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A Study of the Forced Van der Pol Generalized Oscillator with the Renormalization Group Method

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Abstract

In this paper the equation of forced VAN der Pol generalized oscillator is examined with renormalization group method. A brief recall of the renormalization group technique is done. We have applied this method to the equation of forced Van der Pol generalized oscillator to search for its asymptotic solution and its renormalization group equation. The analysis of the numerical simulation graph is done; the method's efficiency is pointed out.

Keywords: forced VAN der Pol generalized oscillator, renormalization group method, renormalization group equation

1. Introduction

The analysis of the asymptotic behavior has played an important role in applied mathematics and theoretical physics. In many cases, the regular perturbation methods become inapplicable than the singular perturbation methods see (Bender & Orszag, 1978; Chen, Goldenfeld, & Oono, 1996; Chiba, 2008b; Hinch, 1991). We can cite the singular perturbation methods for solving ordinary differential equations (**ODE**), methods of multiple scales, WKB (Bender & Orszag, 1978), the method recovery (Roberts, 1985), etc. The renormalization group method which is the subject of this study was compiled by Chen, Goldenfeld, and Oono (1994, 1996) for differential equations of the following form

$$\dot{x} = Fx + \epsilon g(x, t, \epsilon); x \in \mathbb{R}^n \tag{1}$$

where $\epsilon \geq 0$ is a small parameter. They showed that the renormalization group method unifies the singular perturbation methods listed above (Chiba, 2008b). With this method, the renormalization constants of integration can raise divergence. This technique of renormalization does appear the renormalization group equation (**RGE)** of involving the amplitude which stabilizes the limit cycle; it is simple for dynamical system analysis. Chiba (2008b) used the renormalization group method to analyze the model of Kuramoto coupled oscillators.

The Van der Pol equation is a basic model for oscillatory processes in physics, electronics, biology, neurology, sociology and economic (Marios, 2006). In this work we decided to investigate the forced Van der Pol oscillator in its generalized form governed by the dimensionless equation below. We have done the similar work, where the unforced Van der Pol generalized oscillator is studied. We will force the system with a periodic external force of pulsation Ω. In this case $g(x, t, \epsilon)$ is an explicit function of time. This oscillator has been applied for modeling a Bipedal Robot by de Pina Filho and Dutra (2009) and known as Hybrid Van der Pol-Rayleigh oscillators. Sarkar and Bhattacharjee (2010) recently studied the unforced Van der Pol oscillator with another technique of the renormalization group theory to find its limit cycle. The paper is organized as follows.

In the second section a brief recall of the renormalization group technique will be done. In the third section, the method will be applied to the equation of forced Van der Pol generalized oscillator leading to the occurrence of a Hopf's classical bifurcation. In the fourth section our results will be analyzed through the graphs. The conclusions will be presented in the final section.

2. Renormalization Group Method

In this section we recall the outline of the technical group renormalization for (**ODE**).

For more details we refer to (Chiba, 2008b). We consider an (**ODE)** of the form

$$\dot{x} = Fx + \epsilon g(x, t, \epsilon),$$

$$= Fx + \epsilon g_1(x,t) + \epsilon^2 g_2(x,t) + \cdots ;\ x \in \mathbb{R}^n \tag{2}$$

where $0 \le \epsilon \ll 1$. For this system, we assume that:

1) The matrix F is a diagonalizable $n * n$ constant matrix all of whose eigenvalues lie on the imaginary axis.

2) The function $g(x,t,\epsilon)$ is C^∞ class with respect to t, x and ϵ. The formal power series expansion of $g(x,t,\epsilon)$ in ϵ is given as above.

3) Each $g_i(x,t)$ is periodic in $t \in \mathbb{R}$ and polynomial in x.

Firstly we apply the simple development method and secondly the renormalization group method will be applied to break down the divergence. We replace x in Equation (2) by

$$x(t) = x_0(t) + \epsilon x_1(t) + \cdots \tag{3}$$

After development and identification of the coefficients of ϵ we find:

$$\dot{x}_0 = Fx_0, \tag{4}$$

$$\dot{x}_i = F\,x_i + G_i(t, x_0 + x_1 + \cdots + x_{i-1}) \tag{5}$$

where the homogeneous term G_i is a regular function of t, x_0, x_{i-1} with:

$$G_1(t,x_0) = g_1(x_0,t), \tag{6}$$

$$G_2(t,x_0,x_1) = \frac{\partial g_1}{\partial x}(x_0,t)x_1 + g_2(x_0,t). \tag{7}$$

We can verify the Equality (8) below (Chiba, 2008a, **lemma A.2 for the proof**):

$$\frac{\partial G_i}{\partial x_j} = \frac{\partial g_{i-1}}{\partial x_{j-1}} = \frac{\partial g_{i-j}}{\partial x_0}\ ;\ i > j > 0. \tag{8}$$

