpha k)[\exp(-\delta_2 kh) - \exp(-\delta_1 kh)]\delta_1\delta_2 - \\ \rho_2 U_p^2 \gamma^* [\delta_1 \exp(-\delta_2 kh) - \delta_2 \exp(-\delta_1 kh)] = 0\end{gathered} \tag{3.24}$$

where

$$\gamma^* = -\frac{1}{\theta_1}\left[1 - \sqrt{1 + \theta_1^2\left(1 - \frac{U_p^2}{C_{s1}^2}\right)}\right] > 0 \tag{3.25}$$

Let us consider condition $\theta_1 > 1$. Taking into consideration that at sea level $C_{s1} \approx 340$ m/sec and $g \approx 10$ m/sec^2 we shall find $k > 8.64\times10^{-5}$ m^{-1} or $\lambda < 0.72\times10^5$ m. Thus, this condition covers the whole range of wavelengths from capillary to tsunami and the bigger is k the better it is met. For instance, when the value of gravity wave length is $\lambda \approx 100$ m, for air $\theta_1 \approx 0.71\times10^3 >> 1$ while for water $\theta_2 \approx 0.14\times10^5 >> 1$. It is also apparent that $U_p^2/C_{s2}^2 << U_p^2/C_{s1}^2 << 1$ and then it is easy to check that $\delta_1 = \gamma^* = 1$ and $\delta_2 = -1$, following which dispersive Equation (3.24) is reduced and takes the form

$$\left(U_p - \frac{\alpha k}{\rho_1 U_p}\right) th(kh) = -\frac{\rho_2}{\rho_1} U_p \tag{3.26}$$

As we see, the Equation (3.26) does not contain gravity acceleration g and does not have solution when $\alpha = 0$, thus it can be concluded that linear theory of surface capillary-gravity waves is adequate only for capillary waves on which gravity field has no effect. Furthermore it is obvious that when $\alpha = 0$ and $g = 0$ the task loses its physical meaning and therefore the solution of Helmholtz (2.7) does not contain any information. Taking into consideration that $\rho_1 << \rho_2 = \rho$, the solution of the Equation (3.26) shall be

$$U_p = \frac{\omega}{k} = \pm\sqrt{\frac{k\alpha}{\rho} th(kh)} \tag{3.27}$$

Solution (3.27) invalidates the current opinion that capillary waves are generated only in deep water. We see that they are generated in deep ($kh > 1$, $th(kh) \approx 1$) as well as in shallow ($kh < 1$, $th(kh) \approx kh$) water. In the first instance dispersive equation is given by

$$\omega = \pm k\sqrt{\frac{k\alpha}{\rho}} \tag{3.28}$$

And in the second one

$$\omega = \pm k^2\sqrt{\frac{\alpha h}{\rho}} \tag{3.29}$$

Since condition (1.2) limiting the length of capillary waves no longer exists, let us consider perturbation with wavelength $\lambda = 0.1$ m ($k = 62.8$ m^{-1}) for which from formulae (3.28) and (3.29) we will obtain that in deep water ($h \geq 0.5$ m) $\omega = 0.13$ sec^{-1} and $U_p = 2$ cm/sec and in shallow water ($h = 0.05$ m) $\omega = 0.07$ sec^{-1} and $U_p = 1$ cm/sec. As we see, frequency and phase speed of capillary wave drop two times when depth lowers 10 times.

4. Conclusion

The above critical analysis and calculations conclusively prove that the existing theory of surface capillary-gravity waves is incorrect and contradictory. The reason is an essential error made by Kelvin as far back in 1871, excluding assumption on potentiality of motion of liquids in the gravity field of the Earth. Consequently, problems of surface gravity waves should be solved only on the basis of system of hydrodynamic equations. However, here again the scientists make significant mistakes, since they seek to obtain results coinciding with the results of Kelvin. They are conditioned by incorrect understanding of criteria of compressibility and incompressibility of medium. The present paper demonstrates that these contradictions were eliminated after their correct definition had been provided in the paper (Kirtskhalia, 2013). On the basis of this correct definition equations of gravity waves in linear approximation are obtained for each of these media and by sewing (joining) their solutions on the surface of tangential discontinuity dispersive equation is obtained for surface wave. It is shown that the equation is solved only if surface tension of liquid is taken into consideration i.e. it describes only capillary waves on which influence of gravity filed is small to negligible. It was to be expected, since the Equation (3.5), describes acoustic waves (Kirtskhalia, 2012a) and consequently, is suitable only for linear waves amplitudes of which are comparable to amplitudes of acoustic waves. Agreement of result (3.28) with the result of Kelvin (1.1) when $g = 0$, is explained by the fact that in such instance motion of fluid can be considered potential and Kelvin theory gives correct result. Thus, capillary-gravity waves do not exist in linear theory, since surface wave is always capillary and can be propagated in deep as well as shallow water. When considering gravity waves in the system of hydrodynamic equations nonlinear members must be taken into account.

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