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The Galerkin Method for Global Solutions to the Maxwell-Boltzmann System

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Abstract

We prove an existence and uniqueness of solution to the Maxwell- Boltzmann coupled system globally in time. We first of all describe the background space-time and the unknown functions, making some *hypotheses* concerning the potentials of gravitation a and b, which determine the gravitational field g, the distribution function f which is unknown and is subject to the Boltzmann equation, as well as the collision kernel σ which appears in the collision operator. We after clarify the choice of the function spaces and we establish step by step, using Sobolev theorems, all the essential energy estimations leading to the global existence theorem. The method used for the investigation of the global existence combine the Galerkin method which is applied in a particular separable Hilbert space which is a Sobolev space with weight, and the standard theory on the first order differential systems. We then give at the end, the physical significance of our work.

Keywords: quasilinear system, Boltzmann relativistic equation, Maxwell system, charged particles, binary collisions, Galerkin method, global existence

Mathematics Subject Classifications: $83Cxx$

1. Introduction

In this paper we consider the coupled MAXWELL- BOLTZMANN System which is one of the basic systems of the kinetic theory.

The relativistic Boltzmann equation rules the dynamics of a kind of particles subject to mutual collisions, by determining their *distribution function*, which is a non-negative real-valued function of both the position and the momentum of the particles. Physically, this function is interpreted as the *probability of the presence density* of the particles in a given volume, during their collisional evolution. We consider the case of instantaneous, localized, binary and elastic collisions. Here the distribution function is determined by the Boltzmann equation through a non-linear operator called the collision operator. The operator acts only on the momentum of the particles, and describes, at any time; at each point where two particles collide with each other, the effects of the behaviour imposed by the collision to the distribution function, also taking in account the fact that the momentum of each particle is not the same, before and after the collision, only the sum of their two momenta being preserved.

The Maxwell equations are the basic equations of Electromagnetism and determine the electromagnetic field F created by the fast-moving charged particles. We consider the case where the electromagnetic field F is generated, through the Maxwell equations by the Maxwell current defined by the distribution function f of the colliding particles, a charge density e, and a future pointing unit vector u, tangent at any point to the temporal axis.

The system is coupled in the sense that, f, which is subject to the Boltzmann equation generates the Maxwell current in the Maxwell equations, whereas the electromagnetic field F, which is subject to the Maxwell equations is in the Lie derivative of f with respect to the vectors field tangent to the trajectories of the particles.

Some authors studied local in time existence theorems of the relativistic Boltzmann equation as: Bancel (1973), Bancel and Choquet-Bruhat (1973) . Glassey and Strauss (1992) obtained a global result in the case of data near to that of an equilibrium solution with non-zero density.

More recently , Noutchegueme and Dongho (2006) obtained a global existence theorem, *but the method used for the investigation of the local existence theorem for the relativistic Boltzmann equation, through characteristics is not quite clear,* in the sense that the equivalence between the Boltzmann equation and the system obtained by characteristics is not proved. Noutchegueme, Dongho and Takou (2005); Noutchegueme and Ayissi (2010) have used the same method.

Now Mucha (1998) proved only a local existence theorem for the Einstein-Boltzmann system using the Galerkin method, but the process of the investigation of the sequence of approximations is not completely explained.

The objectives of the present work in the particular case of the Bianchi type I space- time are:

- firstly to prove the global existence in time and uniqueness of solution to the coupled Maxwell-Boltzmann system, clarifying things in the method used by Mucha, explaining the choice made for the function spaces, demonstrating completely the main theorems;

- secondly to give a correct method of solving the relativistic Boltzmann equation which ignores the method of characteristics heavily used by us in Noutchegueme & Dongho (2006) , Noutchegueme, Dongho, & Takou (2005) , Noutchegueme & Ayissi, (2010) , as in several other works.

Moreover we take away the hypothesis we require in Noutchegueme, N, & Ayissi, R. (2010) ; that the initial datum f_0 for the Boltzmann equation be invariant under a certain subgroup of O_3, which allows to take F as a real unknown and complicates the investigations.

The paper is organized as follows:

In section 2, we introduce the space-time and we give the unknown functions.

In section 3, we describe theMaxwell-Boltzmann system.

In section 4, we introduce the function spaces and we give the energy estimations.

In section 5, we prove the local existence theorem.

In section 6, we prove the global existence theorem.

2. The Background Space-Time and the Unknown Functions

Greek indexes $\alpha, \beta, \gamma, ...$ range from 0 to 3, and Latin indexes $i, j, k, ...$ from 1 to 3. We adopt the Einstein summation convention:

$$A^\alpha B_\alpha = \sum_\alpha A^\alpha B_\alpha.$$

We consider the collisional evolution of a kind of fast moving massive and charged particles in the time-oriented Bianchi type 1 space-time $\left(\mathbb{R}^4, g\right)$, and denote by $x^\alpha = \left(x^0, x^i\right) = \left(t, x^i\right)$ the usual coodinates in $\mathbb{R}^4$, where $x^0 = t$ represents the time and $\left(x^i\right)$ the space; g stands for the *given* metric tensor of Lorentzian signature $(-, +, +, +)$ which writes:

$$g = -(dt)^2 + a^2(t)\left(dx^1\right)^2 + b^2(t)\left(\left(dx^2\right)^2 + \left(dx^3\right)^2\right) \tag{1}$$

where $a > 0,\ b > 0$ are two continuously differentiable functions on $\mathbb{R}$, whose variable is denoted t.

The expression of the the Levi-Civita connection ∇ associated to g, which is:

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu}\left[\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right],$$

gives directly:

$$\begin{cases} \Gamma^1_{10} = \dfrac{\dot{a}}{a}; \Gamma^2_{20} = \frac{\dot{b}}{b};\ \Gamma^3_{30} = \frac{\dot{b}}{b};\ \Gamma^0_{11} = a\dot{a};\ \Gamma^0_{22} = b\dot{b} = \Gamma^0_{33} \\ \qquad\qquad \Gamma^\lambda_{\alpha\beta} = 0 \ \text{ otherwise,} \end{cases} \tag{2}$$

where the dot stands for the derivative with respect to t. Recall that $\Gamma^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha}$.

We require the assumption that $\frac{\dot{a}}{a}$ and $\frac{\dot{b}}{b}$ be bounded. This implies that there exists a constant $C > 0$ such that:

$$\left|\frac{\dot{a}}{a}\right| \leq C,\ \left|\frac{\dot{b}}{b}\right| \leq C. \tag{3}$$

As a direct consequence, whe have for $t \in \mathbb{R}^+$:

$$a(t) \leq a_0 e^{Ct};\ b(t) \leq b_0 e^{Ct};\ \frac{1}{a}(t) \leq \frac{1}{a_0} e^{Ct};\ \frac{1}{b}(t) \leq \frac{1}{b_0} e^{Ct} \tag{4}$$

where $a_0 = a(0)$; $b_0 = b(0)$.

The massive particles have a rest mass $m > 0$, normalized to the unity, *i.e* $m = 1$. We denote by $T\left(\mathbb{R}^4\right)$ the tangent bundle of $\mathbb{R}^4$ with coordinates $\left(x^\alpha, p^\beta\right)$, where $p = \left(p^\beta\right) = \left(p^0, \overline{p}\right)$ stands for the momentum of each particle and $\overline{p} = \left(p^i\right)$, $i = 1, 2, 3$. Really the charged particles move on the future sheet of the mass-shell or the mass hyperboloid $P\left(\mathbb{R}^4\right) \subset T\left(\mathbb{R}^4\right)$, whose equation is $P_x(p) : g_x(p, p) = g_{\alpha\beta} p^\alpha p^\beta = -1$ or equivalently, using expression (1) of g :

$$p^0 = \sqrt{1 + a^2 \left(p^1\right)^2 + b^2 \left(\left(p^2\right)^2 + \left(p^3\right)^2\right)}, \tag{5}$$

where the choice $p^0 > 0$ symbolizes the fact that, naturally, the particles eject towards the future.

Setting:

$$\varrho = \sqrt{\sum_{i=1}^{3} \left(p^i\right)^2} = \varrho\left(\overline{p}\right),$$

if $\varrho > 1$, the relations (4) and (5) also show that in any interval $[0, T]$, $T > 0$:

$$A p^0 \leq \varrho \leq B p^0, \tag{6}$$

where $A = A(T) > 0, B = B(T) > 0$ are constants.

The invariant volume element in $P_x(p)$ reads:

$$\omega_p = |g|^{\frac{1}{2}} \frac{dp^1 dp^2 dp^3}{p^0},$$

where

$$|g| = \left|det g_{\alpha\beta}\right|.$$

We denote by f the distribution function which measures the probability of the presence of particles in the plasma. f is a non-negative unknown real-valued function of both the position (x^α) and the 4−momentum of the particles $p = (p^\alpha)$, so:

$$f : T\left(\mathbb{R}^4\right) \approx \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^+,\ (x^\alpha, p^\alpha) \longmapsto f(x^\alpha, p^\alpha) \in \mathbb{R}^+.$$

We define a scalar product on $\mathbb{R}^3$ by setting for $p = \left(p^0, \overline{p}\right) = \left(p^0, p^i\right)$ and $q = \left(q^0, \overline{q}\right) = \left(q^0, q^i\right)$:

$$\overline{p}.\overline{q} = a^2 p^1 q^1 + b^2 \left(p^2 q^2 + p^3 q^3\right). \tag{7}$$

In this paper we consider the homogeneous case for which f depends only on the time $x^0 = t$ and $\overline{p}$. According to the Laplace low, the fast moving and charged particles create an unknown electromagnetic field F which is a 2−closed antisymetric form and locally writes:

$$F = F_{\alpha\beta} dx^\alpha \wedge dx^\beta.$$

So in the homogeneous case we consider:

$$F_{\alpha\beta} : \mathbb{R} \to \mathbb{R},\ t \longmapsto F_{\alpha\beta}(t) \in \mathbb{R}.$$

In the presence of the electromagnetic field F, the trajectories $s \longmapsto (x^\alpha(s), p^\alpha(s))$ of the charged particles are no longer the geodesics of space-time $\left(\mathbb{R}^4, g\right)$, but the solutions of the differential system:

$$\frac{dx^\alpha}{ds} = p^\alpha;\ \frac{dp^\alpha}{ds} = P^\alpha \tag{8}$$

where:

$$P^{\alpha} = P(F, f) = -\Gamma^{\alpha}_{\lambda\mu} p^{\lambda} p^{\mu} + e p^{\beta} F^{\alpha}_{\beta}, \tag{9}$$

where $e = e(t)$ denotes the charge density of particles.