In what follow we denote the fundamental matrix e^{Ft} as $X(t)$. Define the function R_i, $h_t^i, i = 1,2, \ldots.$ on $\mathbb{R}^n$ by

$$R_1(y) = lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} [X(s)^{-1} G_1(s, X(s)y)]ds\ , \tag{9}$$

$$h_t^1(y) = X(t)\int_{t_0}^{t}[X(s)^{-1}G_1(s,X(s)y) - R_1(y)]ds, \tag{10}$$

$$R_i(y) = lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}[X(s)^{-1}G_1\left(s, X(s)y, h_s^1(y), \ldots, h_s^{i-1}(y)\right) - X(s)^{-1}\textstyle\sum_{k=1}^{i-1}(Dh_t^k)_y\, R_{i-k}(y)\big]ds, i = 2,3 \ldots \tag{11}$$

$$h_t^i(y) = X(t)\int_{t_0}^{t}[X(s)^{-1}G_1\left(s, X(s)y, h_s^1(y), \ldots, h_s^{i-1}(y)\right) - X(s)^{-1}\textstyle\sum_{k=1}^{i-1}(Dh_t^k)_y\, R_{i-k}(y) - R_i(y)\big]ds, \tag{12}$$

Proposal: Let $x_0(t) = X(t)y$ be the solution to Equation (4) whose initial value is $y \in \mathbb{R}^n$.

Then for an arbitrary time $\zeta \in \mathbb{R}$, and $i = 1, 2, 3 \ldots$, the curve x_i defined by

$$x_i = h_t^i(y) + p_1^i(t,y)(t-\zeta) + p_2^i(t,y)(t-\zeta)^2 + \cdots + p_i^i(t,y)(t-\zeta)^i \tag{13}$$

gives a solution to Equation (5), where the functions $p_1^i \ldots, p_j^i$ are given by:

$$p_1^i(t,y) = X(t)R_i(y) + \textstyle\sum_{k=1}^{i-1}(Dh_t^k)_y\, R_{i-k}(y), \tag{14}$$

$$p_j^i(t,y) = \frac{1}{j}\sum_{k=1}^{i-1} \frac{\partial p_{j-1}^k}{\partial y}(t,y)R_{i-k}(y)\ , \quad (j = 2, 3, \ldots i-1\ ). \tag{15}$$

Further, the functions h_t^i are bounded uniformly in *t*. The solution of the Equation (2) is given by

$$x(t,\zeta,y) = X(t)y + \epsilon\left(h_t^1(y) + X(t)R_i(y)(t-\zeta)\right) + O(\epsilon^2). \tag{16}$$

It is the solution obtained by simple development; it diverges for time long, leading to the need for its renormalization. It should not depend on ζ $\left(\frac{\partial x(t,\zeta,y(\zeta))}{\partial \zeta}|_{\zeta=t} = 0\right)$, then

$$X(t)\frac{dy(t)}{dt} + \epsilon\frac{\partial h_t^1}{\partial y}\frac{\partial y(t)}{\partial t} - \epsilon X(t)R_1(y) = 0. \tag{17}$$

We verify that the Equation (17) admits solution:

$$\frac{dy(t)}{dt} = \epsilon R_1(y) + O(\epsilon^2). \tag{18}$$

Let *y(t)* be a solution of the Equation (18), then the solution of the Equation (2) looked for the renormalization group method is given by:

$$x(t,t,y) = X(t)y(t) + \epsilon h_t^1(y(t)) + O(\epsilon^2)\,. \tag{19}$$

The Equation (18) is the equation of the renormalization group of Equation (2). The calculation for a higher order is in the same way and one obtains the equation of renormalization group of order m as follows:

$$\frac{dy}{dt} = \epsilon R_1(y) + \epsilon^2 R_2(y) + \cdots + \epsilon^m R_m(y),\ y \in \mathbb{R}^n. \tag{20}$$

3. Application to the Forced Van Der Pol Generalized Oscillator

We consider the forced Van der Pol generalized oscillator governing by dimensionless equation as follows

$$\ddot{x} + x - \epsilon(1 - ax^2 - b\dot{x}^2)\dot{x} = Esin(\Omega t); \tag{21}$$

where $a,\ b,\ \epsilon,\ E$ and Ω are positifs control parameters such as ϵ is small. $Esin(\Omega t)$ is external force for pulsation Ω and amplitude E. The internal pulsation is here equal to one. With $E = \epsilon c,\ y = \dot{x}, x = (z + \overline{z})$ and $y = i(z - \overline{z})$, we rewrite the Equation (19) as

$$\begin{cases} \dot{z} = iz + \frac{\epsilon}{2}[(z-\overline{z}) - a(z+\overline{z})^2(z-\overline{z}) + b(z-\overline{z})^3 - icsin(\Omega t)] \\ \dot{\overline{z}} = -i\overline{z} + \frac{\epsilon}{2}[-(z-\overline{z}) + a(z+\overline{z})^2(z-\overline{z}) - b(z-\overline{z})^3 + icsin(\Omega t)], \end{cases} \tag{22}$$

