On Singular Solutions in Spherically Symmetric Static Problem of General Relativity

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Abstract

The paper is concerned with analysis of the singular solutions in spherically symmetric static gravitation problem of the General Theory of Relativity. Classical results obtained for external space and for the incompressible liquid sphere are discussed and generalized for the cases of compressible liquid and elastic solid spheres studied by numerical methods.

Keywords: general relativity, singular solution, spherically symmetric problem

1. Governing Equations

Consider a General Theory of Relativity (GTR) problem for a solid spherical body with density $\rho(r)$ and external radius *R* surrounded by an infinite empty space (Figure 1) with $\rho = 0$ (Synge, 1960). The line element of the inside and outside semi-Riemannian spaces induced by the body gravitational field can be written as

$$ds^2 = gdr^2 + r^2(d\theta^2 + \sin^2\varphi d\varphi^2) - hdt^2 \tag{1}$$

in which $g(r)$ and $h(r)$ are the coefficients of the metric tensor which do not depend on time t. The components of the energy tensor for the static problem are (for the problem under study the mixed components which coincide with the corresponding physical components are used)

$$T_1^1 = \sigma_r, \quad T_2^2 = T_3^3 = \sigma_\theta, \quad T_4^4 = \rho c^2 \tag{2}$$

Here, σ_r and σ_θ are the radial and the circumferential stresses induced in the body by the gravitation field. The energy tensor, T_i^j, must satisfy the following conservation equation (Synge, 1960):

$$(T_1^1)' - \frac{2}{r}(T_2^2 - T_1^1) + \frac{h'}{2h}(T_1^1 - T_4^4) = 0 \tag{3}$$

in which $(...)' = d(...)/dr$. According to the basic idea of GTR, Eq.(3) is satisfied identically if the energy tensor is expressed in terms of the Einstein tensor G_i^j as

$$\chi T_1^1 = G_1^1 = -\frac{1}{g}\left(\frac{h'}{rh} + \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{4}$$

$$\chi T_2^2 = \chi T_3^3 = G_2^2 = -\frac{1}{2g}\left[\frac{h''}{h} - \frac{1}{2}\left(\frac{h'}{h}\right)^2 + \frac{1}{r}\left(\frac{h'}{h} - \frac{g'}{g}\right) - \frac{g'h'}{2gh}\right] \tag{5}$$

$$\chi T_4^4 = G_4^4 = -\frac{1}{g}\left(\frac{1}{r^2} - \frac{g'}{rg}\right) + \frac{1}{r^2} \tag{6}$$

in which

$$\chi = 8\pi\gamma / c^4 \tag{7}$$

and γ is the gravitation constant. Because substitution of Eqs. (4)-(6) in Eq. (3) satisfies this equation identically, only three of four Eqs.(3)-(6) are mutually independent. Traditionally (Synge, 1960), the simplest set of equations including Eqs. (3), (4) and (6) is used to solve particular problems. The remaining equation, Eq. (5), is satisfied identically or specifies G_2^2 if G_1^1 and G_4^4 in Eqs. (4) and (6) has been found. Using Eqs. (2), we can transform Eqs. (3), (4) and (6) to the following form:

$$\sigma_r' - \frac{2}{r}(\sigma_\theta - \sigma_r) + \frac{h'}{2h}(\sigma_r - \rho c^2) = 0 \tag{8}$$

$$\chi\sigma_r = -\frac{1}{rg}\left(\frac{h'}{h} + \frac{1}{r}\right) + \frac{1}{r^2} \tag{9}$$

$$\chi\rho c^2 = \frac{1}{r^2}\left(r - \frac{r}{g}\right)' \tag{10}$$

The boundary conditions

$$\sigma_r(r=0) = \sigma_\theta(r=0)\,,\quad \sigma_r(r=R) = 0 \tag{11}$$

should be supplemented with the regularity condition at the origin $r=0$ and the compatibility condition of the metric tensor for the internal and the external space at $r=R$.

Three equations in Eqs. (8)-(10) include four unknown functions, i.e., two components of the metric tensor g and h, and two stresses σ_r and σ_θ. Thus, the classical set of GTR equations, in the general case, is not complete.