Notice that the differential system (8) shows that the vectors field $X(F)$ defined locally by:

$$X(F) = (p^{\alpha}, P^{\alpha}(F)) \tag{10}$$

where P^{α} is given by (9), is tangent to the trajectories.

The charged particles also create a current $J = \left(J^{\beta}\right)$, $\beta = 0, 1, 2, 3$, called the Maxwell current we take in the form:

$$J^{\beta} = \int_{\mathbb{R}^3} p^{\beta} f \omega_p - e u^{\beta} \tag{11}$$

in which $u = \left(u^{\beta}\right)$ is a unit future pointing time-like vector, tangent to the time axis at any point, which means that $u^0 = 1,\ u^i = u_i = 0,\ \ i = 1, 2, 3$. The particles are then supposed to be spatially at rest.

The electromagnetic field $F = \left(F^{0i}, F_{ij}\right)$, where F^{0i} and F_{ij} stand for the electric and magnetic parts respectively is subject to the Maxwell equations.

3. The Maxwell-Boltzman System in F and f

3.1 The Maxwell System in F

The Maxwell system in F can be written, using the covariant notation:

$$\nabla_{\alpha} F^{\alpha\beta} = J^{\beta} \tag{12}$$

$$\nabla_{\alpha} F_{\beta\gamma} + \nabla_{\beta} F_{\gamma\alpha} + \nabla_{\gamma} F_{\alpha\beta} = 0. \tag{13}$$

(12) and (13) are respectiveley the first and second group of the Maxwell equations, and ∇_{α} stands for the convariant derivative in g. In (12), J^{β} represents the Maxwell current we take in the form (11). Now the well-known identity $\nabla_{\alpha}\nabla_{\beta} F^{\alpha\beta} = 0$ imposes, given (12) that the current j^{β} is always subject to the conservation law:

$$\nabla_{\beta} J^{\beta} = 0. \tag{14}$$

However using $\beta = 0$ in (12), we obtain:

$\nabla_{\alpha} F^{\alpha 0} = \partial_{\alpha} F^{\alpha 0} + \Gamma^{\alpha}_{\alpha\lambda} F^{\alpha 0} + \Gamma^{0}_{\alpha\lambda} F^{\alpha\lambda} = 0$, since $F = F(t)$, $F^{\alpha\lambda} = -F^{\lambda\alpha}$, and by (2) $\Gamma^{\alpha}_{\alpha i} = 0$.

So (12) implies that:

$$J^0 = 0. \tag{15}$$

By (15), the expression (11) of J^{β} in which we set $\beta = 0$ then allows to compute e and gives, since $u^0 = 1$:

$$e(t) = \int_{\mathbb{R}^3} f a b^2 d\overline{p}, \tag{16}$$

which shows that f determines e.

The second set (13) of the Maxwell equations is identically satisfied since $F = F(t)$, and the first set reduces to $\partial F_{ij} = 0$. Then F_{ij} is constant and:

$$F_{ij} = F_{ij}(0) = \varphi_{ij}. \tag{17}$$

This physically shows that the magnetic part of F does not evolve and stays in its primitive state. It remains to determine the electric part F^{0i}.

Writing the equation (11) for $\beta = i$, using (2), $\omega_p = |g|^{\frac{1}{2}} \frac{dp^1 dp^2 dp^3}{p^0}$ and $u^i = 0$ implies that

$$J^i = \int_{\mathbb{R}^3} \frac{p^i f(t, \overline{p}) a b^2}{p^0} d\overline{p}. \tag{18}$$

By (12) , we obtain the linear o.d.e in F^{0i} which writes

$$\dot{F^{0i}} + \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) F^{0i} = \int_{\mathbb{R}^3} \frac{p^i f(t, \overline{p})\, ab^2}{p^0} d\overline{p}. \tag{19}$$

In (19) , the expression $H = -\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) = -\frac{1}{2} g^{kl} \partial_0 g_{kl}$ represents the second fundamental form in $\left(\mathbb{R}^3, g\right)$. Really H is the trace of the 2− symetric tensor $K = \left(K_{ij}\right)$ where $K_{ij} = -\frac{1}{2}\partial_0 g_{ij}$. H is called the *middle curvature* of $\left(\mathbb{R}^4, g\right)$. Since a and b are given, so is H.

3.2 The Boltzmann Equation in f

The relativistic Boltzmann equation in f, for charged particles in the Bianci type 1 space-time can be written:

$$L_X f = Q(f, f) \tag{20}$$

where L_X is the Lie derivative of f with respect to the vectors field $X(F)$ defined by (10) and $Q(f, f)$ the collision operator we now introduce.

According to Lichnerowicz and Chernikov, we consider a scheme, in which, at a given position $\left(t, x^i\right)$, only two particles collide each other, without destroying each one, the collision affecting only the momentum of each particle, which changes after shock, only the sum of the two momenta being preserved. If p, q stand for the two momenta before the shock, and p', q' for the two momenta after the shock, then we have:

$$p + q = p' + q'.$$

The collision operator Q is then defined, using functions f and g on $\mathbb{R}^3$ and the above notations, by:

$$Q(f, g) = Q^+(f, g) - Q^-(f, g) \tag{21}$$

where:

$$Q^+(f, g) = \int_{\mathbb{R}^3} \omega_{\overline{q}} \int_{S^2} f(\overline{p}')\, g(\overline{q}')\, \sigma(t, \overline{p}, \overline{q}, \overline{p}', \overline{q}', \Omega)\, d\Omega \tag{22}$$

$$Q^-(f, g) = \int_{\mathbb{R}^3} \omega_{\overline{q}} \int_{S^2} f(\overline{p})\, g(\overline{q})\, \sigma(t, \overline{p}, \overline{q}, \overline{p}', \overline{q}', \Omega)\, d\Omega \tag{23}$$

whose elements we now introduce step by step, specifying properties and hypotheses we adopt:

•S^2 is the unit sphere of $\mathbb{R}^3$, whose area element is denoted $d\Omega$;

• σ is a non-negative continuous real-valued function of all its arguments, called the *collision kernel* or the *cross-section* of the collisions, on which we require the boundedness and Lipschitz continuity assumptions, in which $C_1 > 0$ is a constant:

$$\begin{cases} 0 \leq \sigma(t, \overline{p}, \overline{q}, , \Omega) \leq C_1 \\ \left|\sigma(t, \overline{p}_1, \overline{q}, \overline{p}', \overline{q}', \Omega) - \sigma(t, \overline{p}_2, \overline{q}, \overline{p}', \overline{q}', \Omega)\right| \leq C_1 \left\|\overline{p}_1 - \overline{p}_2\right\| \end{cases} \tag{24}$$

where $\|\overline{p}\| = \left(\sum_{i=1}^{3} \left(p^i\right)^2\right)^{\frac{1}{2}} = \varrho$ is the norm in $\mathbb{R}^3$.

• The conservation law $p + q = p' + q'$ splits into:

$$p^0 + q^0 = p'^0 + q'^0 \tag{25}$$

$$\overline{p} + \overline{q} = \overline{p}' + \overline{q}'. \tag{26}$$

(22) expresses, using (5) , the conservation of the quantity:

$$\tilde{e} = \sqrt{1 + a^2 (p^1)^2 + b^2 \left((p^2)^2 + (p^3)^2\right)} + \sqrt{1 + a^2 (q^1)^2 + b^2 \left((q^2)^2 + (q^3)^2\right)} \tag{27}$$

called the elementary energy of the unit rest mass particles; we can interpret (23) by setting (Glassey & Strauss,1992):

$$\begin{cases} \overline{p}' = \overline{p} + c(\overline{p}, \overline{q}, \Omega)\, \Omega \\ \overline{q}' = \overline{q} - c(\overline{p}, \overline{q}, \Omega)\, \Omega \end{cases} \quad \left(\Omega \in S^2\right) \tag{28}$$

in which $c(\overline{p}, \overline{q}, \Omega)$ is a real-valued function. We prove, by a direct calculation, using (5) to express p'^0, q'^0 in terms of $\overline{p}', \overline{q}'$ and next (25) to express $\overline{p}', \overline{q}'$ in terms of $\overline{p}, \overline{q}$, that equation (22) leads to a quadratic equation in c, which solves to give the only non trivial solution:

$$c(\overline{p}, \overline{q}, \Omega) = \frac{2p^0 q^0 \tilde{e} \Omega \cdot \left(\hat{\overline{q}} - \hat{\overline{p}}\right)}{(\tilde{e})^2 - [\Omega \cdot (\overline{p} + \overline{q})]^2} \tag{29}$$

in which $\hat{\overline{p}} = \frac{\overline{p}}{p^0}$, $\tilde{e}$ is given by (24) and the dot $(\cdot)$ is the scalar product defined by (7).

It then appears, using (25) that the functions in the integrals (19) and (20) depend only on $\overline{p}, \overline{q}, \Omega$, and that these integrals with respect to $\overline{q}$ and Ω give functions $Q^+(f, g)$ and $Q^-(f, g)$ of the single variable $\overline{p}$.