The two equations of the system are nearly identical, the problem amounts to solving one of them. With

$$z(t) = z_0 + \epsilon z_1 + \cdots \tag{23}$$

we find

$$\dot{z_0} = iz_0\,, \tag{24}$$

$$\dot{z_1} = i\,z_1 + G_1(t, z_0), \tag{25}$$

From zero order we have:

$$z_0 = qe^{it} = qZ(t), \tag{26}$$

with q the integration constant of Equation (24). Expressions (9) and (10) give

$$R_1 = \frac{1}{2}\,q(1 - (a+3b)|q|^2) - ic\,lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}[Z(s)^{-1}\sin{(\Omega s)}]ds\,, \tag{27}$$

$$h_t^1(y) = \frac{i}{4}\left[(a-b)\left(q^3e^{3it} + \frac{1}{2}\,\overline{q}^3e^{-3it}\right) + \left((a+3b)q\overline{q}^2 - \overline{q}\right)e^{-it}\right] - \frac{ice^{it}}{2}\int_{t_0}^{t}[Z(s)^{-1}sin\,(\Omega s)]ds\,, \tag{28}$$

where t_0 is an initial time and $Z(s) = e^{is}$. We find after computation:

$$R_1(q) = \frac{1}{2}\,q(1 - (a+3b)|q|^2)\,, \tag{29}$$

$$h_t^1(y) = \frac{i}{4}\left[(a-b)\left(q^3e^{3it} + \frac{1}{2}\,\overline{q}^3e^{-3it}\right) + \left((a+3b)q\overline{q}^2 - \overline{q}\right)e^{-it}\right] + -\frac{ice^{it}}{2}I\,, \tag{30}$$

With

$$I = \frac{1}{2}\left\{\frac{cos(1-\Omega)t}{1-\Omega} - \frac{cos(1+\Omega)t}{1+\Omega}\right\} - \frac{i}{2}\left\{\frac{sin(1-\Omega)t}{1-\Omega} - \frac{sin(1+\Omega)t}{1+\Omega}\right\}, i^2 = -1;\ 1 \neq \Omega. \tag{31}$$

According to the proposal and the above results we find:

$$z(t,\zeta,y) = Z(t)q + \epsilon\left(h_t^1(q) + Z(t)R_i(q)(t-\zeta)\right) + O(\epsilon^2), \tag{32}$$

which diverges for long t because of the last term. Using the notion of renormalization constant of integration $\left(\frac{\partial x(t,\zeta,y(\zeta))}{\partial\zeta}|_{\zeta=t} = 0\right)$ mentioned in the previous section and taking $q(\zeta) = r(\zeta)e^{i\Theta(\zeta)}$, we find

$$\begin{gathered} x = 2r\,cos\big(t + \Theta(\zeta)\big) - \frac{r\epsilon}{2}sin(t + \Theta(\zeta)) + \frac{Esin(\Omega t)}{1-\Omega^2} + \\ \frac{\epsilon}{2}\left\{\left(\frac{b-a}{2}\right)r^3\,sin3(\,t + \Theta(\zeta)) + (a+3b)\,r^3\,sin(\,t + \Theta(\zeta))\right\} + O(\epsilon^2). \end{gathered} \tag{33}$$

$$\begin{cases} \frac{dr}{d\zeta} = \frac{\epsilon r}{2}\,(1 - (a+3b)r^2) + O(\epsilon^2) \\ \frac{d\Theta(\zeta)}{d\zeta} = 0 + O(\epsilon^2)\,. \end{cases} \tag{34}$$

The first equation of system (34) gives the stable cycle limit radius $r_s = \frac{1}{\sqrt{(a+3b)}}$, with $(a+3b) \neq 0$.

For $a = 0, b = \frac{1}{3}$ we get the Rayleigh forced oscillator equation and the Equation (31) takes the form

$$x(r_s = 1, t, \Theta) = 2\cos(t+\Theta) + \frac{E\sin(\Omega t)}{1-\Omega^2} + \frac{\epsilon}{12}\sin 3(t+\Theta) + O(\epsilon^2). \tag{35}$$

Also for $a = 1, b = 0$ we deal with the forced Van der Pol oscillator equation and the Equation (33) becomes

$$x(r_s = 1, t, \Theta) = 2\cos(t+\Theta) + \frac{E\sin(\Omega t)}{1-\Omega^2} - \frac{\epsilon}{4}\sin 3(t+\Theta) + O(\epsilon^2). \tag{36}$$

When we cancel the external force ($E = 0$), the Equation (36) reduces to the results found by Hasan (1981) and recently by Sarkar and Bhattacharjee (2010).