2. Solution for the External Space

Solution for the external space $r \geq R$ was found, as known, by K.Schwarzschild in 1916. For the empty space, $\rho = 0$ and $\sigma_r = \sigma_\theta = 0$. Then, Eq.(8) is satisfied identically and the remaining equations, Eqs. (9) and (10), can be reduced to

$$\frac{h_e'}{h_e} = \frac{1}{r}(g_e - 1)\;,\quad \left(r - \frac{r}{g_e}\right)' = 0 \tag{12}$$

and include two unknown functions h_e and g_e. Subscript "*e*" specifies the external field. Integration of the second equation in Eqs.(12) yields

$$g_e = \frac{1}{1 + C_1/r} \tag{13}$$

Here, C_1 is the constant of integration which can be found from the compatibility condition using the solution for the internal field. Having found g_e, we can determine h_e from the fist equation in Eqs.(12) and get

$$h_e = C_2\left(1 + \frac{C_1}{r}\right) \tag{14}$$

in which C_2 is the integration constant. Because $h_e(r \to \infty) = 1$, we have $C_2 = 1$.

3. Solution of the Internal Problem for the Liquid Sphere

Consider the internal problem for which $r \leq R$. Presume that the matter of the spherical body is the perfect liquid for which

$$\sigma_r = \sigma_\theta = -p(r) \tag{15}$$

and *p* is the pressure. Thus, we have three equations, Eqs. (8)-(10), for three unknown functions g, *h* and *p*, and the governing set of equations is complete, if the dependence $\rho(p)$ is known. For the stresses in Eqs.(14), the governing equations, Eqs. (8) and (9), reduce to

$$p' + \frac{h_i'}{2h_i}(p + \rho c^2) = 0\,,\quad \chi p = \frac{1}{rg_i}\left(\frac{h_i'}{h_i} + \frac{1}{r}\right) - \frac{1}{r^2} \tag{16}$$

in which subscript "*i*" corresponds to the internal space. The boundary condition in Eqs.(11) is

$$p(r=R) = 0 \tag{17}$$

3.1 Incompressible Liquid

Presume that the liquid density does not depend on pressure, i.e., that $\rho = \rho_0$ (line 1 in Figure 1). This problem has been also solved by *K*. Schwarzschild. For $\rho = \rho_0$, the solution of Eq. (10) satisfying the regularity condition at $r=0$ has the following form:

$$g_i = \frac{1}{1-(\chi/3)\rho_0 c^2 r^2} \tag{18}$$

Substituting Eq. (18) in the second equation of Eqs.(16), we get

$$\frac{h_i'}{h_i} = \chi r \frac{3p+\rho_0 c^2}{3-\chi\rho_0 c^2 r^2} \tag{19}$$

To determine constant C_1 entering Eqs.(13) and (14) for the metric coefficients of the external field, we must apply the compatibility condition according to which $g_e(R) = g_i(R)$. As a result, Eqs. (13) and (18) can be reduced to

$$g_e = \frac{1}{1-r_g^0/r} \tag{20}$$

$$g_i = \frac{1}{1-r_g^0 r^2/R^3} \tag{21}$$

in which $r_g^0 = (\chi/3)\rho_0 c^2 R^3$. Introducing the classical mass of the spherical body as $m_0 = (4/3)\pi R^3 \rho_0$ and using Eq. (7), we arrive at

$$r_g^0 = 2m_0\gamma/c^2 \tag{22}$$

Thus, $C_1 = -r_g^0$ in Eqs.(13) and (14).

Equation (22) specifies the so-called gravitational radius. Singular GTR solutions are expected if the external radius *R* of the sphere with mass m_0 becomes equal to the gravitational radius, as follows from Eqs. (20) and (21). Consider possible singularities.

As follows from Eq. (20), the metric coefficient of the external space can be singular if $r = r_g^0$. However, applying the obvious condition $g_i > 0$ to Eq. (21), we arrive at the following inequality:

$$R^3 > r_g^0 r^2$$

which is valid for any r. Taking $r = R$, we get $R > r_g^0$. So, the surface with the gravitation radius is located inside the body, whereas Eq.(20) is valid outside the body. Thus, no singularity can exist in the external field.

For the internal field, the metric coefficient specified by Eq.(21) becomes infinitely high if $R = r_g^0$ and $r = R$, i.e., on the surface of the sphere with radius r_g^0.