Now using the usual properties of the determinants, the jacobian of the change of variables $(\overline{p}, \overline{q}) \mapsto (\overline{p}', \overline{q}')$ defined by (25)is computed to be:

$$\frac{\partial (p', q')}{\partial (p, q)} = -\frac{p'^0 q'^0}{p^0 q^0}. \tag{30}$$

Since $f = f(t, \overline{p})$, using (5), the Boltzmann Equation (17) takes the form:

$$\frac{\partial f}{\partial t} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i} = \frac{1}{p^0} Q(f, f). \tag{31}$$

3.3 The Coupled System

From (9), using (2) we obtain

$$\frac{P^i}{p^0} = -2\Gamma^i_{0i} p^i - e\left[F^{0i} + g^{ii}\frac{p^k F_{ik}}{p^0}\right],\ i = 1, 2, 3. \tag{32}$$

Setting $E^i = F^{0i}$ and using (16), (19), (31) and (32) the Maxwell-Boltzmann system (F, f) reduces to the following form

$$\dot{E}^i = HE^i + \int_{\mathbb{R}^3} \frac{p^i f(t, \overline{p})\, ab^2}{p^0} d\overline{p} \tag{33}$$

$$\frac{\partial f}{\partial t} + \left[-2\Gamma^i_{0i} p^i + \left(-E^i + g^{ii}\frac{p^k \varphi_{ik}}{p^0}\right)\int_{\mathbb{R}^3} f(t, \overline{p})\, ab^2 d\overline{p}\right]\frac{\partial f}{\partial p^i} = \frac{1}{p^0} Q(f, f) \tag{34}$$

which is an integro-differential system to solve in what is to follow.

4. Function Spaces and Energy Estimations

We define now the function spaces in which we are searching the solution to the Maxwell-Boltzmann system. We also establish some useful energy estimations.

Definition 1. $L^1_2\left(0, T, \mathbb{R}^3\right)$ We define

$$L^1_2\left(\mathbb{R}^3\right) = \left\{g : \mathbb{R}^3 \to \mathbb{R},\ (1 + \varrho)\, g \in L^1\left(\mathbb{R}^3\right)\right\}$$

where $\varrho = \|\overline{p}\|, \overline{p} = \left(p^i\right) \in \mathbb{R}^3$.

Let $T > 0$ be given, then

$$L^1_2\left(0, T, \mathbb{R}^3\right) = \left\{h\ :\ [0, T] \times \mathbb{R}^3 \to \mathbb{R},\ h(t, .) \in L^1_2\left(\mathbb{R}^3\right)\right\}.$$

$L^1_2\left(0, T, \mathbb{R}^3\right)$ is a Banach space endowed with the norm:

$$|||h|||_{L^1_2\left(0, T, \mathbb{R}^3\right)} = \sup_{t \in [0, T]} \|(1 + \varrho)\, h(t, .)\|_{L^1\left(\mathbb{R}^3\right)}.$$

Definition 2. $H^m_s\left(0, T, \mathbb{R}^3\right)$

Let $T > 0,\ m \in \mathbb{N},\ s \in \mathbb{R}$ be given.

We define $H_s^m\left(\mathbb{R}^3\right)$ as

$$H_s^m\left(\mathbb{R}^3\right)=\left\{h\,:\,\mathbb{R}^3\to\mathbb{R},\ (1+\varrho)^{s+|\beta|}\,\partial_{\overline{p}}^{\beta}h\in L^2\left(\mathbb{R}^3\right),|\beta|\leq m\right\}.$$

$H_s^m\left(\mathbb{R}^3\right)$ will be endowed with the norm

$$\|h\|_{H_s^m\left(\mathbb{R}^3\right)}=\underset{0\leq|\beta|\leq m}{Max}\left\|(1+\varrho)^{s+|\beta|}\,\partial_{\overline{p}}^{\beta}h\right\|_{L^2\left(\mathbb{R}^3\right)}.$$

$\overline{H}_s^m\left(\mathbb{R}^3\right)$ will be the completion of $H_s^m\left(\mathbb{R}^3\right)$ in the norm $\|.\|_{H_s^m\left(\mathbb{R}^3\right)}$.

A function $y\in H_s^m\left(0,T,\mathbb{R}^3\right)$ if for all $t\in[0,T]$ and all $|\beta|\leq m$

$$(1+\varrho)^{s+|\beta|}\,\partial_{\overline{p}}^{\beta}y\,(t,\cdot)\in L^2\left(\mathbb{R}^3\right)$$

and

$$t\mapsto(1+\varrho)^{s+|\beta|}\,\partial_{\overline{p}}^{\beta}y\,(t,.)$$

is a continuous function from $[0,T]$ to $L^2\left(\mathbb{R}^3\right)$.

Endowed with the norm

$$|||y|||_{H_s^m\left(0,T,\mathbb{R}^3\right)}=\underset{0\leq|\beta|\leq m}{max}\,\underset{t\in[0,T]}{sup}\left\|(1+\varrho)^{s+|\beta|}\,\partial_{\overline{p}}^{\beta}y\,(t,\cdot)\right\|_{L^2\left(\mathbb{R}^3\right)},$$

$H_s^m\left(0,T,\mathbb{R}^3\right)$ is a Banach space.

For $r>0$ be given, we define

$$H_{s,r}^m=\left\{y\in H_s^m\left(0,T,\mathbb{R}^3\right),y\geq 0\text{ a.e, }\ |||y|||_{H_s^m\left(0,T,\mathbb{R}^3\right)}\,H_s^m\left(0,T,\mathbb{R}^3\right)\leq r\right\}.$$

Endowed with the induced distance by the norm $|||\cdot|||_{H_s^m\left(0,T,\mathbb{R}^3\right)}$, $H_{s,r}^m$ is a complete metric subspace of $H_s^m\left(0,T,\mathbb{R}^3\right)$.

Remark 2. If $m=0$, then $y\in H_s^m\left(\mathbb{R}^3\right)\Longleftrightarrow(1+\varrho)^s\,y\in L^2\left(\mathbb{R}^3\right)$, so $H_s^0\left(\mathbb{R}^3\right)$ will be denoted $L_s^2\left(\mathbb{R}^3\right)$.

Remark 3. **The reasons for the choice of the function space $H_d^m\left(\mathbb{R}^3\right)$ for $m=3$ and $d>\frac{5}{2}$.**

The objective of the present work being the existence of solution to the Maxwell-Boltzmann system, and particularly the Boltzmann equation (28), we are searching a function $f=f\,(t,\overline{p})$ which is continuously differentiable, in particular we can search $f=f\,(t,\cdot)$ belonging to the space $C_{\mathrm{b}}^1\left(\mathbb{R}^3\right)$.

We want to use the Faedo-Galerkin method which is applied for separable Hilbert spaces. That is the case for the Sobelev spaces $H^m\left(\mathbb{R}^3\right),m\in\mathbb{N}$.

We need then to find an integer m such that

$$H^m\left(\mathbb{R}^3\right)\hookrightarrow C_{\mathrm{b}}^1\left(\mathbb{R}^3\right).$$

But we know by the Sobolev theorems that

$$W_p^m\left(\mathbb{R}^n\right)\hookrightarrow\mathcal{E}^k\left(\mathbb{R}^n\right),\ \ m>k+\frac{n}{p}.$$

Since in our case we have $n=3,\ p=2,\ k=1\left(\ W_2^m=H^m\right)$, we must choose m such that

$$m>1+\frac{3}{2}\ =\frac{5}{2}.$$

The smallest interger m satisfying $m>\frac{5}{2}$ is naturally $m=3$.

Consequently we have

$$H_d^3\left(\mathbb{R}^3\right)\hookrightarrow H^3\left(\mathbb{R}^3\right)\hookrightarrow C_{\mathrm{b}}^1\left(\mathbb{R}^3\right).$$

Furthermore if

$$d > \frac{5}{2}$$

then

$$H_d^m\left(\mathbb{R}^3\right) \hookrightarrow L_d^2\left(\mathbb{R}^3\right) \hookrightarrow L_2^1\left(\mathbb{R}^3\right). \tag{35}$$

It then results that

$$H_d^m\left(\mathbb{R}^3\right) \cap L_2^1\left(\mathbb{R}^3\right) = H_d^m\left(\mathbb{R}^3\right).$$

In fact if $f \in H_d^m\left(\mathbb{R}^3\right)$, then

$$\|f\|_{L_2^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (1+\varrho)|f|\, d\overline{p} = \int_{\mathbb{R}^3} (1+\varrho)^{1-d+d} |f|\, d\overline{p} =$$

$$\int_{\mathbb{R}^3} (1+\varrho)^{1-d} (1+\varrho)^d |f|\, d\overline{p}.$$

So by the Cauchy-Schwarz inequality we obtain

$$\|f\|_{L_2^1(\mathbb{R}^3)} \leq \left(\int_{\mathbb{R}^3} (1+\varrho)^{2-2d}\, d\overline{p}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} (1+\varrho)^{2d} |f|^2\, d\overline{p}\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{\mathbb{R}^3} (1+\varrho)^{2-2d}\, d\overline{p}\right)^{\frac{1}{2}} \|f\|_{L_d^2(\mathbb{R}^3)}.$$

But using polar coordinates, the integral $\int_{\mathbb{R}^3} (1+\varrho)^{2-2d}\, d\overline{p}$ converges if and only if it is the case for the integral $\int_0^{+\infty} (1+r)^{2-2d} r^2 dr$.

Since $\int_0^{+\infty} (1+r)^{2-2d} r^2 dr \underset{r\to+\infty}{\sim} \int_0^{+\infty} r^{4-2d} dr$, using also the fact that $\int_0^{+\infty} r^{4-2d} dr$ is convergent if in the primitive containing r^{5-2d} we have $5-2d<0$, we conclude that if

$$d > \frac{5}{2}$$

then

$$\|f\|_{L_2^1(\mathbb{R}^3)} \leq C\|f\|_{L_d^2(\mathbb{R}^3)} \leq C\|f\|_{H_d^3(\mathbb{R}^3)}.$$

We can now state the following result which in what follows will be fundamental.

Proposition 1. *Let $d > \frac{5}{2}$, $\|\sigma\|_{L^1(\mathbb{R}^3\times S^2)} \in L^\infty\left(\mathbb{R}^3\right)$ and $\left(\partial^\beta\sigma\right)(1+|\overline{p}|)^{|\beta|-1} \in L^\infty\left(\mathbb{R}^3\times\mathbb{R}^3\times S^2\right)$, $|\beta| \leq 3$ be given. If $f,\ g \in H_d^3\left(\mathbb{R}^3\right)$ then $\frac{1}{p^0}Q(f,g) \in H_d^3\left(\mathbb{R}^3\right)$ and we have*

$$\left\|\frac{1}{p^0}Q(f,g)\right\|_{H_d^3\left(\mathbb{R}^3\right)} \leq C\|f\|_{H_d^3\left(\mathbb{R}^3\right)}\|g\|_{H_d^3\left(\mathbb{R}^3\right)} \tag{36}$$

where $C = C(T) > 0$.