Finally for $a = 1, b = 1$ we have the forced Van der Pol generalized oscillator equation and the Equation (34) becomes

$$x\left(r_s = \frac{1}{2}, t, \Theta\right) = \cos(t+\Theta) + \frac{E\sin(\Omega t)}{1-\Omega^2} + O(\epsilon^2). \tag{37}$$

The integration of the Equations system (34) gives us:

$$\begin{cases} r(t) = \dfrac{r_0 e^{\epsilon\frac{t}{2}}}{\sqrt{1+r_0{}^2(a+3b)(1-e^{\epsilon t})}} + O(\epsilon^2 t) \\ \Theta(t) = \Theta_0 + O(\epsilon^2 t). \end{cases} \tag{38}$$

For $\epsilon = 0$ we have a cycle limit radius r:

$$\begin{cases} r(t) = r = \dfrac{r_0}{\sqrt{1+r_0{}^2(a+3b)}}, \\ \Theta(t) = \Theta_0. \end{cases} \tag{39}$$

It becomes, for $r_0 = 1$:

$$\begin{cases} r(t) = r = \dfrac{1}{\sqrt{1+(a+3b)}}, \\ \Theta(t) = \Theta_0. \end{cases} \tag{40}$$

4. Analysis Results

In this section we will go through the graphical analysis of the numerical simulation figures below. These graphs are obtained on one hand by direct simulation of Equation (21) for some parameters values and on the other hand by simulation of the solution asymptotic Equation (33) found by the renormalization group method, for the same values of these parameters with the logician MATHEMATICA.

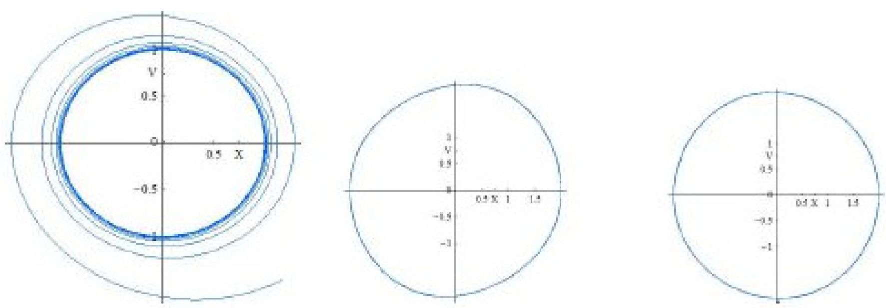

Figure 1. Phase diagram: Van der Pol generalized oscillator, Van der Pol oscillator and Rayleigh oscillator, for $\epsilon = 0.1, E = 0, \Omega = 0$

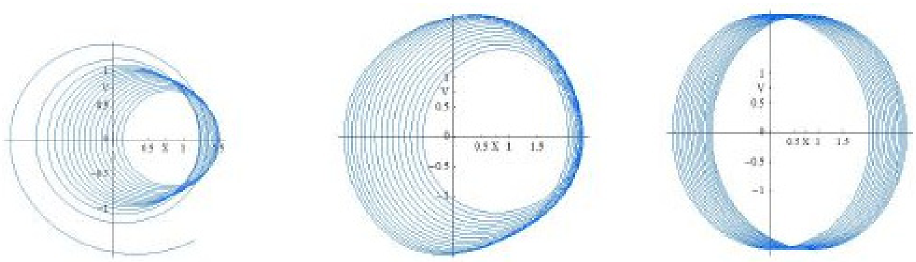

Figure 2. Phase diagram: forced Van der Pol generalized oscillator, forced Van der Pol oscillator and forced Rayleigh oscillator, *for* $\epsilon = 0.1, E = 1, \Omega = 0.01$

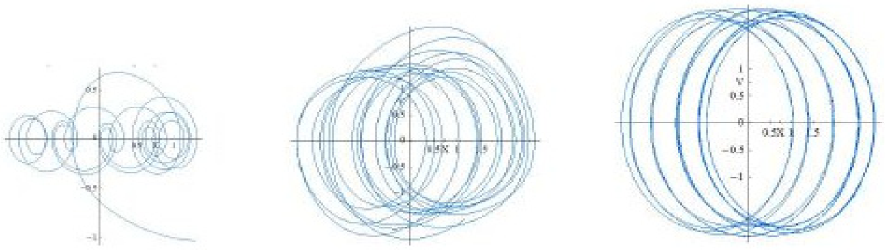

Figure 3. Phase diagram: forced Van der Pol generalized oscillator, forced Van der Pol oscillator and forced Rayleigh oscillator, for $\epsilon = 0.1, E = 1, \Omega = 0.1$

For each one of the Figures 1, 2, and 3, we have, on the left, the phase diagram of Van der Pol generalized oscillator, in the middle, the phase diagram of Van der Pol oscillator and on the right, the phase diagram of Rayleigh oscillator. These figures show us, progressively when one increases the magnitude of E and Ω, the phase portrait goes from periodic condition or almost periodic to chaotic condition. They show us also, the effects of the control parameters a and b on the system.