To proceed, determine the pressure p in the liquid. Substituting Eqs. (19) and (21) in the first equation of Eqs. (16), we arrive at the following equation for p:

$$p' + \frac{\rho_0 c^2 r_g^0 r}{2R^3(1-r_g^0 r^2/R^3)}\left(1+\frac{3p}{\rho_0 c^2}\right)\left(1+\frac{p}{\rho_0 c^2}\right) = 0 \tag{23}$$

The solution of this equation is

$$p(r) = -\rho_0 c^2 \frac{\sqrt{1-r_g^0 r^2/R^3} - C_3}{\sqrt{1-r_g^0 r^2/R^3} - 3C_3} \tag{24}$$

Here, C_3 is the integration constant. It can be found from the boundary condition in Eq. (17) which yields

$$C_3 = \sqrt{(1-r_g^0/R}$$

Then, Eq. (24) takes the following final form (Synge, 1960):

$$\overline{p}(\overline{r}) = -\frac{\sqrt{1-\overline{r}_g^0\overline{r}^2} - \sqrt{1-\overline{r}_g^0}}{\sqrt{1-\overline{r}_g^0\overline{r}^2} - 3\sqrt{1-\overline{r}_g^0}} \tag{25}$$

in which

$$\overline{r} = \frac{r}{R}, \quad \overline{r}_g^0 = \frac{r_g^0}{R}, \quad \overline{p} = \frac{p}{\rho_0 c^2} \tag{26}$$

Note that Eq. (25) is not valid if the external radius R is equal to the gravitational radius r_g^0. Indeed, taking $r = R$ in Eq. (24), we arrive at

$$p(R) = -\rho_0 c^2 \frac{\sqrt{1 - r_g^0 / R} - C_3}{\sqrt{1 - r_g^0 / R} - 3C_3}$$

If $R = r_g^0$, we get $p(R) = -\rho_0 c^2 / 3$ and the boundary condition in Eq. (17) cannot be satisfied. Moreover, the obtained result for the pressure has no physical meaning, because positive pressure requires negative density. Thus, the solution of GTR boundary problem does not exist for the liquid sphere whose radius is equal to the gravitational radius. Hence, Eq. (21) has no singularity at $r = R$, and we can conclude that the metric coefficients in Eqs.(20) and (21) cannot be singular.

Consider the possible singularity of the field variable p. As known (Misner et al., 1973) the pressure $\bar{p}$ specified by Eq. (25) can become infinitely high at the sphere center. Taking $r = 0$ in Eq. (25), we get

$$\bar{p}(0) = -\frac{1 - \sqrt{1 - \bar{r}_g^0}}{1 - 3\sqrt{1 - \bar{r}_g^0}}$$

The denominator of this expression can become zero at $\bar{r}_g^0 = 8/9$. Figure 2 shows the dependences of the normalized pressure $\bar{p}$ on $\bar{r}$ for two $\bar{r}_g^0$ values (solid lines). As can be seen, the pressure at the sphere center dramatically increases while $\bar{r}_g^0$ approaches 8/9, i.e., while the sphere radius R reduces to $1.125\, r_g^0$.

Now a natural question arises as to what happens if $\bar{r} < 8/9$. Find the radius $r = r_s$ at which the denominator of Eq. (25) becomes zero, i.e., the pressure becomes infinitely high. The result is (Weinberg, 1972)

$$\bar{r}_s = \sqrt{9 - 8 / \bar{r}_g^0} \tag{27}$$

For $\bar{r}_g^0 = 8/9$, we get $\bar{r}_s = 0$, whereas for $\bar{r}_g^0 < 8/9$, Eq. (27) gives $\bar{r}_s > 0$, and the pressure singularity appears at the points which do not coincide with the sphere center. For the radial coordinates which are less than $\bar{r}_s$, the normalized pressure $\bar{p}$ becomes negative which means that either pressure or density are negative. So, GTR equations do not give a feasible solution for the points with the radial coordinate $\bar{r} \le \bar{r}_s$.

Thus, we can conclude that the pressure in the incompressible liquid sphere becomes singular at the sphere center if the sphere radius R reaches $1.125 r_g^0$. If R is less than $1.125 r_g^0$, the solution does not exist.

3.2 Compressible Liquid

First, we derive the equation which is valid for an arbitrary density of the sphere material. For any function $\rho(r)$, generalize Eq.(22) for the gravitational radius as

$$r_g = 2m\gamma / c^2 \tag{28}$$

in which the sphere mass is

$$m = 4\pi \int_0^R \rho(r) r^2 dr \tag{29}$$

Substituting ρ from Eq. (10) in Eq. (29), calculating the integral in Eq. (29) and taking into account Eq. (7) for constant χ and Eq. (28) for r_g, we arrive at the following general relation:

$$\bar{r}_g = \frac{r_g}{R} = 1 - 1/g_R \tag{30}$$

where $g_R = g(R)$. Determining the constant in Eq.(13) from the compatibility condition $g_e(R) = g_R$ and using Eq. (30), we get for the external field

$$g_e = \frac{1}{1 - \bar{r}_g / \bar{r}} \tag{31}$$

Note that Eqs. (30) and (31) are valid irrespective of the nature of the sphere matter.