Moreover

$$\left\|\frac{1}{p^0}Q(f,f) - \frac{1}{p^0}Q(g,g)\right\|_{H_d^3\left(\mathbb{R}^3\right)} \leq$$

$$C\left(\|f\|_{H_d^3\left(\mathbb{R}^3\right)} + \|g\|_{H_d^3\left(\mathbb{R}^3\right)}\right)\|f-g\|_{H_d^3\left(\mathbb{R}^3\right)}.$$

proof. (Mucha, 1998). □

Proposition 2. *Let $d > \frac{5}{2}$, $f \in H_d^0\left(\mathbb{R}^3\right)$ be given. Then $J^i \in H_d^0\left(\mathbb{R}^3\right)$ and $\left|J^i\right| \leq C(a_0, b_0, T)\|f\|_{H_d^3\left(\mathbb{R}^3\right)}$, $i = 1, 1, 3$.*

proof. We have $\left|J^i\right| \leq ab^2 \int_{\mathbb{R}^3} \frac{|p^i| f(t,\overline{p})}{p^0} d\overline{p} \leq \frac{ab^2}{\sqrt{g_{ii}}} \int_{\mathbb{R}^3} (1+\varrho) f d\overline{p} \leq C \|f\|_{L^2_d(\mathbb{R}^3)}$ if $d > \frac{5}{2}$ as we have seen it previously, where by (4) , $C = C(a_0, b_0, T)$. So $J^i \in H^0_d\left(\mathbb{R}^3\right)$ and $\left|J^i\right| \leq C(a_0, b_0, T) \|f\|_{H^3_d\left(\mathbb{R}^3\right)}$, $i = 1, 1, 3$. □

Remark 4. The hypothesis of proposition 1 concerning the collision kernel σ is a supplementary hypothesis for the investigation of the solution to the Boltzmann equation.

In what is to follow, we are searching the local existence and the uniqueness of the solution to the Maxwell-Boltzmann system in a function space which we will precise, applying the standard theory of first order differential systems to the system (33) − (34) .

The framework we will refer to for f is $\overline{H}^3_d\left(\mathbb{R}^3\right)$. The framework we will refer to for $\overline{E} = \left(E^i\right)$ is $\mathbb{R}^3$, whose the norm is denoted $\|\cdot\|$ (or $\|\cdot\|_{\mathbb{R}^3}$) .

$$C\left(\left[[0,T]\,;\mathbb{R}^3\right]\right) = \left\{\overline{E} : [0,T] \longrightarrow \mathbb{R}^3 \text{ continuous and bounded}\right\}.$$

$C\left(\left[[0,T]\,;\mathbb{R}^3\right]\right)$ is a Banach space for the norm:

$$\left|\left|\left|\overline{E}\right|\right|\right| = \sup\left\{\overline{E}(t)\,,\ t \in [0,T]\right\}.$$

We will establish the local existence theorem using $H^3_d\left(\mathbb{R}^3\right)$, which shall subsist in $\overline{H}^3_d\left(\mathbb{R}^3\right)$ by completion.

- We consider on $\mathbb{R}^3 \times H^3_d\left(\mathbb{R}^3\right)$ the norm:

$$\left\|\left(\overline{E}, f\right)\right\| = \left\|\overline{E}\right\| + \|f\|. \tag{37}$$

- we consider on $C\left(\left[[0,T]\,;\mathbb{R}^3\right]\right) \times H^3_d\left(0,T,\mathbb{R}^3\right)$ the norm:

$$\left|\left|\left|\left(\overline{E}, f\right)\right|\right|\right| = \left|\left|\left|\overline{E}\right|\right|\right| + |||f|||. \tag{38}$$

- We will consider the Cauchy problem for the system (33) − (34) for the initial data

$$E^i(0) = E^i_0,\ f(0,\overline{p}) = f_0 \tag{39}$$

where f_0 is given in $H^3_{d,r}\left(0,T,\mathbb{R}^3\right)$ and $E^i_0 \in \mathbb{R},\ i = 1,2,3.$

5. Local Existence Theorem

Theorem 1. *Let $\widetilde{f} \in H^3_{d,r}\left(0,T,\mathbb{R}^3\right)$ be given, and $\widetilde{\overline{E}} = \left(\widetilde{E^i}\right) \in \mathbb{R}^3, i = 1,2,3$* ***be fixed****. Then the linearized partial differential equation*

$$\frac{\partial f}{\partial t} + \left[-2\Gamma^i_{0i}p^i + \left(-\widetilde{E}^i + g^{ii}\frac{p^k\varphi_{ik}}{p^0}\right)\int_{\mathbb{R}^3}\widetilde{f}(t,\overline{p})\,ab^2 d\overline{p}\right]\frac{\partial f}{\partial p^i} = \frac{1}{p^0}Q\left(\widetilde{f},\widetilde{f}\right) \tag{40}$$

whose unknown is f, with $f(0,\overline{p}) = f_0$ has in $H^3_d\left(\mathbb{R}^3\right)$ a local unique and bounded $*-$weakly solution.

proof. We give the method, for the other details, See Mucha (1998) .

We use the Faedo-Galerkin method in the function space $H^3_d\left(\mathbb{R}^3\right)$.

We choose an Hilbertian orthonormal base $(w_k) \subset H^3_d\left(\mathbb{R}^3\right)$ and we take $\widetilde{f} \in H^3_{d,r}\left(\mathbb{R}^3\right)$.

We are searching a solution f of the linearized Boltzmann Equation (40) as a limit of sequence of approximations

$$f^N = \sum_{k=1}^{N} c_k(t)\,w_k,\quad N \in \mathbb{N}^* \tag{41}$$

where the components $c_k(t)$ are differentiable with respect to t and are given as solutions of N linear ordinary differential equations

$$\left(\partial_t f^N / w_k\right) + \left(\frac{\widetilde{P}^i}{p^0}\partial_{p^i} f^N / w_k\right) = \left(\frac{1}{p^0}Q\left(\widetilde{f},\widetilde{f}\right)/w_k\right) \tag{42}$$

and where $(/)$ stands for the scalar product in $L_d^2\left(\mathbb{R}^3\right)$.

The initial data are

$$c_k(0) = (f_0/w_k) \tag{43}$$

in which f_0 stands for the initial datum of the Boltzmann Equation (40).

We are looking for an estimation of $\left\|f^N\right\|_{H_d^3(\mathbb{R}^3)}$ independant on N.

Multiplying the relation (42) by $c_k(t)$ and summing for k going to 1 from N, we find:

$$\left(\partial_t f^N/f^N\right) + \left(\frac{\widetilde{P^i}}{p^0}\partial_{p^i} f^N/f^N\right) = \left(\frac{1}{p^0}Q\left(\widetilde{f},\widetilde{f}\right)/f^N\right). \tag{44}$$

We also have using (41) that $f^N \in H_d^3\left(\mathbb{R}^3\right)$.

We now observe by (42) that

$$\left(\partial_t f^N + \tfrac{\widetilde{P^i}}{p^0}\partial_{p^i} f^N - \tfrac{1}{p^0}Q\left(\widetilde{f},\widetilde{f}\right)/w_k\right) = 0,\ \forall k \in \mathbb{N}^*,$$

so $V = \partial_t f^N + \frac{\widetilde{P^i}\left(\widetilde{\widetilde{E}},\widetilde{f}\right)}{p^0}\partial_{p^i} f^N - \frac{1}{p^0}Q\left(\widetilde{f},\widetilde{f}\right)$ is orthogonal in $H_d^3\left(\mathbb{R}^3\right)$ to the subspace generated by the base $(w_k)_{k\in\mathbb{N}^*}$ which is dense in $H_d^3\left(\mathbb{R}^3\right)$, so is V for the whole space $H_d^3\left(\mathbb{R}^3\right)$. Thus $V = 0$ and consequently

$$\frac{\partial f^N}{\partial t} + \frac{\widetilde{P^i}\left(\widetilde{\widetilde{E}},\widetilde{f}\right)}{p^0}\frac{\partial f^N}{\partial p^i} = \frac{1}{p^0}Q\left(\widetilde{f},\widetilde{f}\right).$$

f^N is built as solution of (40) which verifies following (41) – (43) :

$$f^N(0) = \sum_{k=1}^{N} c_k(0)\, w_k = \sum_{k=1}^{N}(f_0/w_k)\, w_k.$$

By energy estimations, we find

$$\frac{d}{dt}\left(\left\|(1+|\overline{p}|)^{d+|\beta|}\,\partial^\beta f^N\right\|_{L^2(\mathbb{R}^3)}\right) \leq$$
$$C\left(\left\|(1+|\overline{p}|)^{d+|\beta|}\,\partial^\beta f^N\right\|_{L^2(\mathbb{R}^3)}\right) + C\left\|\frac{1}{p^0}Q\left(\widetilde{f},\widetilde{f}\right)\right\|_{H_d^3(\mathbb{R}^3)} \tag{45}$$

Using proposition 1 and the Gronwall inequality, we find

$$\left\|f^N\right\|_{H_d^3(\mathbb{R}^3)} \leq C.$$

Accordingly, we can choose a weak convergent subsequence of the bounded sequence of approximations $\left(f^N\right)$ in the reflexif space $H_d^3\left(\mathbb{R}^3\right)$

$$f^{N_k} \rightharpoonup f,\ \ k \to +\infty$$

where f is the unique solution of (40) in $H_d^3\left(\mathbb{R}^3\right)$ such that $f(0) = f_0$.