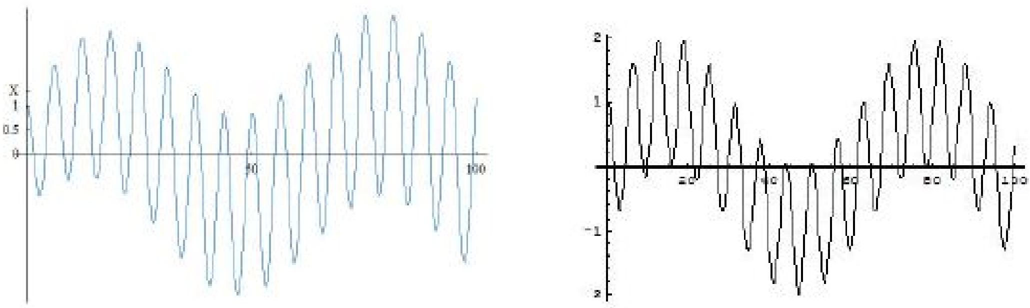

Figure 4. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.1, E = 1, a = 1, b = 1$

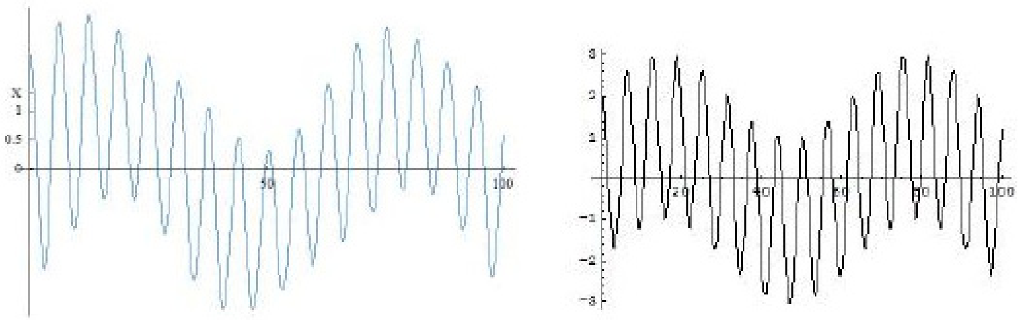

Figure 5. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.1, E = 1, a = 1, b = 0$

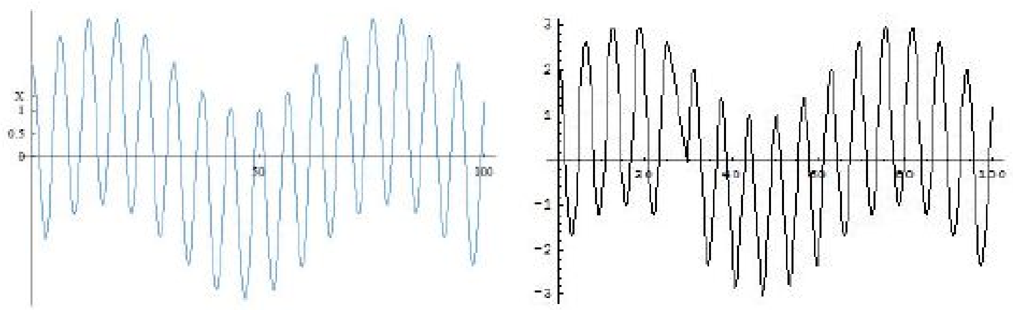

Figure 6. $x(t)$, for : $\epsilon = 0.1, \Omega = 0.1, E = 1, a = 0, b = \frac{1}{3}$

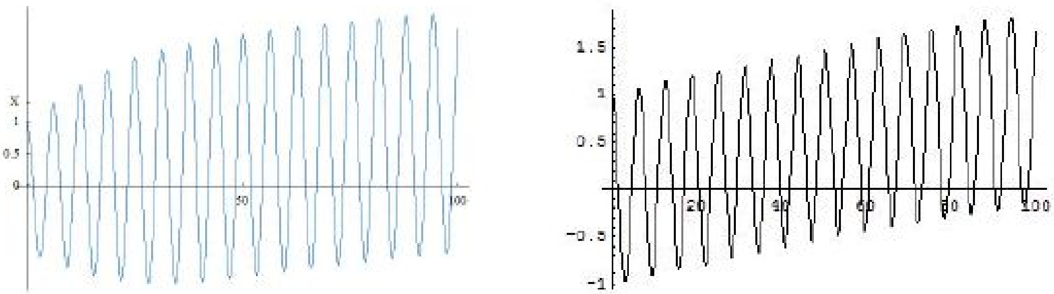

Figure 7. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.01, E = 1, a = 1, b = 1$

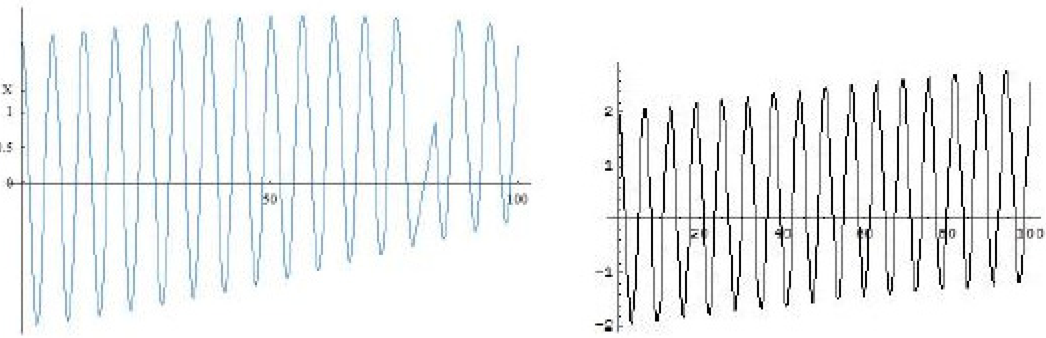

Figure 8. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.01, E = 1, a = 1, b = 0$