Return to the liquid sphere and consider the liquid whose density depends on pressure. For the simplest linear law (line 2 in Figure 1) we have

$$\rho = kp \tag{32}$$

in which k is some constant coefficient. For the density specified by Eq. (31), the first equation of Eqs. (16) becomes

$$p' + \frac{h'}{2h}(1+kc^2)p = 0$$

Integration yields

$$p = C_3 h^{-\frac{1}{2}(1+kc^2)} \tag{33}$$

in which C_3 is the integration constant. Because h is not zero at $r = R$, the boundary condition in Eq. (17) yields $C_3 = 0$ and $p = 0$ inside the sphere. Thus, GTR equations do not allow us to obtain the solution for the liquid with density specified by Eq. (32). This result was first obtained by J. R. Oppenheimer and G. M. Volkoff (1939).

Generalize Eq. (32) as (see line 3 in Figure 1)

$$\rho = \rho_0(1+kp) \tag{34}$$

For the density in Eq. (34), the first equation of Eq. (16) becomes

$$p' + \frac{h_i'}{2h_i}[\rho_0 c^2 + p(1+k\rho_0 c^2)] = 0 \tag{35}$$

The solution of this equation which satisfies the boundary condition in Eq. (17) is

$$\overline{p} = \frac{1}{1+\bar{k}}\left[\left(\frac{h_R}{h_i}\right)^{\frac{1}{2}(1+\bar{k})} - 1\right] \tag{36}$$

in which $\bar{k} = k\rho_0 c^2$, $\overline{p} = p/\rho_0 c^2$ and $h_R = h_i(R)$. As can be seen, the pressure becomes infinitely high at the point where $h_i(\overline{r}) = 0$.

To study the liquid sphere with the density specified by Eq. (34), apply Eq. (10) for the metric coefficient and Eq. (35) for the pressure. First, substitute ρ from Eq. (34) in Eq. (10) and use Eqs. (22) and (26) to get

$$\overline{r}g_i' - g_i + g_i^2[1 - 3\overline{r}_g^0\overline{r}^2(1+\bar{k}\overline{p})] = 0 \tag{37}$$

To derive the second equation, substitute h_i from Eq. (19) in Eq. (35). Taking into account Eqs. (7), (22) and (26), we arrive at

$$\overline{r}\,\overline{p}' + \frac{1}{2}(3\overline{r}_g^0\overline{r}^2 g_i\overline{p} + g_i - 1)[1 + (1+\bar{k})\overline{p}] = 0 \tag{38}$$

Here and in Eq. (37), $(...)' = d(...)/d\overline{r}$. The boundary conditions are

$$\overline{p}(\overline{r} = 1) = 0, \quad g_i(\overline{r} = 0) = 1 \tag{39}$$

Because the set of nonlinear equations, Eqs. (37) and (38) can hardly be solved analytically, apply the finite difference numerical method and use MAPLE-7 for calculation.

It should be taken into account that parameter $\overline{r}_g^0$ entering Eqs. (37) and (38) and specified by Eq. (22) is not the gravitational radius for the sphere with variable density. To solve Eqs. (37) and (38), we preset some $\overline{r}_g^0$ value and integrate these equations with the boundary conditions in Eqs. (39). The result of integration allows us to find $g_R = g_i(r = R)$ and to use Eq. (30) to determine the corresponding gravitational radius $\overline{r}_g$.

Numerical solution for the liquid sphere with variable density has been obtained by Misner et al. (1973) with the aid of the shooting method in which the pressure at the sphere center has been taken arbitrary and integration is performed up to the radius at which the pressure becomes zero (this radius has been identified with the external radius of the sphere). This method cannot be applied to the problem under consideration, i.e., to study the pressure which can be infinitely high at the sphere center. It should be also noted that numerical method can hardly be used to obtain the singular solution. To identify such solutions, Eqs. (37) and (38) have been solved for the incompressible liquid for which the analytical solution exists. Circles in Figure 2 correspond to numerical integration and are in good agreement with the exact analytical solution (lines). For $\overline{r}_g^0 = 8/9$, for which the analytically found pressure at the sphere center becomes infinitely high, the loss of convergence of the numerical procedure has been observed. Thus, the parameter $\overline{r}_g^0$ for which numerical integration of Eqs. (37) and (38) does not converge has been identified as the maximum possible parameter and the corresponding value of the gravitational radius $\overline{r}_g$ is found from Eq. (30).