Consequently

$$\|f\|_{H_d^3(\mathbb{R}^3)} \leq C.$$

□

Theorem 2. *Let* $\widetilde{\widetilde{E}} = \left(\widetilde{E^i}\right) \in \mathbb{R}^3, i = 1,2,3$ *be given. Then the Boltzmann equation*

$$\frac{\partial f}{\partial t} + \left[-2\Gamma^i_{0i}p^i + \left(-\widetilde{E^i} + g^{ii}\frac{p^k\varphi_{ik}}{p^0}\right)\int_{\mathbb{R}^3} f(t,\overline{p})\, ab^2 d\overline{p}\right]\frac{\partial f}{\partial p^i} = \frac{1}{p^0}Q(f,f)$$

has in $H_d^3\left(\mathbb{R}^3\right)$ *a local unique ∗-weak solution* f *such that* $f(0) = f_0$.

proof. We use the Banach fixed point theorem in $H^3_d\left(\mathbb{R}^3\right)$ for the map:

$$\widetilde{f} \in H^3_{d,r}\left(\mathbb{R}^3\right) \longmapsto \Xi\left(\widetilde{f}\right) = f$$

where f satifies equation (40).

• We would like firstly to prove that we can choose $\|f_0\|_{H^3_{d,r}(\mathbb{R}^3)}$ and $T > 0$ such that

$$\widetilde{f} \in H^3_{d,r}\left(\mathbb{R}^3\right) \Rightarrow \Xi\left(\widetilde{f}\right) = f \in H^3_{d,r}\left(\mathbb{R}^3\right). \tag{46}$$

In what is to follow, $C = C\left(a_0, b_0, r, T, \left|E^i_0\right|, \left|\varphi_{ij}\right|\right)$.

Let us assume that $\widetilde{f} \in H^3_{d,r}\left(\mathbb{R}^3\right)$. Applying directly the Gronwall inequality to (45), using proposition 1, we find

$$\left\|(1+|\overline{p}|)^{d+|\beta|}\,\partial^\beta f^N\right\|_{L^2(\mathbb{R}^3)} \leq e^C\left\{\|f_0\|_{H^3_d(\mathbb{R}^3)} + Cr^2 t\right\}.$$

Reporting the above inequality in (45) yields

$$\frac{d}{dt}\left(\left\|(1+|\overline{p}|)^{d+|\beta|}\,\partial^\beta f^N\right\|_{L^2(\mathbb{R}^3)}\right) \leq C\left\{e^C\left\{\|f_0\|_{H^3_d(\mathbb{R}^3)} + Cr^2 t\right\}\right\} + Cr^2.$$

Integrating then over $[0,t]$ and using $t \leq T$ implies that

$$\left\|(1+|\overline{p}|)^{d+|\beta|}\,\partial^\beta f^N\right\|_{L^2(\mathbb{R}^3)} \leq \|f_0\|_{H^3_d(\mathbb{R}^3)} + Ce^C\left(\|f_0\|_{H^3_d(\mathbb{R}^3)}\,T + \frac{Cr^2T^2}{2}\right) + Cr^2 T.$$

If we take

$$\begin{cases} \|f_0\|_{H^3_d(\mathbb{R}^3) \leq \frac{r}{2}} \\ C\left(e^C+1\right)\left(\frac{rT}{2} + \frac{Cr^2T^2}{2} + Cr^2T\right) \leq \frac{r}{2}, \end{cases}$$

then

$$\left\|f^N\right\|_{H^3_d(\mathbb{R}^3)} \leq r,\ \ N \in \mathbb{N}^*.$$

Because $f^{N_k} \rightharpoonup f,\ \ k \to +\infty$ in $H^3_d\left(\mathbb{R}^3\right)$, it results that

$$\|f\|_{H^3_d(\mathbb{R}^3)} \leq r.$$

• Let now $\widetilde{f_1}, \widetilde{f_2} \in H^3_{d,r}\left(\mathbb{R}^3\right)$ be given, and f_1, f_2 be two solutions of (40). Then:

$$\begin{cases} \frac{\partial f_1}{\partial t} + \left[-2\Gamma^i_{0i}p^i + \left(-\widetilde{E}^i + g^{ii}\frac{p^k\varphi_{ik}}{p^0}\right)\int_{\mathbb{R}^3}\widetilde{f_1}\left(t,\overline{p}\right)ab^2 d\overline{p}\right]\frac{\partial f_1}{\partial p^i} = \frac{1}{p^0}Q\left(\widetilde{f_1}, \widetilde{f_1}\right) \\ \frac{\partial f_2}{\partial t} + \left[-2\Gamma^i_{0i}p^i + \left(-\widetilde{E}^i + g^{ii}\frac{p^k\varphi_{ik}}{p^0}\right)\int_{\mathbb{R}^3}\widetilde{f_2}\left(t,\overline{p}\right)ab^2 d\overline{p}\right]\frac{\partial f_2}{\partial p^i} = \frac{1}{p^0}Q\left(\widetilde{f_2}, \widetilde{f_2}\right). \end{cases}$$

Let $G = f_1 - f_2$ and $\widetilde{G} = \widetilde{f_1} - \widetilde{f_2}$.

Then we get

$$\frac{\partial G}{\partial t} + \frac{\widetilde{P^i}\left(\widetilde{E}, \widetilde{G}\right)}{p^0}\frac{\partial G}{\partial p^i} = \frac{1}{p^0}Q\left(\widetilde{f_1}, \widetilde{G}\right) - \frac{1}{p^0}Q\left(\widetilde{G}, \widetilde{f_2}\right).$$

Invoking (45), applying the Gronwall inequality, using proposition 1 and remembering that $G(0,\overline{p}) = 0$, we obtain:

$$\|G\|_{H^3_d(\mathbb{R}^3) \leq C\left(a_0,b_0,r,T,\left|E^i\right|,\left|\varphi_{ij}\right|\right)T\|\widetilde{G}\|_{H^3_d}\left(\mathbb{R}^3\right)}. \tag{47}$$

where $C_2 = C\left(a_0, b_0, r, T, \left|E^i\right|, \left|\varphi_{ij}\right|\right)T$ is a positive constant.

Since the function $T \mapsto C\left(a_0, b_0, r, T, \left|E^i\right|, \left|\varphi_{ij}\right|\right)T$ is continuous at the point $T = 0$, because $C\left(a_0, b_0, r, T, \left|E^i\right|, \left|\varphi_{ij}\right|\right)$ is also a polynomial in T, we can choose $T > 0$ small enough such that

$$C\left(a_0, b_0, r, T, \left|E^i\right|, \left|\varphi_{ij}\right|\right) < 1. \tag{48}$$

The relations (46) , (47) and (48) show clearly that

$H^3_{d,r}\left(\mathbb{R}^3\right) \to H^3_{d,r}\left(\mathbb{R}^3\right)$, $\widetilde{f} \longmapsto \Xi\left(\widetilde{f}\right) = f$ is a contracting map, so by the Banach theorem Ξ has a unique fixed point $f = \widetilde{f}$ and the proof of theorem 2 is complete.

Let $I_1 = I_1\left(t, \overline{E}, f\right)$ denotes the r.h.s of (33) , i.e

$$I_1\left(t, \overline{E}, f\right) = \left(HE^i + \int_{\mathbb{R}^3} \frac{p^i f\left(t, \overline{p}\right) ab^2}{p^0} d\overline{p}\right),\ i = 1, 2, 3.$$

We prove:

Proposition 3. *Let* $\overline{E}_1, \overline{E}_2 \in \mathbb{R}^3,\ f_1, f_2 \in H^3_d\left(\mathbb{R}^3\right)$ *be given. Then*

$$\left\|I_1\left(t, \overline{E_1}, f_1\right) - I_1\left(t, \overline{E_2}, f_2\right)\right\|_{\mathbb{R}^3} \leq C_3\left(\left\|\overline{E}_1 - \overline{E}_2\right\| + \|f_1 - f_2\|\right) \tag{49}$$

where $C_3 = C\left(3\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) + 2ab + b^2\right)$.

proof. We have using (33)

$I_1\left(t, \overline{E_1}, f_1\right) - I_1\left(t, \overline{E_2}, f_2\right) =$

$$\left(H\left(E^i_1 - E^i_2\right) + \int_{\mathbb{R}^3} \frac{p^i\left(f_1\left(t, \overline{p}\right) - f_2\left(t, \overline{p}\right)\right) ab^2}{p^0} d\overline{p}\right).$$

So by (3) and proposition 2

$\left|H\left(E^i_1 - E^i_2\right) + \int_{\mathbb{R}^3} \frac{p^i(f_1(t,\overline{p}) - f_2(t,\overline{p}))ab^2}{p^0} d\overline{p}\right| \leq$

$$\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)\left|E^i_1 - E^i_2\right| + \frac{Cab^2}{\sqrt{g_{ii}}}\|f_1 - f_2\|_{L^2_d(\mathbb{R}^3)}.$$

It follows that

$\left|H\left(E^1_1 - E^1_2\right) + \int_{\mathbb{R}^3} \frac{p^1(f_1(t,\overline{p}) - f_2(t,\overline{p}))ab^2}{p^0} d\overline{p}\right|$

$$C\left(\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) + b^2\right)\left(\left|E^1_1 - E^1_2\right| + \|f_1 - f_2\|_{L^2_d(\mathbb{R}^3)}\right)$$

and

$\left|H\left(E^i_1 - E^i_2\right) + \int_{\mathbb{R}^3} \frac{p^i(f_1(t,\overline{p}) - f_2(t,\overline{p}))ab^2}{p^0} d\overline{p}\right| \leq$

$$C\left(\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) + ab\right)\left(\left|E^i_1 - E^i_2\right| + \|f_1 - f_2\|_{L^2_d(\mathbb{R}^3)}\right),\ i = 2, 3.$$

Thus

$\left\|I_1\left(t, \overline{E_1}, f_1\right) - I_1\left(t, \overline{E_2}, f_2\right)\right\|_{\mathbb{R}^3} \leq$

$$C\left(3\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) + 2ab + b^2\right)\left(\left\|\overline{E}_1 - \overline{E}_2\right\|_{\mathbb{R}^3} + \|f_1 - f_2\|_{H^3_d(\mathbb{R}^3)}\right).$$

□

Proposition 4. *The system* (33) − (34) *is equivalent to the system* $(S_1) - (S_2)$ *given below:*

$$(S)\quad \begin{cases} \frac{dE^i}{dt} = HE^i + \int_{\mathbb{R}^3} \frac{p^i f(t,\overline{p})ab^2}{p^0} d\overline{p} & (S_1) \\ \frac{df}{dt} = \frac{1}{p^0} Q\left(f, f\right) & (S_2) \\ E^i\left(0\right) = E^i_0, F_{ij}\left(0\right) = \varphi_{ij}, i, j = 1, 2, 3. \end{cases}$$

proof. It will be sufficient to show that (34) is equivalent to (S_2).