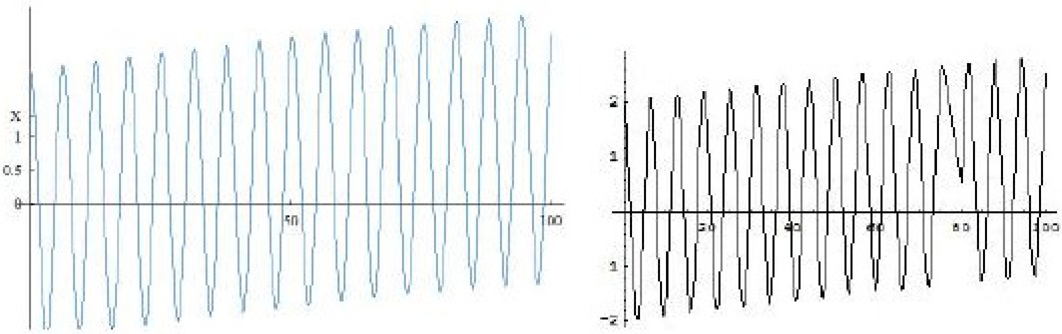

Figure 9. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.01, E = 1, a = 0, b = \frac{1}{3}$

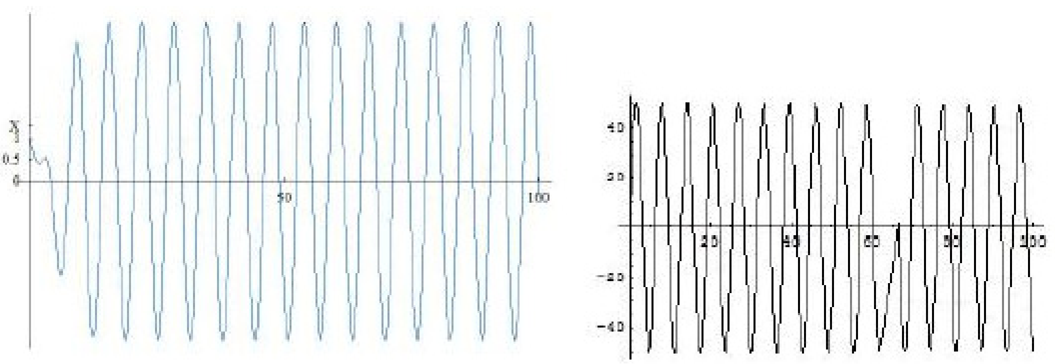

Figure 10. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.99, E = 1, a = 1, b = 1$

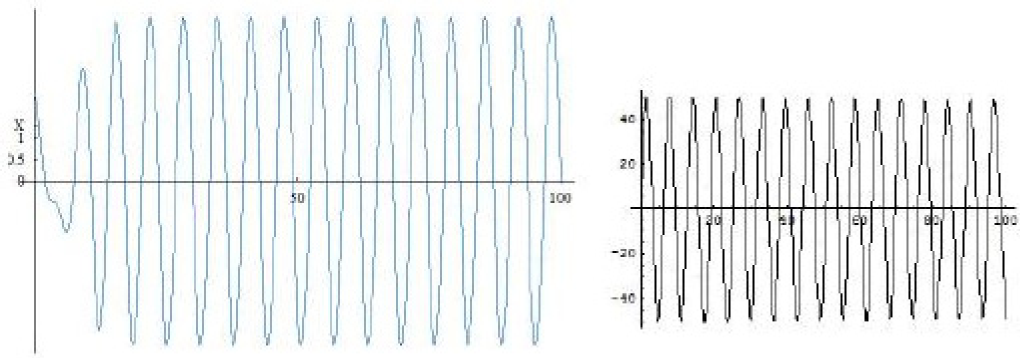

Figure 11. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.99, E = 1, a = 1, b = 0$

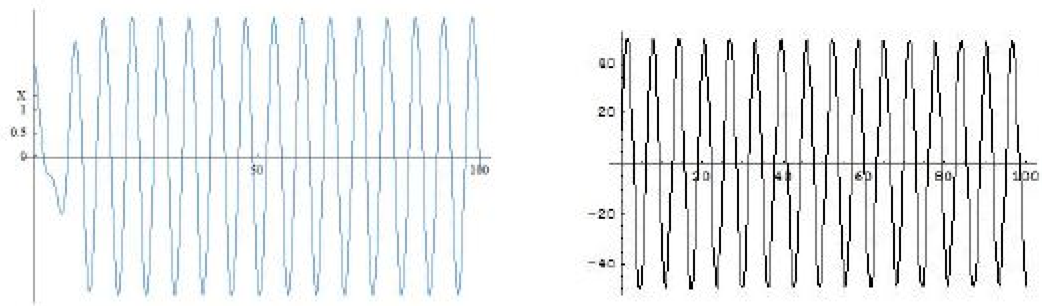

Figure 12. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.99, E = 1, a = 0, b = \frac{1}{3}$