The results of calculation for $\overline{r}_g^0 = 0.5$ and two *k* values ($\overline{k} = 0$ and $\overline{k} = 1$) are presented in Figures 3 and 4 showing the dependences of the normalized pressure and the metric coefficient on the radial coordinate for the incompressible ($\overline{k} = 0$) and compressible ($\overline{k} = 1$) liquids. The dependences of the maximum values of $\overline{r}_g^0$ and $\overline{r}_g$ (beyond which the integration process does not converge) on parameter $\overline{k}$ are shown in Figure 5. As can be seen, the dependence of the density on pressure in accordance with Eq. (34) reduces the normalized gravitational radius. It is important to mention that $\overline{r}_g$ is the ratio of r_g to the sphere radius R. So, the reduction of $\overline{r}_g$ means that for the compressible liquid, the pressure becomes singular at the sphere center $(r_s = 0)$ if the sphere radius R is considerably higher than the gravitational radius, e.g., for $\overline{k} = 10$ we get $\overline{r}_g = 0,21$ and $R = 4.76 r_g$.

4. Solution of the Internal Problem for the Elastic Sphere

Consider the elastic sphere and introduce infinitely small strains ε_r and ε_θ in the radial and circumferential directions linked with the corresponding stresses by Hooke's law (Wang, 1953), i.e.,

$$\varepsilon_r = \frac{1}{G(2G+3\lambda)}\left[(G+\lambda)\sigma_r - \lambda\sigma_\theta\right], \quad \varepsilon_\theta = \frac{1}{2G(2G+3\lambda)}\left[(2G+\lambda)\sigma_\theta - \lambda\sigma_r\right] \tag{40}$$

Here

$$G = \frac{E}{2(1+\nu)}\;, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{41}$$

in which *E* is the Young's modulus and ν is the Poisson's ratio. For small strains, the radial and the circumferential metric coefficients can be presented as

$$g_i^\varepsilon = g_i(1+\varepsilon_r)^2 \approx g_i(1+2\varepsilon_r)\,, \quad r_\varepsilon = r(1+\varepsilon_\theta) \tag{42}$$

in which index "ε" corresponds to the deformed space.

As noted in Section 1, the GTR equations, Eqs. (8)-(10), do not allow us to solve the problem, because they include the stresses σ_r and σ_θ which are not known, and Eqs. (40) cannot help, because they introduce two unknown strains instead of two stresses. So, we need an additional equation for the strains. In the theory of elasticity, this equation has the form

$$(r\varepsilon_\theta)' = \varepsilon_r \tag{43}$$

and is known as the compatibility equation. However, we cannot use this equation, because it is derived under the condition that the space inside the solid is Euclidean before and after the deformation, whereas for the problem under study, the space is semi-Riemannian. To derive the necessary equation, we use the invariant condition for the Einstein tensor proposed by Vasiliev and Fedorov (2006). As known, the Einstein tensor in Eqs. (4)-(6) allows us to satisfy identically the conservation equation, Eq. (3). It is natural to presume that this property of the Einstein tensor is valid not only for the initial space, but for the deformed space as well. Thus, the only one independent in the problem under study space component of the Einstein tensor, G_1^1 (the component G_2^2 as noted, follows from Eq. (5), should not change under deformation, i.e.,

$$G_1^1(g_i, r) = G_1^1(g_i^\varepsilon, r_\varepsilon) \tag{44}$$

in which $G_1^1(g_i, r)$ is specified by Eq. (4), whereas $G_1^1(g_i^\varepsilon, r_\varepsilon)$ can be found from Eq. (4) if we change g and r to g_i^ε and r_ε in accordance with Eqs. (42) and neglect the terms which are nonlinear with respect to the strains. The resulting equation is

$$G_1^1(g_i^\varepsilon, r_\varepsilon) = G_1^1(g_i, r) - \frac{2}{r^2 g_i}\left[r\varepsilon_\theta'\left(1 + r\frac{h_i'}{2h_i}\right) + g_i\varepsilon_\theta - \varepsilon_r\left(1 + r\frac{h_i'}{h_i}\right)\right]$$

Substitution in Eq. (44) yields

$$r\varepsilon_\theta'\left(1 + \frac{rh_i'}{2h_i}\right) + g_i\varepsilon_\theta = \varepsilon_r\left(1 + \frac{rh_i'}{h_i}\right) \tag{45}$$

For Euclidean space, $h_i = 1$, $g_i = 1$ and Eq.(45) reduces to Eq.(43) which is the compatibility equation of the classical theory of elasticity.