In fact, the trajectories of the charged particles are given by the relations (8) which we recall write:

$$\frac{dx^\alpha}{ds} = p^\alpha; \frac{dp^\alpha}{ds} = P^\alpha.$$

Since $x^0 = t$, the equation $\frac{dx^\alpha}{ds} = p^\alpha$ for $\alpha = 0$ gives

$$\frac{dt}{ds} = p^0$$

and allows to take t as the parameter. Writing the trajectories for $\alpha = i$ and using the above equality leads to:

$$\frac{dp^i}{dt} = \frac{ds}{dt} \times \frac{dp^i}{ds} = \frac{1}{p^0}\frac{dp^i}{ds} = \frac{P^i}{p^0}.$$

Now, on the trajectories $\left(t, p^i(t)\right)$, we have $f = f\left(\left(t, p^i(t)\right)\right)$, so

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial p^i}\frac{dp^i}{dt} = \frac{\partial f}{\partial t} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i}$$

and (31) allows us to conclude that

$\frac{\partial f}{\partial t} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i} = \frac{1}{p^0}Q(f, f)$ is equivalent to $\frac{df}{dt} = \frac{1}{p^0}Q(f, f)$.

Since (31) and (34) are equivalent, we conclude the proof. □

We are setting in what is to follow:

$$I_2\left(t, \overline{E}, f\right) = \frac{1}{p^0}Q(f, f), \quad I = (I_1, I_2).$$

Notice that I is the r.h.s of the system $(S_1) - (S_2)$.

We have for $\overline{E}_1, \overline{E}_2 \in \mathbb{R}^3$, $f_1, f_2 \in H^3_d\left(\mathbb{R}^3\right)$, using proposition 1 :

$$\left\|I_2\left(t, \overline{E_1}, f_1\right) - I_2\left(t, \overline{E_2}, f_2\right)\right\|_{H^3_d(\mathbb{R}^3)} \leq C_4\left(\left\|\overline{E}_1 - \overline{E}_2\right\|_{\mathbb{R}^3} + \|f_1 - f_2\|_{H^3_d(\mathbb{R}^3)}\right) \tag{50}$$

where $C_4 = 4\pi ab^2(\|f_1\| + \|f_2\|)$.

Theorem 3. *Let $\left(\overline{E_0}, f_0\right) \in \mathbb{R}^3 \times H^3_d\left(\mathbb{R}^3\right)$ be given. Then:*
there exists a real number $\delta > 0$ such that the system $(S_1) - (S_2)$ has a unique solution

$$\left(\overline{E}, f\right) \in C\left(\left[[0, \delta]; \mathbb{R}^3\right]\right) \times H^3_d\left(0, \delta, \mathbb{R}^3\right)$$

satisfying $\left(\overline{E}, f\right)(0) = \left(\overline{E_0}, f_0\right)$. Moreover, f satisfies the relation:

$$|||f||| = \sup\{\|f(t)\|, t \in [0, \delta]\} \leq |||f_0|||. \tag{51}$$

proof. We apply the standard theory on the first order differential systems to $(S_1) - (S_2)$.

Since $a, b, \frac{\dot{a}}{a}, \frac{\dot{b}}{b}, \sigma$ are continuous functions of t,so is the function

$$I\left(t, \overline{E}, f\right) = \left(HE^i + \int_{\mathbb{R}^3}\frac{p^i f(t, \overline{p})\, ab^2}{p^0}d\overline{p}, \frac{1}{p^0(t)}Q(f, f)\right),\ i = 1, 2, 3.$$

By continuity of the functions $z = a, b, \frac{\dot{a}}{a}, \frac{\dot{b}}{b}$ at $t = 0$, there exists a real number $\delta_0 > 0$ such that:

$$t \in]-\delta_0, \delta_0[\implies |z(t)| \leq |z(0)| + 1.$$

The above relation implies, using (3) and (4) to bound $z = a, b, \frac{\dot{a}}{a}, \frac{\dot{b}}{b}$ that:

$$t \in]-\delta_0, \delta_0[\implies |z(t)| \leq (a_0 + b_0 + 1)C + 1.$$

Next, set $B(f_0,1)=\left\{f\in H_d^3\left(\mathbb{R}^3\right),\|f-f_0\|<1\right\}$.

Then:

$$f\in B(f_0,1)\Longrightarrow\|f\|\le\|f_0\|+1.$$

Now consider the neighborhood $V_0=\left]-\delta_0,\delta_0\right[\times\mathbb{R}^3\times B(f_0,1)$ of $\left(0,\overline{E_0},f_0\right)$ in the Banach space $\mathbb{R}\times\mathbb{R}^3\times H_d^3\left(\mathbb{R}^3\right)$ and take

$$\left(t,\overline{E_1},f_1\right),\left(t,\overline{E_2},f_2\right)\in V_0.$$

We deduce from the inequalities (49), (50), the definitions of C_3,C_4, the implications $t\in\left]-\delta_0,\delta_0\right[\Longrightarrow|z(t)|\le(a_0+b_0+1)C+1$ and $f\in B(f_0,1)\Longrightarrow\|f\|\le\|f_0\|+1$, that there exists an absolute constant

$$C_5=C_5\left(a_0,b_0,f_0,\left|E_0^i\right|,\left|\varphi_{ij}\right|\right)$$

such that:

$$\left\|I\left(t,\overline{E_1},f_1\right)-I\left(t,\overline{E_2},f_2\right)\right\|_{\mathbb{R}^3\times H_d^3(\mathbb{R}^3)}\le C_5\left(\left\|\overline{E}_1-\overline{E}_2\right\|_{\mathbb{R}^3}+\|f_1-f_2\|_{H_d^3(\mathbb{R}^3)}\right)\tag{52}$$

which shows that $I=(I_1,I_2)$ is locally Lipschitzian in $\left(\overline{E},f\right)$ with repect to the norm of the Banach space $\mathbb{R}^3\times H_d^3\left(\mathbb{R}^3\right)$. The existence of a unique solution $\left(\overline{E},f\right)$ of the differential system $(S_1)-(S_2)$ on a interval $[0,\delta]$, $\delta>0$, such that $\left(\overline{E},f\right)(0)=\left(\overline{E_0},f_0\right)$ is guaranteed by the standard theory on the first order differential systems.

The relation $|||f|||=\sup\{\|f(t)\|,\ t\in[0,\delta]\}\le|||f_0|||$ is established in Noutchegueme & Dongho (2006).

As a direct consequence, we can deduce that there exists a real number $\delta>0$ such that the Maxwell-Boltzmann system (12) − (13) − (31) has a unique solution (F,f) on $[0,\delta]$ satisfying

$$F^{0i}(0)=E_0^i;\ F_{ij}(0)=\varphi_{ij};\ f(0)=f_0.$$

□

6. Global Existence Theorem

6.1 The Method

Denote by $[0,T[$ the maximal existence domain of solution, denoted here by $\left(\widetilde{\overline{E}},\widetilde{f}\right)$ and given by theorem 3, of the system $(S_1)-(S_2)$, with the initial data $\left(\widetilde{\overline{E_0}},f_0\right)\in\mathbb{R}^3\times H_d^3\left(\mathbb{R}^3\right)$. We want to prove that $T=+\infty$.

- If we already have $T=+\infty$, then the problem of global existence is solved.
- We are going to show that, if we suppose $T<+\infty$, then the solution $\left(\widetilde{\overline{E}},\widetilde{f}\right)$ can be extended beyond T, which contradicts the maximality of T.
- The strategy is as follows: suppose $0<T<+\infty$ and let $t_0\in[0,T]$. We will show that there exists a strictly positive number $\delta>0$ **independant of** t_0 such that the system $(S_1)-(S_2)$ on $[t_0,t_0+\delta]$, with the initial data $\left(\widetilde{\overline{E}}(t_0),\widetilde{f}(t_0)\right)$ at $t=t_0$, has a unique solution $\left(\overline{E},f\right)$ on $[t_0,t_0+\delta]$. Then, by taking t_0 sufficiently close to T, for instance t_0 such that $0<T-t_0<\frac{\delta}{2}$, hence $T<t_0+\frac{\delta}{2}$, we can extend the solution $\left(\widetilde{\overline{E}},\widetilde{f}\right)$ to $\left[0,t_0+\frac{\delta}{2}\right]$,which contains strictly $[0,T[$, and this contradicts the maximality of T. In order to simplify the notations, it will be enough if we could look for a number δ such that $0<\delta<1$.
- In what follows we fix a number $r>0$ and we take f_0 such that $\|f_0\|\le r$ or equivalently such that $f_0\in H_{d,r}^3\left(\mathbb{R}^3\right)$.

By (51) we have:

$$\left\|\widetilde{f}_{t_0}\right\|\le\|f_0\|.\tag{53}$$

We also deduce from (53) using (51) that any solution f of the Boltzmann equation on $[t_0,t_0+\delta]$ such that $f(t_0)=\widetilde{f}(t_0)$, satisfies:

$$\|f(t)\|\le r,\ t\in[t_0,t_0+\delta].\tag{54}$$

Notice that (54) shows that a solution $\left(\overline{E},f\right)$ of the system $(S_1)-(S_2)$ on $[t_0,t_0+\delta]$, $\delta>0$, such that $\left(\overline{E},f\right)(t_0)=\left(\widetilde{\overline{E}}(t_0),\widetilde{f}(t_0)\right)$ satisfies:

$$\left(\overline{E},f\right)\in C\left([t_0,t_0+\delta];\mathbb{R}^3\right)\times H_{d,r}^3\left(t_0,t_0+\delta,\mathbb{R}^3\right).\tag{55}$$

In what follows, $[0, T[$, $T > 0$ is the maximal existence domain of solution $\left(\widetilde{\overline{E}}, \widetilde{f}\right)$ of $(S_1) - (S_2)$ such that $\left(\widetilde{\overline{E}}, \widetilde{f}\right)(0) = \left(\widetilde{\overline{E}_0}, \widetilde{f_0}\right) \in \mathbb{R}^3 \times H^3_{d,r}\left(\mathbb{R}^3\right)$.