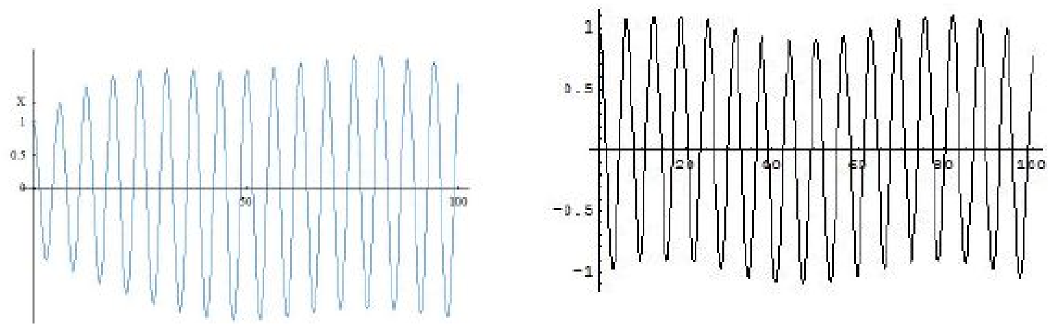

Figure 13. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.1, E = 0.1, a = 1, b = 1$

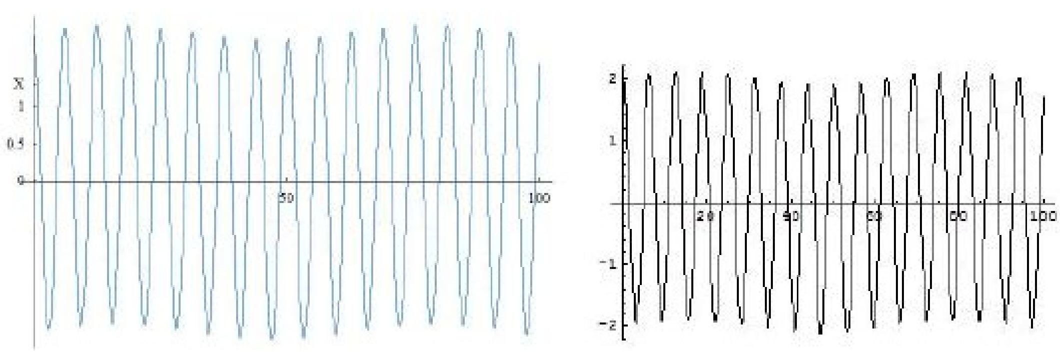

Figure 14. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.1, E = 0.1, a = 1, b = 0$

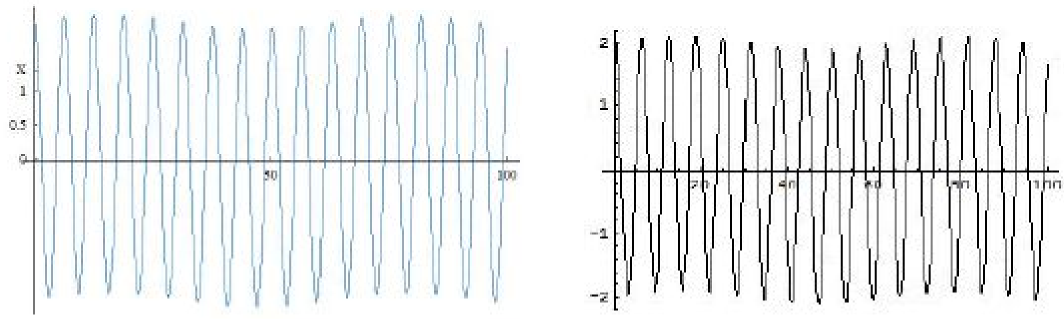

Figure 15. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.1, E = 0.1, a = 0, b = \frac{1}{3}$

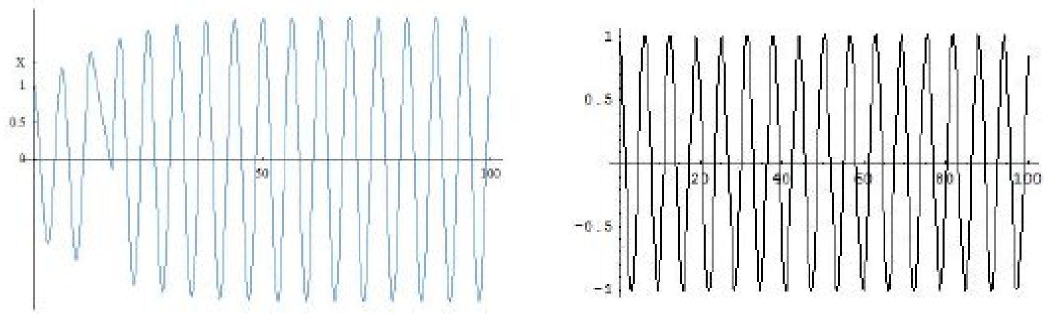

Figure 16. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.1, E = 0.01, a = 1, b = 1$