Thus, we have now four equations, Eqs. (8)-(10) and Eq. (45). Presume that the sphere density is constant, so that $\rho = \rho_0$. Then, Eq. (9) transformed with the aid of Eq. (7) for χ and Eq. (21) for g_i which is valid for any material with constant density ρ_0 yields

$$\frac{h_i'}{h_i} = \frac{\overline{r}\,\overline{r}_g^0}{1-\overline{r}^2\overline{r}_g^0}\left(1-\frac{3\sigma_r}{\rho_0 c^2}\right) \tag{46}$$

Substituting this expression in Eq. (45) and using again Eq. (21) for g_i, we arrive at the following final equation generalizing the compatibility equation, Eq.(43), of the theory of elasticity:

$$(r\varepsilon_\theta)' - \varepsilon_r = 2\varphi(\overline{r})\left[\varepsilon_r - \varepsilon_\theta - \frac{\overline{r}^2}{2}\varepsilon_\theta'\left(1-\frac{3\sigma_r}{\rho_0 c^2}\right) - 3\varepsilon_r\frac{\sigma_r}{\rho_0 c^2}\right] \tag{47}$$

in which, as earlier, $\overline{r} = r/R$, $(...)' = d(...)/d\overline{r}$ and

$$\varphi(\overline{r}) = \frac{\overline{r}\,\overline{r}_g^0}{2(1-\overline{r}^2\overline{r}_g^0)} \tag{48}$$

Now, Eq. (47) should be written in terms of stresses with the aid of Eqs.(40). Introducing the following notations

$$\overline{\sigma}_r = \frac{\sigma_r}{\rho_0 c^2}, \quad \overline{s} = \frac{2}{\rho_0 c^2}(\sigma_\theta - \sigma_r) \tag{49}$$

we can present the resulting equation as

$$\begin{gathered}\overline{r}(2G+\lambda)\overline{s}\,' + (2G+3\lambda)\overline{s} + 4G\,\overline{r}\overline{\sigma}_r' + \\ +\varphi(\overline{r})\{2(2G+3\lambda)\overline{s} + \overline{r}(1-3\overline{\sigma}_r)[(2G+\lambda)\overline{s}\,' + 4G\overline{\sigma}_r'] + 12\overline{\sigma}_r(G\overline{\sigma}_r - \lambda\overline{s})\} = 0\end{gathered} \tag{50}$$

To derive the second equation, apply Eq. (8). Substituting h_i from Eq.(46) and using notations in Eqs. (49), we finally arrive at

$$\overline{r}\overline{\sigma}_r' - \overline{s} - \varphi(\overline{r})(1-3\overline{\sigma}_r)(1-\overline{\sigma}_r) = 0 \tag{51}$$

Recall that $\varphi(\overline{r})$ entering Eqs. (50) and (51) is given by Eq. (48).

The obtained set of equations, Eqs.(50) and (51), includes two unknown functions $\overline{\sigma}_r(\overline{r})$ and $\overline{s}(\overline{r})$. The boundary conditions for the solid sphere specified by Eqs. (11). Using notations (49), we get

$$\overline{s}(\overline{r}=0) = 0, \quad \overline{\sigma}_r(\overline{r}=1) = 0 \tag{52}$$

Consider two particular cases.

First, assume that $\overline{r}_g^0$ is small in comparison with unity. Then, we can take $\varphi(\overline{r}) = 0$ in Eq. (50) and simplify it as

$$\overline{r}(2G+\lambda)\overline{s}\,' + (2G+3\lambda)\overline{s} + 4G\overline{r}\overline{\sigma}_r{}' = 0 \tag{53}$$

To simplify respectively Eq.(51), we first neglect $\overline{\sigma}_r$ in comparison with unity, so that

$$\overline{r}\overline{\sigma}_r - \overline{s} - \varphi(\overline{r}) = 0$$

Second, we neglect the term with $\overline{r}_g^0$ in the denominator of Eq.(48) for $\varphi(r)$. The final equation becomes

$$\overline{r}\overline{\sigma}_r{}' - \overline{s} - \frac{1}{2}\overline{r}^2\overline{r}_g^0 = 0 \tag{54}$$

Naturally, we cannot neglect the term with $\overline{r}_g^0$ in this equation, because the gravitation disappears in this case.