We prove the following result which shows helpful in what is to follow:

Lemma 1. *$t \longmapsto \widetilde{\overline{E}}(t)$ is uniformly bounded over* $[0, T]$.

proof. Consider (S_1) in which we set $\overline{E} = \widetilde{\overline{E}}$ and $f = \widetilde{f}$ i.e, the equality:

$$\dot{\widetilde{E^i}} = H\widetilde{E^i} + \int_{\mathbb{R}^3} \frac{p^i \widetilde{f}(t, \overline{p})\, ab^2}{p^0} d\overline{p}, \text{ on } [0, T]. \qquad \text{(a)}$$

We have, using $H = -\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)$, (3), (4) and (51) with $f = \widetilde{f}$:

$$|H| \leq C; \left|\int_{\mathbb{R}^3} \frac{p^i \widetilde{f}(t, \overline{p})\, ab^2}{p^0} d\overline{p}\right| \leq C^i_6, \qquad (b)$$

where $C^i_6 = C^i_6(a_0, b_0, r, T)$.

We then deduce from (a), using (b) that:

$$\left|\dot{\widetilde{E^i}}\right| \leq C\left|\widetilde{E^i}\right| + C^i_6,\ i = 1, 2, 3. \qquad (c)$$

Now, integrating (c) over $[0, t]$ $t \in [0, T[$, yiels, using $\widetilde{E^i}(0) = E^i_0$:

$$\left|\widetilde{E^i}(t)\right| \leq \left(\left|E^i_0\right| + C^i_6 T\right) + C\int_0^t \left|\widetilde{E^i}\right|(s)\, ds,\ t \in [0, T],\ i = 1, 2, 3.$$

The lemma then follows from Gronwall inequality which gives the following relation:

$$\left|\widetilde{E^i}(t)\right| \leq \left(\left|E^i_0\right| + C^i_6 T\right) e^{CT},\ t \in [0, T],\ i = 1, 2, 3. \tag{56}$$

This hence ends the proof of lemma 3.

6.2 Global Existence of Solutions

Firstly, we consider, for $t_0 \in [0, T]$ and $\delta > 0$,

$$\left(\overline{\overline{E}}, \overline{f}\right) \in C\left([t_0, t_0 + \delta]; \mathbb{R}^3\right) \times H^3_{d,r}\left(t_0, t_0 + \delta, \mathbb{R}^3\right).$$

Then, we built from system $(S_1) - (S_2)$ by setting in its r.h.s $I = (I_1, I_2)$ which is given by

$$I = (I_1, I_2) = \left(HE^i + \int_{\mathbb{R}^3} \frac{p^i f(t, \overline{p})\, ab^2}{p^0} d\overline{p}, \frac{1}{p^0(t)} Q(f, f)\right),\ i = 1, 2, 3:$$

$f = \overline{f}$ in I_1 and $\overline{E} = \overline{\overline{E}}$ in I_2, the following differential system:

$$\frac{dE^i}{dt} = \overline{I_1}\left(t, \overline{E}, \overline{f}\right) \tag{57}$$

$$\frac{df}{dt} = \overline{I_2}\left(t, \overline{\overline{E}}, f\right) \tag{58}$$

where

$$\overline{I_1}\left(t, \overline{E}, \overline{f}\right) = HE^i + \int_{\mathbb{R}^3} \frac{p^i \widetilde{f}(t, \overline{p})\, ab^2}{p^0} d\overline{p},\ i = 1, 2, 3. \tag{59}$$

$$\overline{I_2}\left(t, \overline{\overline{E}}, f\right) = \frac{1}{p^0(t)} Q(f, f). \tag{60}$$

It is important to notice that on the trajectories, the equation (60) is equivalent to the equation (34) which writes in our situation:

$$\frac{\partial f}{\partial t}+\left[-2\Gamma^i_{0i}p^i+\left(-\overline{E^i}+g^{ii}\frac{p^k\varphi_{ik}}{p^0}\right)\int_{\mathbb{R}^3}f\left(t,\overline{p}\right)ab^2d\overline{p}\right]\frac{\partial f}{\partial p^i}=\frac{1}{p^0}Q\left(f,f\right).$$

We claim:

Proposition 5. *Let* $t_0\in[0,T[\,,\ \delta\in\,]0,1[\,,$ *and* $\left(\overline{\overline{E}},\overline{f}\right)\in C\left([t_0,t_0+\delta]\,;\mathbb{R}^3\right)\times H^3_{d,r}\left(t_0,t_0+\delta,\mathbb{R}^3\right)$ *be given. Then, the differential system* (57) – (58) *has a unique solution* $\left(\overline{E},f\right)\in C\left([t_0,t_0+\delta]\,;\mathbb{R}^3\right)\times H^3_{d,r}\left(t_0,t_0+\delta,\mathbb{R}^3\right)$ *such that* $\left(\overline{E},f\right)(t_0)=\left(\widetilde{\overline{E}}(t_0)\,,\widetilde{f}(t_0)\right).$

proof. • We consider equation (57) in $\overline{E}$, with $\overline{I_1}$ defined by (59) in which $\overline{f}$ is fixed. Since $a,b,\dot{a},\dot{b},\frac{1}{a},\frac{1}{b},\overline{f}$ are continuous functions of t, so is $\overline{I_1}$. Next, we deduce from (49) in which we set $f_1=f_2=\overline{f}$ that:

$$\left\|I_1\left(t,\overline{E_1},\overline{f}\right)-I_1\left(t,\overline{E_2},\overline{f}\right)\right\|_{\mathbb{R}^3}\leq C_3\left\|\overline{E}_1-\overline{E}_2\right\| \tag{61}$$

where

$$C_3=C\left(3\left(\frac{\dot{a}}{a}+2\frac{\dot{b}}{b}\right)+2ab+b^2\right). \tag{62}$$

Now we can use (3) and (4) to bound $z=a,b,\frac{\dot{a}}{a},\frac{\dot{b}}{b}$ and we obtain, for $t\in[t_0,t_0+\delta]\,,$ then $t\leq t_0+\delta\leq T+1$:

$$|z(t)|\leq(C+a_0+b_0)\,e^{C(T+1)}+1,t\in[t_0,t_0+\delta]\,,\ z=a,b,\frac{\dot{a}}{a},\frac{\dot{b}}{b}. \tag{63}$$

We then deduce from (63) that:

$$C_3\leq C_3',\ \text{where}\ C_3'=C_1'\left(a_0,b_0,T\right). \tag{64}$$

By (61) and (64) , $\overline{I_1}$ is (globally) Lipschitzian with respect to the $\mathbb{R}^3-$ norm and the *local existence* of a solution $\overline{E}$ of (57) such that $\overline{E}(t_0)=\widetilde{\overline{E}}(t_0)$is guaranteed by the standard theory of first order differential systems.

Now, since $\overline{E}$ satisfies (57) in which $\overline{I_1}$ is given by (59) , following the same way as in the proof of lemma 3, substituting $\overline{E}$ to $\widetilde{\overline{E}}$, $\overline{f}$ to $\widetilde{f}$, using (63) and integrating this time over $[t_0,t_0+t]\,,\ t\in[0,\delta[\,,$ leads to:

$$\left|E^i(t_0+t)\right|\leq\left(\left|\widetilde{E^i}(t_0)\right|+C^i_7T\right)+C\int_{t_0}^{t_0+t}\left|E^i\right|(s)\,ds,\ t\in[0,\delta]\,,\ i=1,2,3$$

where $C^i_7=C^i_7\left(a_0,b_0,T,r\right)$. However, by lemma 3, and more precisely (56) , we have, since $t_0\,[0,T[$: $\left|\widetilde{E^i}(t_0)\right|\leq\left(\left|E^i_0\right|+C^i_6T\right)e^{CT}$. Then, by Gronwall inequality:

$$\left|E^i(t_0+t)\right|\leq\left(\left(\left|E^i_0\right|+C^i_6T\right)e^{CT}+C^i_7T\right)e^{C(T+1)},\ t\in[0,\delta]\,,\ i=1,2,3$$

which shows that, every solution $\overline{E}$ of (57) is *uniformly bounded*. By the standard theory on first order differential systems, the solution $\overline{E}$ is defined all over $[t_0,t_0+\delta]$ and $\overline{E}\in C\left([t_0,t_0+\delta]\,;\mathbb{R}^3\right)$.

• Next, we have proved in theorem 2 that the single equation (58) in f, which also writes

$$\frac{\partial f}{\partial t}+\left[-2\Gamma^i_{0i}p^i+\left(-\overline{E^i}+g^{ii}\frac{p^k\varphi_{ik}}{p^0}\right)\int_{\mathbb{R}^3}f\left(t,\overline{p}\right)ab^2d\overline{p}\right]\frac{\partial f}{\partial p^i}=\frac{1}{p^0}Q\left(f,f\right),$$

has a unique solution $f\in H^3_{d,r}\left(t_0,t_0+\delta,\mathbb{R}^3\right)$, substituting t_0 to 0,$t_0+\delta$ to T, such that $f(t_0)=\widetilde{f}(t_0)$ .This completes the proof of proposition 5.

In what is to follow we set:

$$X_\delta=C\left([t_0,t_0+\delta]\,;\mathbb{R}^3\right)\times H^3_{d,r}\left(t_0,t_0+\delta,\mathbb{R}^3\right). \tag{65}$$

X_δ is a complete metric subspace of the Banach space $C\left([t_0,t_0+\delta]\,;\mathbb{R}^3\right)\times H^3_d\left(t_0,t_0+\delta,\mathbb{R}^3\right)$. Proposition 5 allows us to define the map:

$$g\ :\ X_\delta\longrightarrow X_\delta,\ \left(\overline{\overline{E}},\overline{f}\right)\longmapsto\left(\overline{E},f\right). \tag{66}$$

We prove: □

Proposition 6. *Let* $t_0 [0, T]$.