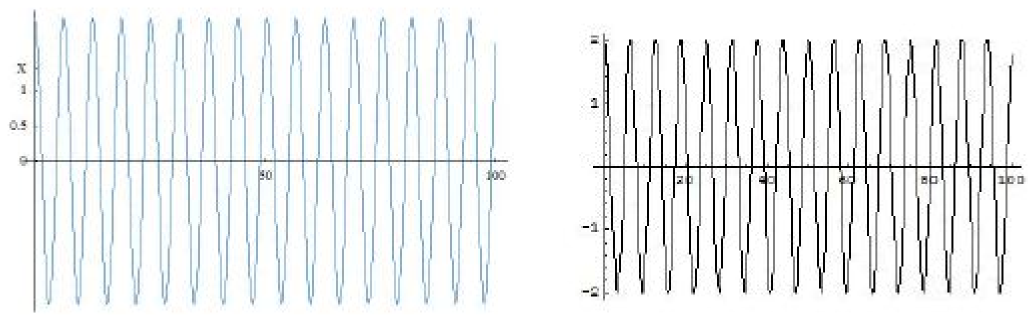

Figure 17. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.01, E = 0.01, a = 1, b = 0$

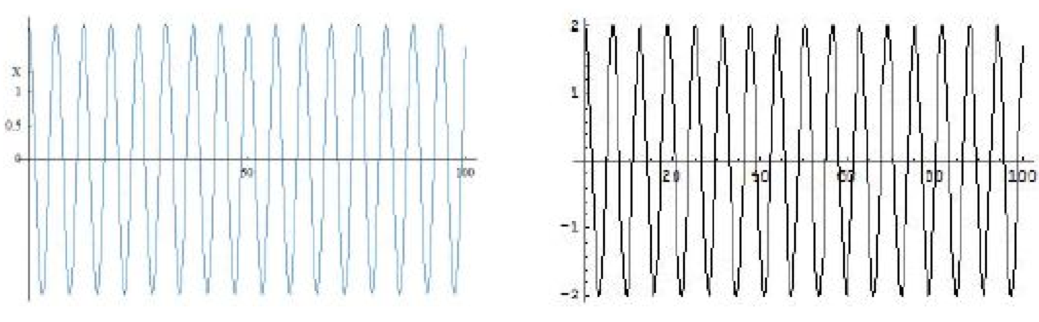

Figure 18. $x(t)$, for: $\epsilon = 0.1, \Omega = 0.01, E = 0.01, a = 1, b = 0$

As for the figures from (4) to (18) we have the graph of the exact solution of Equation (21) on the left and the graph of the approximate solution Equation (33) on the right. The chaotic behavior noticed in Figure (3) is confirmed in real space through the behavior of the curves in the Figures 4, 5, and 6 respectively (forced Van der Pol generalized, forced Van der Pol and forced Rayleigh oscillators). Furthermore the quasi-periodic oscillation is noticed in the behavior of the curves of the Figures 7, 8, and 9.

The second term ($\frac{Esin(\Omega t)}{1-\Omega^2}$) of the solution for Equation (33) shows the appearance of the resonance for $\Omega \approx 1$. This behavior is illustrated by the Figures 10, 11, and 12 where the dynamic system's amplitude is increasing. We see through each figure that the approximate solution found approaches more or less the exact solution, which justifies that ours results are optimal.

The equations of system (34) show us that the phase initial of dynamic system is a constant and the amplitude r is a function of both the time and the control parameters of system. We chose this initial constant equal to zero to simplify our simulation. Also, the first equation of the equations system (34) above, gives the stable cycle limit radius that is only a function of the parameters a and b. They show also that there is the occurrence of the Hopf's classical bifurcation.

5. Conclusion

We recalled the outline of the method of the renormalization group method for ordinary differential equations **(ODE)** which provides in addition to the solution, the renormalization group **(EGR)** which leads to the determination of the amplitude of the stable limit cycle. An application of this method for a forced Van der Pol generalized oscillator equation is made and the approximate solution found is valid for any order of the variable t. The numerical simulation of the forced Van der Pol generalized oscillator equation on one hand and the approximate solution found on the other hand for some values of control parameters show the method efficiency and the validity of the approximate solution found. We have noticed that these control parameters play a key role in the dynamic of the system. One notices also the primary resonance presence near the $\boldsymbol{\Omega = 1}$ zone and that the system presents a classical Hopf's bifurcation through the renormalization equation.

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