Expressing $\overline{s}$ from Eq.(54) and substituting it in Eq.(53), we get

$$\overline{r}\overline{\sigma}_r{}'' - 4\overline{\sigma}_r{}' = \frac{\overline{r}^2\overline{r}_g^0(6G+5\lambda)}{2(2G+\lambda)}$$

The solution of this equation satisfying the boundary conditions in Eqs.(52) is

$$\overline{\sigma}_r = -\frac{\overline{r}_g^0(6G+5\lambda)}{20(2G+\lambda)}(1-\overline{r}^2) \tag{55}$$

This solution corresponds to the classical elasticity and gravitation theories. Thus, for small $\bar{r}_g^0$, GTR reduces to the classical (Newton) gravitation theory.

Second, consider the perfect incompressible liquid with density ρ_0 discussed in Section 3.1. For the perfect liquid, the shear modulus $G = 0$. Then, Eq.(50) reduces to

$$\bar{r}\bar{s}' + 3\bar{s} + \varphi(\bar{r})[6\bar{s} + \bar{r}(1 - 3\bar{\sigma}_r)\bar{s}' - 12\bar{s}\bar{\sigma}_r] = 0$$

This equation has only trivial solution $\bar{s} = 0$ (this conclusion follows from the derivation of this equation). Thus, Eqs.(49) yields $\bar{\sigma}_r = \bar{\sigma}_\theta = -p$ and Eq.(51) coincides with the foregoing Eq.(23) for the perfect liquid which is considered in Section 3.1.

Return to the elastic sphere described by Eqs.(50) and (51). For numerical calculation, presume that the Poisson's ratio $\nu = 0$. Then, in accordance with Eqs.(41), $\lambda = 0$ and Eq.(50) takes the form

$$\bar{r}\bar{s}' + \bar{s} + 2\bar{r}\bar{\sigma}_r' + \varphi(\bar{r})[2\bar{s} + \bar{r}(1 - 3\bar{\sigma}_r)(\bar{s}' + 2\bar{\sigma}_r') + 6\bar{\sigma}_r^2] = 0 \quad (56)$$

The results of numerical integration of Eqs.(51) and (56) are presented in Figure 6 which demonstrates the dependences of $\bar{\sigma}_r$ on the radial coordinate for various values of the gravitational radius $\bar{r}_g^0$. The curve corresponding to $\bar{r}_g^0 = 8/9$, in contrast to the solution for the liquid in Eq.(25), does not demonstrate any singularity. However, this singularity appears at the sphere center while $\bar{r}_g^0$ approaches unity. Dependence of stress $\bar{\sigma}_r(\bar{r} = 0)$ on $\bar{r}_g^0$ is shown in Figure 7 with a solid line. Dashed line corresponds to the classical solution in Eq. (55). As can be seen, the stress corresponding to GTR rapidly increases in the vicinity of $\bar{r}_g^0 = 1$. As follows from the calculation, for $\bar{r}_g^0 = 0.99$, we have $\bar{\sigma}_r(0) = 4.6$ and for $\bar{r}_g^0 = 0.999$, the stress is $\bar{\sigma}_r(0) = 30.5$. In the vicinity of the gravitation radius, numerical integration does not converge, and we can expect that the solution is singular if the sphere radius can reach the gravitational radius. Return to Eq. (30) which is valid for any material of the sphere. As can be seen, $\bar{r}_g$ can reach unity, if the metric coefficient g_R on the surface of the sphere becomes infinitely high.

5. Conclusion

As follows from the foregoing analysis, singular metric coefficient cannot appear in the spherically symmetric static GTR problem for the liquid sphere (compressible or incompressible). The singularity can take place only for the field variable, i.e., for the pressure at the sphere center, and not for the metric coefficient, because the sphere radius cannot reach the gravitational radius. For this reason, the metric coefficient of the external field cannot be singular for the liquid sphere.

However, for the solid sphere, the field variables, i.e. the stresses, and the metric coefficient of the internal field can become singular because the gravitational radius is the limiting point for the sphere radius. The same is true for the metric coefficient of the external field surrounding the solid sphere.

The foregoing conclusions are valid only for the static problem under consideration.