There exists a number $\delta \in]0, 1[$, ***independant of*** t_0, *such that the system* $(S_1) - (S_2)$ *has a unique solution* $\left(\overline{E}, f\right) \in X_\delta$ *such that* $\left(\overline{E}, f\right)(t_0) = \left(\widetilde{\overline{E}}(t_0), \widetilde{f}(t_0)\right)$. *proof.* We will prove that there exists a number $\delta \in [0, 1]$, ***independant of*** t_0, such that the map g, defined by (66) is a contraction of the complete metric space X_δ defined by (65), which will then have a fixed point $\left(\overline{E}, f\right)$ solution of the system $(S_1) - (S_2)$.

With the initial data $\left(\widetilde{\overline{E}}(t_0), \widetilde{f}(t_0)\right)$ at $t = t_0$, the differential system (57) − (58) with $\overline{I_1}, \overline{I_2}$ given by (59) − (60) is equivalent to the integral system:

$$E^i(t_0 + t) = \widetilde{\overline{E}}(t_0) + \int_{t_0}^{t_0+t} \left\{ HE^i + \int_{\mathbb{R}^3} \frac{p^i \overline{f}(t, \overline{p})\, ab^2}{p^0} d\overline{p} \right\}(\tau)\, d\tau, \tag{67}$$

$$f(t_0 + t) = \widetilde{f}(t_0) + \int_{t_0}^{t_0+t} \frac{1}{p^0} Q(f, f)(\tau)\, d\tau,\ t \in [0, \delta],\ i = 1, 2, 3. \tag{68}$$

To $\left(\overline{\overline{E_j}}, \overline{f_j}\right) \in X_\delta,\ j = 1, 2$, corresponds the solutions $\left(\overline{E_j}, f_j\right) \in X_\delta,\ j = 1, 2$, whose existence is proved in proposition 5. We now write the integral system (67) − (68) for $j = 1$ and $j = 2$, and taking the differences, using notations (57) − (58), we obtain the following, with $i = 1, 2, 3$:

$$\left(E_1^i - E_2^i\right)(t_0 + t) = \int_{t_0}^{t_0+t} \left[\overline{I_1}\left(t, \overline{E_1}, \overline{f_1}\right) - \overline{I_1}\left(t, \overline{E_2}, \overline{f_2}\right)\right](\tau)\, d\tau, \tag{69}$$

$$(f_1 - f_2)(t_0 + t) = \int_{t_0}^{t_0+t} \left[\overline{I_2}\left(t, \overline{\overline{E}}_1, f_1\right) - \overline{I_2}\left(t, \overline{\overline{E}}_2, f_2\right)\right](\tau)\, d\tau. \tag{70}$$

Now we can deduce from (49) in which we set $f_1 = \overline{f_1},\ f_2 = \overline{f_2}$:

$$\left\| I_1\left(t, \overline{E_1}, f_1\right) - I_1\left(t, \overline{E_2}, f_2\right)\right\|_{\mathbb{R}^3} \leq C_3' \left(\left\|\overline{E}_1 - \overline{E}_2\right\| + \left\|\overline{f_1} - \overline{f_2}\right\| \right)(\tau) \tag{71}$$

where $C_3' = C_3'(a_0, b_0, T)$ is still given by (64).

Next, since $\left(\overline{E_j}, f_j\right) \in X_\delta$, we deduce from (50) in which we set $\overline{E_1} = \overline{\overline{E_1}},\ \overline{E_2} = \overline{\overline{E_2}}$ and using C_4 still given in (50), $\left\|f_j(t)\right\| \leq \left|\left|\left|f_j\right|\right|\right| \leq r,\ j = 1, 2$:

$$\left\| I_2\left(t, \overline{\overline{E}}_1, f_1\right) - I_2\left(t, \overline{\overline{E}}_2, f_2\right)\right\|_{H_d^3(\mathbb{R}^3)} \leq C_4' \left(\left\|\overline{\overline{E}}_1 - \overline{\overline{E}}_2\right\|_{\mathbb{R}^3} + \|f_1 - f_2\|_{H_d^3(\mathbb{R}^3)} \right)(\tau) \tag{72}$$

where $C_4' = C_4'(a_0, b_0, T, r)$.

Already notice that the constants C_3' and C_4' are absolute constants independent of t_0.

Now using the inequalities (71) and (72), we deduce from (69) and (70), using the norm $|||.|||$ and since $t \in [0, \delta]$:

$$\left|\left|\left|\overline{E}_1 - \overline{E}_2\right|\right|\right| \leq C_3'\delta \left(\left|\left|\left|\overline{E}_1 - \overline{E}_2\right|\right|\right| + \left|\left|\left|\overline{f_1} - \overline{f_2}\right|\right|\right| \right) \tag{73}$$

$$|||f_1 - f_2||| \leq C_4'\delta \left(\left|\left|\left|\overline{\overline{E}}_1 - \overline{\overline{E}}_2\right|\right|\right| + |||f_1 - f_2||| \right). \tag{74}$$

Now add (73) and (74) to obtain:

$\left|\left|\left|\overline{E}_1 - \overline{E}_2\right|\right|\right| + |||f_1 - f_2||| \leq$

$$\left(C_3' + C_4'\right)\delta \left(\left|\left|\left|\overline{E}_1 - \overline{E}_2\right|\right|\right| + |||f_1 - f_2||| \right) + \left(C_3' + C_4'\right)\delta \left(\left|\left|\left|\overline{\overline{E}}_1 - \overline{\overline{E}}_2\right|\right|\right| + \left|\left|\left|\overline{f_1} - \overline{f_2}\right|\right|\right| \right). \tag{75}$$

If we take δ such that:

$$0 < \delta < \inf \left\{ 1, \frac{1}{4\left(C_3' + C_4'\right)} \right\}, \tag{76}$$

(76) implies in particular $0 < \left(C_3' + C_4'\right)\delta < \frac{1}{4}$, from which we deduce, by sending the first term of r.h.s of (75) to l.h.s that:

$$\frac{3}{4}\left|\left|\left|\overline{E}_1 - \overline{E}_2\right|\right|\right| + |||f_1 - f_2||| \leq \frac{1}{4}\left(\left|\left|\left|\overline{\overline{E}}_1 - \overline{\overline{E}}_2\right|\right|\right| + \left|\left|\left|\overline{f_1} - \overline{f_2}\right|\right|\right|\right) \tag{77}$$

and (77) gives:

$$\left|\left|\left|\overline{E}_1 - \overline{E}_2\right|\right|\right| + |||f_1 - f_2||| \leq \frac{1}{3}\left(\left|\left|\left|\overline{\overline{E}}_1 - \overline{\overline{E}}_2\right|\right|\right| + \left|\left|\left|\overline{f_1} - \overline{f_2}\right|\right|\right|\right). \tag{78}$$

(78) shows that $g : X_\delta \longrightarrow X_\delta, \left(\overline{\overline{E}}, \overline{f}\right) \longmapsto \left(\overline{E}, f\right)$ is a contracting map in the complete metric space X_δ which then has a unique fixed point $\left(\overline{E}, f\right)$, solution of the integral system $(67) - (68)$ and hence, of the differential system $(S_1) - (S_2)$ such that $\left(\overline{E}, f\right)(t_0) = \left(\widetilde{\overline{E}}(t_0), \widetilde{f}(t_0)\right)$.

Consequently, for $t_0\ [0, T[$ *, there exists a number* $\delta \in\ ]0, 1[$ *,* ***independant of*** t_0*, such that the system* $(33) - (34)$ *has a unique solution* $\left(\overline{E}, f\right) \in X_\delta$ *such that* $\left(\overline{E}, f\right)(t_0) = \left(\widetilde{\overline{E}}(t_0), \widetilde{f}(t_0)\right)$. This completes the proof of proposition 6.

Based on the method detailed in subsection 6.1, we have proved the following result:

Theorem 4. *Let* $\overline{E_0} = \left(E_0^i\right) \in \mathbb{R}^3$, $\varphi_{ij} \in \mathbb{R}$, $f_0 \in H_d^3\left(\mathbb{R}^3\right)$ *be given, such that* $\|f_0\| \leq r$*, where* $r > 0$ *is a given real number. Then:*

$1\cdot$*the differential system* $(33) - (34)$ *has a unique global solution* $\left(\overline{E}, f\right)$ *defined all over the interval* $[0, +\infty[$ *and such that* $\left(\overline{E}, f\right)(0) = \left(\overline{E_0}, f_0\right)$,

2. *the Maxwell-Boltzmann system* $(12) - (13)) - (31)$ *has a unique global solution* (F, f) *defined all over the interval* $[0, +\infty[$ *and satisfying:* $\overline{E}(0) = \overline{E_0}$, $F_{ij}(0) = \varphi_{ij}$, $f(0) = f_0$.

7. Conclusions

The physical significance of the work we did in the present paper, is the study of the global dynamics of a kind of fast moving, massive and charged particles. We have coupled the Boltzmann equation to the Maxwell equations for the Electromagnetic field, represented by the unknown$F$. Notice that this present work follows our paper titled "The Faedo-Galerkin method for the relativistic Boltzmann equation ", where the unknown was the distribution function f, subject to the Boltzmann equation. This time,the electromagnetic field F, also becomes an unknown function, subject to the Maxwell equations.

We also introduce new function spaces, which allow, using the Galerkin method, the Sobolev inequalities, and some useful energy estimations, to obtain regular solutions, as their uniqueness for some given initial data.

In our future investigations, we intend to study the asymptotic behaviour and the geodesic completeness of our regular global solution. We also intend to couple the Maxwell-Boltzmann system to the Euler equations in presence of a massive scalar field, this investigation seems to have a great interest in the sence that, it leads to a heavy system of physical constraints which we find in several natural phenomena.

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