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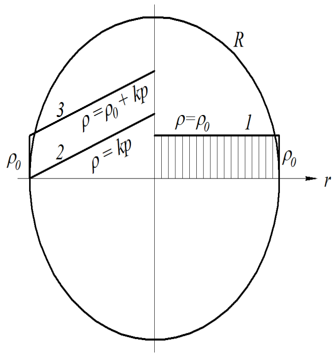

Figure 1. Dependences of the density on pressure

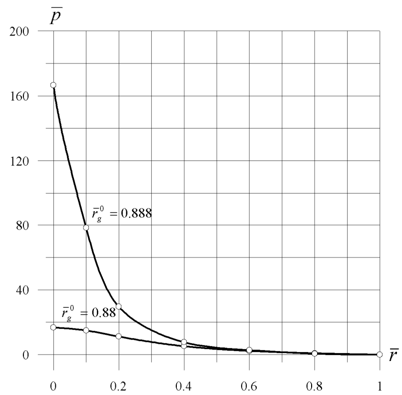

Figure 2.
Dependences of the normalized pressure on the radial coordinate for $\bar{r}_g^0=0.88$ and $\bar{r}_g^0=0.888$
— analytical solution
○ numerical solution

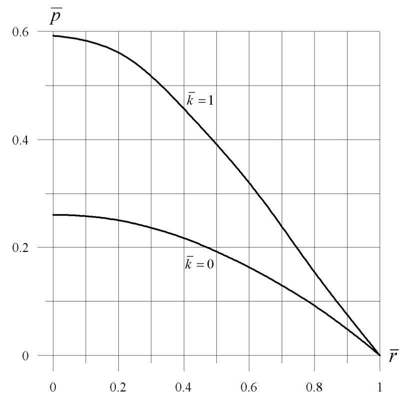

Figure 3.
Dependences of the normalized pressure on the radial coordinate for $\bar{r}_g^0=0.5$ and $\bar{k}=0$, $\bar{k}=1$

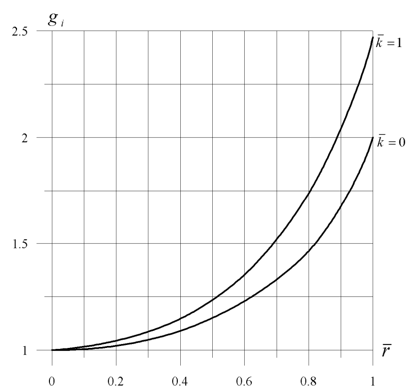

Figure 4.
Dependences of the metric coefficient on the radial coordinate for $\bar{r}_g^0=0.5$ and $\bar{k}=0$, $\bar{k}=1$

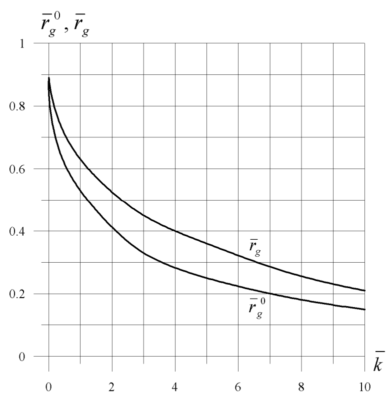

Figure 5.
Dependences of $\bar{r}_g^0$ and $\bar{r}_g$ on parameter $\bar{k}$

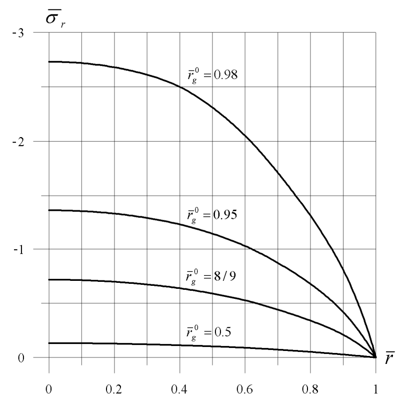

Figure 6.
Dependences of the normalized radial stress on the radial coordinate for various $\bar{r}_g^0$ values

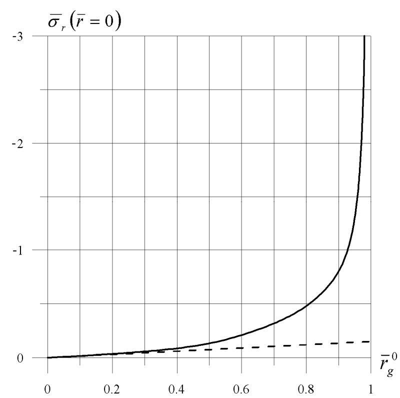

Figure 7.
Dependences of the radial stress at the sphere center on the gravitational radius
—— GTR solution
– – – Classical solution